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Appendix:
Chapterwise Solved January 2019 JEE Main Questions (All Sets) A.1–A.4

1 Thermometry, Thermal Expansion and Calorimetry

HEAT

Heat is the form of energy that is transferred between two bodies or between adjacent parts of a body at different temperatures. It is energy in transit whenever temperature differences occur. Once it is transferred, it becomes the internal energy of the receiving body. It should be clearly understood that the word "heat" is meaningful as long as the energy is being transferred. Thus, expressions like "heat in a body" or "heat of a body" are meaningless. When we say that a body is heated, it means that its molecules begin to move with greater kinetic energy.

TEMPERATURE AND ZEROTH LAW OF THERMODYNAMICS

It is observed that an object with a higher temperature comes in contact with an object of lower temperature, it will transfer heat to the lower temperature object. The objects will approach the same temperature and in the absence of loss to other objects, they will maintain a single constant temperature. Therefore, thermal equilibrium is attained. Two systems are said to be in thermal equilibrium if there is no net flow of heat between them when they are brought into thermal contact. The zeroth law of thermodynamics deals with the concept of thermal equilibrium. The Thermodynamics Zeroth Law states that if two systems are at the same time in thermal equilibrium with a third system, they are in equilibrium with each other.

If two objects A and B are separately in thermal equilibrium with a third object C, then A and B are in thermal equilibrium with each other. Practically this means all three objects are at the same temperature and it forms the basis for comparison of temperatures.

In less formal language, the message of the zeroth law is: "Every body has a property called **temperature**. When two bodies are in thermal equilibrium, their temperatures are equal and vice versa." We can use a thermoscope (the third body T) as a thermometer being confident that its readings will have physical meaning. All we have to do is to calibrate it.

Important Points:

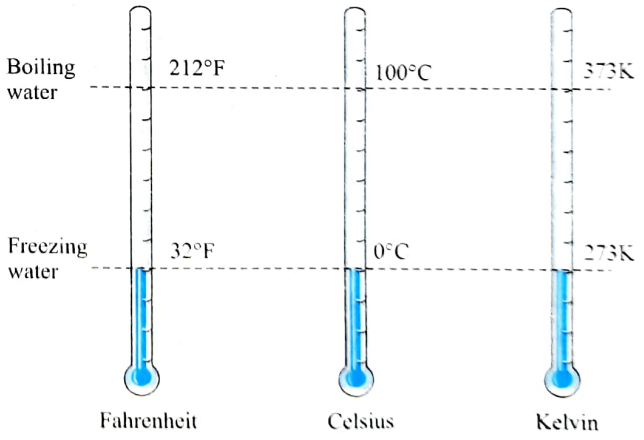
- Temperature is one of the seven fundamental quantities with dimension $[\theta]$.
- It is a scalar physical quantity with SI unit Kelvin.
- When heat is given to a body and its state does not change, the temperature of the body rises and if heat is taken from a body its temperature falls i.e., temperature can be regarded as the effect of cause "heat".
- According to kinetic theory of gases, temperature (macroscopic physical quantity) is a measure of average translational kinetic energy of a molecule (microscopic physical quantity).

  Temperature $\propto$ kinetic energy $\quad \left[\text{As } E = \frac{3}{2}RT\right]$
- Although the temperature of a body can be raised without limit, it cannot be lowered without limit and theoretically limiting low temperature is taken to be zero on the Kelvin scale.

SCALES OF TEMPERATURE

To construct a scale of temperature, two fixed points are taken. First fixed point is the freezing point (ice point) of water, it is called lower fixed point (LFP). Second fixed point is the boiling point (steam point) of water, it is called upper fixed point (UFP).

Centigrade or Celsius (°C), Fahrenheit (°F) and Kelvin (K), are commonly used temperature scales.



Celsius scale: In this scale LFP (ice point) is taken 0° and UFP (steam point) is taken 100°. The temperatures measured on this scale are in degree Celsius (°C).

Fahrenheit scale: This scale of temperature has LFP as 32°F and UFP as 212°F. The change in temperature of 1°F corresponds to a change of less than 1° on Celsius scale.

Kelvin scale: The Kelvin temperature scale is also known as thermodynamic scale. The triple point of water is also selected to be the zero of scale of temperature. The temperatures measured on this scale are in Kelvin (K).

Different measuring scales

Scale	Symbol for each degree	Lower fixed point (LFP)	Upper fixed point (UFP)	Number of divisions on the scale
Celsius	°C	0°C	100°C	100
Fahrenheit	°F	32°F	212°F	180
Kelvin	K	273.15 K	373.15 K	100

Temperature on one scale can be converted into other scale by using the following identity:

$$\frac{\text{Reading on any scale} - \text{LFP}}{\text{UFP} - \text{LFP}} = \text{Constant for all scales}$$

All these temperatures are related to each other by the following relationship:

$$\frac{T_C - 0}{100} = \frac{T_F - 32}{212 - 32} = \frac{T_K - 273.15}{373.15 - 273.15} = \frac{T - LFP}{UFP - LFP}$$

where T = temperature at any scale

The relation between Celsius, Fahrenheit and Kelvin scale can be written as:

$$\frac{T_C}{5} = \frac{T_F - 32}{9} = \frac{T_K - 273}{5}$$

The Celsius and Kelvin scales have different zero points but the same size degrees. Therefore any temperature difference is the same on the Celsius and Kelvin scales $(T_2 - T_1)$ °C = $(T_2 - T_1)$K.

Note that we do not use a degree mark in reporting Kelvin temperatures. It is 300 K (not 300°K), and it is read "300 kelvins" (not "300 degree Kelvin").

The Fahrenheit scale, used in the United States, employs a smaller degree than the Celsius scale and a different zero of temperature. The relation between the Celsius and Fahrenheit scales is

$$T_F = \frac{9}{5}T_C + 32°, \text{ where } T_F \text{ is Fahrenheit temperature}$$

THERMOMETRY

An instrument used to measure the temperature of a body is called a thermometer. As the temperature is measured by the value of the thermodynamic property of a substance i.e., a property which varies linearly with the temperature, two fixed points are needed to define a temperature scale.

The linear variation in some physical properties of a substance with change of temperature is the basic principle of thermometry and these properties are defined as thermometric properties (x) of the substance. x may be

(i) length of liquid in capillary
(ii) pressure of gas at constant volume
(iii) volume of gas at constant pressure
(iv) resistance of a given platinum wire

In old thermometry, two arbitrarily fixed points, ice and steam point (freezing point and boiling point at 1 atm), are taken to define the temperature scale.

In Celsius scale, freezing point of water is assumed to be 0°C while boiling point is taken as 100°C and the temperature interval between these is divided into 100 equal parts.

So if the thermometric properties at temperature 0°C, 100°C and T_0°C are x_0, x_{100} and, x, respectively, then by linear variation ($y = mx + c$) we can say that

$$0 = mx_0 + c \quad \ldots(i)$$
$$100 = mx_{100} + c \quad \ldots(ii)$$
$$T_c = mx + c \quad \ldots(iii)$$

From these equations $\frac{T_C - 0}{100 - 0} = \frac{x - x_0}{x_{100} - x_0}$

$$\therefore \quad T_C = \frac{x - x_0}{x_{100} - x_0} \times 100°C$$

In modern thermometry, instead of two fixed points, only one reference point is chosen i.e., the triple point of water. The other is itself 0 K where the value of thermometric property is assumed to be zero. The triple point of water on Celsius scale is 0.01°C. Thus the absolute temperature T for triple point of water will be given by

$$T = t_c + 273.15 = 0.01 + 273.15 = 273.16 \text{ K}$$
$$\therefore \quad T = 273.16 \text{ K (triple point of water)}$$

So if the values of thermometric property at 0 K, 273.16 K and T_K K are 0, x_{Tr} and x, respectively, then by linear variation ($y = mx + c$) we can say that

$$0 = m \times 0 + c \quad \ldots(i)$$
$$273.16 = m \times x_{Tr} + c \quad \ldots(ii)$$
$$T_K = m \times x + c \quad \ldots(iii)$$

From these equations $\frac{T_K}{273.16} = \frac{x}{x_{Tr}}$

$$\therefore \quad T_K = 273.16\left[\frac{x}{x_{Tr}}\right]\text{kelvin}$$

ILLUSTRATION 1.1

Suppose you come across old scientific notes that describe a temperature scale called Z on which the boiling point of water is 65.0°Z and the freezing point is −14.0°Z.

To what temperature on the Fahrenheit scale would a temperature of T = −98.0°Z correspond? Assume that the Z scale is linear; that is, the size of a Z degree is the same everywhere on the Z scale.

Sol. For making a conversion factor between two temperature scales we use two known (benchmark) temperatures, such as the boiling and freezing points of water.

The number of degrees between the known temperatures of one scale is equivalent to the number of degrees between them on the other scale.

The unknown temperature T on the Z scale (T = −98.0°Z) is closer to the freezing point (−14.0°Z) than to the boiling point (65.0°Z), we use the freezing point as reference point .The difference of T with respect to the freezing point (−14.0°Z).

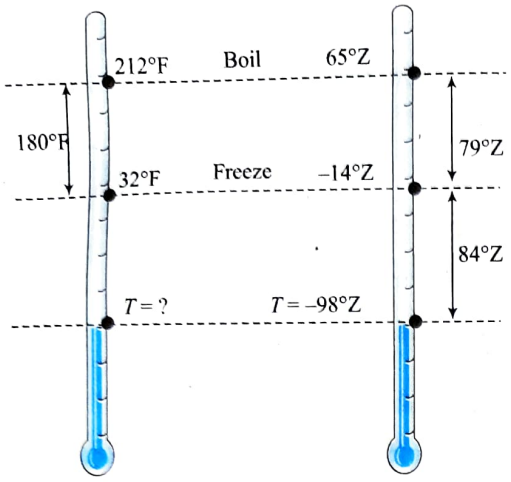
$$\Delta T = (-14.0°Z) - (-98.0°Z) = 84.0°Z$$

Next, we set up a conversion factor between the Z and Fahrenheit scales to convert this difference. To do so, we use both known temperatures on the Z scale and the corresponding temperatures on the Fahrenheit scale.

On the Z scale, the difference between the boiling and freezing points is $\Delta T_Z = (65.0°Z) - (-14.0°Z) = 79.0°Z$

On the Fahrenheit scale, the difference between the boiling and freezing points is $\Delta T_F = (212°\,F) - (32°\,F) = 180°F$.

Thus, a temperature difference of 79.0°Z is equivalent to a temperature difference of 180 °F (figure), and we can use the ratio (180 °F)/(79.0 °Z) as our conversion factor.



Now, since T is below the freezing point by 84.0°Z on Z scale, hence it must also be below the freezing point on the Fahrenheit scale by $(84.0°Z)\dfrac{180°F}{79.0°Z} = 191°F$.

Since the freezing point is at 32.0°F, we have

$$T_F = 32°F - 191°F = -159°F$$

Alternatively:

Using $\dfrac{T_F - 32}{212 - 32} = \dfrac{T_Z - LFP}{UFP - LFP}$

$$\frac{F-32}{212-32} = \frac{(-98)-(-14)}{65-(-14)} \Rightarrow F = -159°F$$

ILLUSTRATION 1.2

What will be the values of the following temperatures on the Kelvin scale: **(a)** 37°C, **(b)** 80°F, **(c)** –196°C?

Sol.

(a) Temperature on Kelvin scale T_K is related to temperature T_C on Celsius scale as

$$T_K = T_c + 273$$

or $T_K = 37 + 273 = 310$ K

(b) Temperatures T_K on Kelvin scale and T_F on Fahrenheit scale are related as

$$\frac{T_K - 273}{373 - 273} = \frac{T_F - 32}{212 - 32}$$

or $$\frac{T_K - 273}{100} = \frac{T_F - 32}{180}$$

or $$T_K = \frac{5}{9}(T_F - 32) + 273$$

here $T_F = 80°F$; thus

$$T_K = \frac{5}{9}(80 - 32) + 273 = 299.66 \text{ K}$$

(c) Again from relation used in part (a)

$$T_K = T_C + 273 = -196 + 273 = 77 \text{ K}$$

ILLUSTRATION 1.3

Two ideal gas thermometers A and B use oxygen and hydrogen, respectively. The following observations are made:

Temperature	Pressure thermometer A	Pressure thermometer B
Triple point of water	1.250×10^5 Pa	0.200×10^5 Pa
Normal melting point of sulphur	1.797×10^5 Pa	0.287×10^5 Pa

(a) What is the absolute temperature of normal melting point of sulphur as read by thermometer A and B?

(b) What do you think is the reason for slightly different answers from A and B?

Sol.

(a) For thermometer A,

$$T_{tr} = 273.16 \text{ K}, \; P_{tr} = 1.250 \times 10^5 \text{ Pa}$$

We have $T = \dfrac{P}{P_{tr}} \times T_{tr} = \dfrac{1.797 \times 10^5}{1.250 \times 10^5} \times 273 = 392.46 \text{ K}$

For thermometer B, $T_{tr} = 273.16$ K, $P_{tr} = 0.200 \times 10^5$ Pa

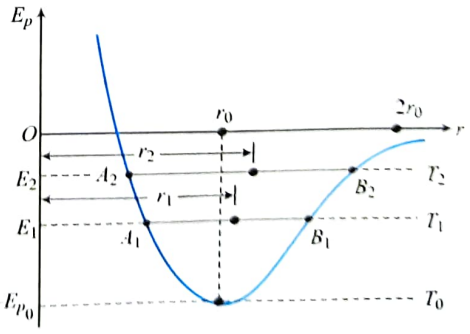
We have $T = \dfrac{P}{P_{tr}} \times t_{tr} = \dfrac{0.287 \times 10^5 \times 273.16}{0.200 \times 10^5} = 391.98 \text{ K}$

(b) The slight difference in the temperatures as read by two thermometers is due to the fact that oxygen and hydrogen do not behave like an ideal gas.

THERMAL EXPANSION

When matter is heated without any change in its state, it usually expands. According to atomic theory of matter, asymmetry in potential energy curve is responsible for thermal expansion. As with rise in temperature the amplitude of vibration and hence energy of atoms increases, hence the average distance between the atoms increases. So the matter as a whole expands.

On the atomic scale, thermal expansion can be explained due to an increase in the interatomic separation with the increase in temperature. The potential energy versus interatomic separation curve for an actual crystal is asymmetric as is shown in figure.



If we plot the total (vibration) energies, i.e., E_1 and E_2 of the atoms for temperatures T_1 and T_2 on this curve.

(i) At $T = T_0 = 0$ K, the atoms have only potential energy ($-E_{p_0}$). In this case, $r = r_0$ and the atoms are at fixed lattice points. They are not allowed to vibrate as no other value of r is allowed.

(ii) At $T = T_1$, the atoms gain vibrational energy E_1 and as a result of this vibrate between the points A_1 and B_1. In this case, the average distance between the atoms r_1 where $r_1 > r_0$. As a result of this, the solid expands.

(iii) $T = T_2$, the energy of the atoms further increases by an amount E_2. Consequently, the atoms vibrate between the points A_2 and B_2. The average distance between the atoms is r_2 where $r_2 > r_1$. Obviously, the solid has expanded further.

Thus, due to the non-symmetrical shape of the potential energy curve, the average distance between the positions of equilibrium of the atoms of a solid increases with an increase in temperature which results in the thermal expansion of the solid.

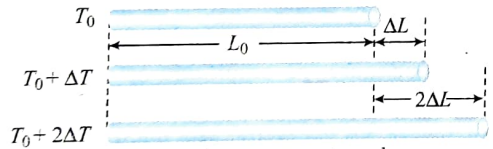
Important Points:

- Thermal expansion is minimum in case of solids but maximum in case of gases because intermolecular force is maximum in solids but minimum in gases.
- Solids can expand in one dimension (linear expansion), two dimensions (superficial expansion) and three dimensions (volume expansion) while liquids and gases usually suffer change in volume only.

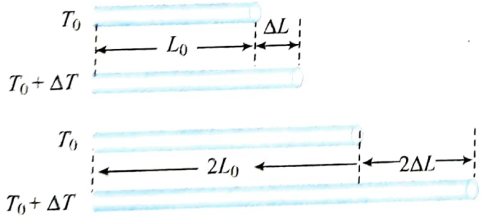
LINEAR EXPANSION

When a solid is heated and its length increases, then the expansion is called linear expansion.

Suppose a rod of material has a length L_0 at some initial temperature T_0. When the temperature changes by ΔT, the length changes by ΔL. Experiments show that if ΔT is not too large (say, less than 100°C or so), ΔL is **directly proportional** to ΔT. If two rods made of the same material have the same temperature change, but one is twice as long as the other, then the change in its length is also twice. It means ΔL must also be proportional to L_0.



(a) For moderate temperature changes, ΔL is directly proportional to ΔT.



(b) ΔL is also directly proportional to L_0

Introducing a proportionality constant α (it describes the thermal expansion properties of a particular material and is called the **coefficient of linear expansion**), we may express these relationships in an equation:

Change in length $\Delta L = \alpha L_0 \Delta T$

(L_0 = Original length, ΔT = Temperature change)

Its final length L at a temperature $T = T_0 + \Delta T$

$L = L_0 (1 + \alpha \Delta T)$

Above equation can be used for both thermal expansion, (when the temperature of the material increases) and thermal contraction (when its temperature decreases).

We define the **average coefficient of linear expansion** as

$$\alpha = \frac{\Delta L}{L_0 \Delta T}$$

Experiments show that α is constant for small change in temperature. The unit of α is $°C^{-1}$ or K^{-1}. Its dimension is $[\theta^{-1}]$.

Variation of α with temperature and distance:

If α varies with distance, $\alpha = ax + b$

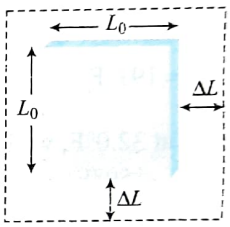
Then total expansion, $\Delta L = \int (ax + b) \cdot dx \cdot \Delta T$

If α varies with temperature, $\alpha = f(T)$, then

$$\Delta L = \int \alpha L_0 dT = \int f(T) L_0 dT$$

SUPERFICIAL (AREAL) EXPANSION

When the temperature of a two-dimensional object is changed, its area changes, then the expansion is called superficial expansion. A two-dimensional object, such as a thin metallic plate, changes area when its temperature is raised or lowered.



For an isotropic material the area expansion may be expressed in terms of the linear expansion coefficient α. Consider a rectangular area of dimensions l_1 and l_2.

The area, $A_0 = l_1 \times l_2$

Differentiating both sides w.r.t. T, we get

$$\frac{dA}{dT} = l_1 \frac{dl_2}{dT} + l_2 \frac{dl_1}{dT}$$

$$\frac{dA}{dT} = l_1(\alpha l_2) + l_2(\alpha l_1) = \alpha l_1 l_2 + \alpha l_1 l_2$$

$$dA = 2\alpha l_1 l_2 \, dT \quad \ldots(i)$$

If α is constant over the temperature range of under consideration then equation (i) can be integrated.

$$\int_{A_i}^{A_f} dA = 2\alpha l_1 l_2 \int_{T_i}^{T_f} dT \Rightarrow \Delta A = 2\alpha A_0 \, \Delta T$$

(A_0 = Original area, ΔT = Temperature change)

or $\Delta A = \beta A_0 \Delta T$, where β is coefficient of superficial expansion twice the coefficient of linear expansion α.

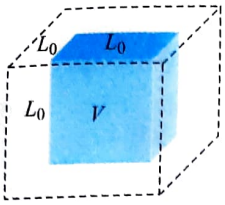
Final area of the plate, $A = A_0 (1 + \beta \Delta T)$

The average co-efficient of superficial expansion, $\beta = \dfrac{\Delta A}{A_0 \Delta T}$

The unit of β is $°C^{-1}$ or K^{-1}. Its dimension is $[\theta^{-1}]$.

VOLUME OR CUBICAL EXPANSION

When a solid is heated and its volume increases, then the expansion is called volume or cubical expansion.



Consider a rectangular parallelepiped of sides l_1, l_2, and l_3.
Its initial volume, $V_0 = l_1 \times l_2 \times l_3$
Differentiating both sides w.r.t. T, we get

$$\frac{dV}{dT} = l_2 l_3 \frac{dl_1}{dT} + l_1 l_3 \frac{dl_2}{dT} + l_1 l_2 \frac{dl_3}{dT}$$

$$\frac{dV}{dT} = l_2 l_3 (\alpha l_1) + l_1 l_3 (\alpha l_2) + l_1 l_2 (\alpha l_3) = 3\alpha l_1 l_2 l_3$$

$$dV = 3\alpha V_0 \, dT \qquad \ldots(i)$$

If γ is constant over the temperature range of under consideration, then equation (i) can be integrated.

$$\int_{V_i}^{V_f} dV = 3\alpha V_0 \int_{T_i}^{T_f} dT \Rightarrow \Delta V = 3\alpha V_0 \, \Delta T$$

or $\Delta V = \gamma V_0 \Delta T$, where γ is coefficient of cubical expansion, thrice the coefficient of linear expansion α.
Final area of the parallelepiped, $V = V_0 (1 + \gamma \Delta T)$

The average coefficient of cubical expansion, $\gamma = \dfrac{\Delta V}{V_0 \Delta T}$ and unit is same as α i.e., $^\circ C^{-1}$ or K^{-1}.

Important Points:

- Values of α, β, γ are independent of the units of length, area and volume respectively.
- The three coefficients of expansion are not constant for a given solid. Their values depend on the temperature range in which they are measured.
- The coefficient α, β and γ for a solid are related to each other as follows:

  $$\alpha = \frac{\beta}{2} = \frac{\gamma}{3} \Rightarrow \alpha : \beta : \gamma = 1 : 2 : 3$$

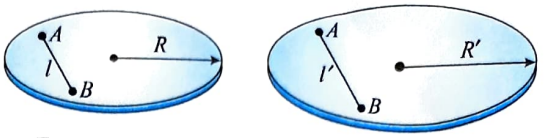
  Hence for the same rise in temperature
  Percentage change in area = 2 × percentage change in length
  Percentage change in volume = 3 × percentage change in length
- For anisotropic solids $\gamma = \alpha_x + \alpha_y + \alpha_z$ where α_x, α_y, and α_z represent the mean coefficients of linear expansion along three mutually perpendicular directions.

The relation of linear expansion $L = L_0 (1 + \alpha \Delta T)$ is not only valid for the length of an object but it can also be used to obtain the enclosed length of an object or distance between two points on an object. For example consider the disc of radius R as shown in figure. If temperature of this disc increases, then its size will also increase, and we can write the final radius of the disc as

$$R' = R(1 + \alpha \Delta T) \qquad \ldots(i)$$

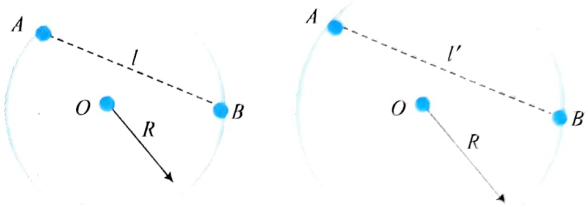
Similarly we can find the distance between any two points on a disc. Also separation between two points A and B situated on, the disc we can write

$$l' = l(1 + \alpha \Delta T)$$



Temperature = T Temperature = $T + \Delta T$

Let us take a metal rod bent in the form of a circle of radius R. We consider two points on it marked as A and B. Let the separation between the two is l at a temperature T. If the temperature of rod is increased by ΔT. It will expand and hence the length l will also increase and become l'. The separation between A and B follows the similar law of linear expansion.



Temperature = T Temperature = $T + \Delta T$

Thus l' can be given as $l' = l(1 + \alpha \Delta T)$
And new radius of the circle, $R' = R(1 + \alpha \Delta T)$
where α is the coefficient of linear expansion of material of rod.

Similarly, the same result can also be used for lengths and dimensions of an object enclosed by a material like radius of a hole in a disc or radius of a spherical cavity in a solid object.

ILLUSTRATION 1.4

A brass rod of length 50 cm and diameter 3.0 cm is joined to a steel rod of the same length and diameter. What is the change in length of the combined rod at 250°C, if the original lengths are at 40.0°C? Is there any stress developed at the junction? The ends of the rod are free to expand.

(Coefficient of linear expansion of brass is $2.0 \times 10^{-5}/^\circ C$, and that of steel is $1.2 \times 10^{-5}/^\circ C$.)

Sol. Change in temperature of each rod, $\Delta T = 250 - 40 = 210^\circ C$
Clearly, change in length of the brass rod

$$\Delta L_b = \alpha_0 L \, \Delta T = (2.0 \times 10^{-5}) \times 50 \times 210 = 0.21 \text{ cm}$$

And change in length of steel rod

$$\Delta L_S = \alpha_S L \, \Delta T = (1.2 \times 10^{-5}) \times 50 \times 210 = 0.126 \text{ cm}$$

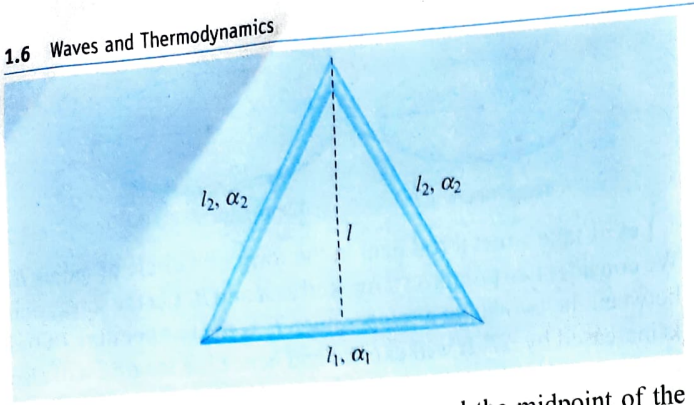
Since the ends of the rods are free to expand, change in the length of the combined rod

$$\Delta L = \Delta L_b + \Delta L_S = 0.21 + 0.13 = 0.34 \text{ cm}$$

Further, as the combined rod is free to expand, no stress is developed at the junction.

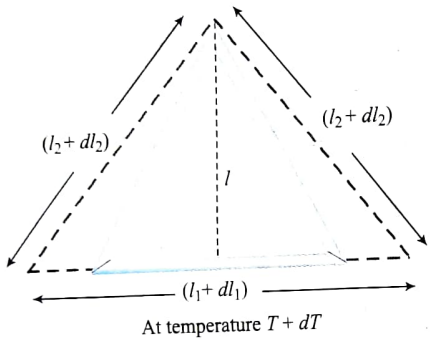
ILLUSTRATION 1.5

An isosceles triangle is formed with a thin rod of length l_1 and coefficient of linear expansion α_1, as the base and two thin rods each of length l_2 and coefficient of linear expansion α_2 as the two sides. If the distance between the apex and the midpoint of the base remain unchanged as the temperature varies, show that $\dfrac{l_1}{l_2} = 2\sqrt{\dfrac{\alpha_2}{\alpha_1}}$.



Sol. The distance between the apex and the midpoint of the base, using Pythagoras theorem, $l = \sqrt{(l_2)^2 - \left(\frac{l_1}{2}\right)^2}$

or $l^2 = (l_2)^2 - \left(\frac{l_1}{2}\right)^2$ ...(i)



At temperature $T + dT$

Differentiating Eq. (i) w.r.t. temperature T

$$0 = 2l_2 \times \frac{dl_2}{dT} - 2\left(\frac{l_1}{2}\right)\cdot\frac{1}{2}\times\frac{dl_1}{dT}$$

$$\Rightarrow \quad \frac{l_1}{2}\times l_1\alpha_1 = 2l_2 \times l_2\alpha_2$$

$$\Rightarrow \quad \left(\frac{l_1}{l_2}\right)^2 = 4\frac{\alpha_2}{\alpha_1} \Rightarrow \frac{l_1}{l_2} = 2\sqrt{\frac{\alpha_2}{\alpha_1}}$$

ILLUSTRATION 1.6

A metal rod A of length 25 cm expands by 0.05 cm when its temperature is raised from 0°C to 100°C. Another rod B of a different metal of length 40 cm expands by 0.04 cm for the same rise in temperature. A third rod C of length 50 cm made up of pieces of rods A and B placed end to end expands by 0.03 cm on heating from 0°C to 50°C. Find the length of each portion of composite rod C.

Sol. From the given data for rod A, we have $\Delta L = \alpha_A L\,\Delta T$

or $\alpha_A = \frac{\Delta L}{L\Delta T} = \frac{0.05}{25\times100} = 2\times10^{-5}/°C$

For rod B, we have $\Delta L = \alpha_B L\Delta T$

or $\alpha_B = \frac{\Delta L}{L\Delta T} = \frac{0.04}{40\times100} = 10^{-5}/°C$

If rod C is made of segments of rod A and B of lengths l_1 and l_2, respectively, then we have at 0°C

$l_1 + l_2 = 50\,\text{cm}$ ...(i)

At $T = 50°C$ $\quad l_1' + l_2' = 50.03$ cm

Thus $\alpha_A l_1\Delta T + \alpha_B l_2\Delta T = 0.03$ cm

or $2\times10^{-5}\times l_1\times50 + 10^{-5}\times l_2\times50 = 0.03$ cm

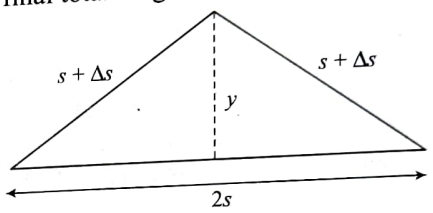
or $2l_1 + l_2 = \frac{0.03}{50}\times10^5 = 60$ cm ...(ii)

Solving Eqs. (i) and (ii), we get $l_1 = 10$ cm and $l_2 = 40$ cm.

ILLUSTRATION 1.7

A steel rail 30 m long is firmly attached to the roadbed only at its ends. The sun raises the temperature of the rail by 50°C, causing the rail to buckle. Assuming that the buckled rail consists of two straight parts meeting in the centre, calculate how much the centre of the rail rises. Coefficient of linear expansion of steel is 12×10^{-6}/K.

Sol. As indicated in the given figure, we let the initial length be $2s$ and the final total length be $2(s + \Delta s)$.



The height of the centre of the buckled rail is denoted by y. Assuming that the standard coefficient α of linear expansion can be used (in spite of the fact that the ends are anchored), we have $\Delta s = \alpha s\Delta T$.

By the Pythagorean theorem

$$y = \sqrt{(s+\Delta s)^2 - s^2} = \sqrt{2s\Delta s + (\Delta s)^2} = s\sqrt{2\alpha\Delta T + (\alpha\Delta T)^2}$$

With $s = 15.0$ m, $\alpha = 12\times10^{-6}$/K, and $\Delta T = 50$ K, we obtain

$$y = (15.0\text{ m})\sqrt{(12\times10^{-4}) + (6\times10^{-4})^2} = 0.52\text{ m}$$

BI-METALLIC STRIP

Two strips of equal lengths but of different materials (different coefficient of linear expansion) when join together, it is called "bi-metallic strip", and can be used in thermostat to break or make electrical contact. This strip has the characteristic property of bending on heating due to unequal linear expansion of the two metals. The strip will bend with metal of greater α on outer side i.e., convex side.

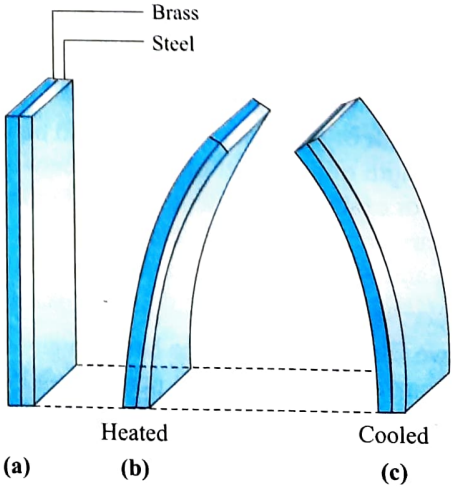
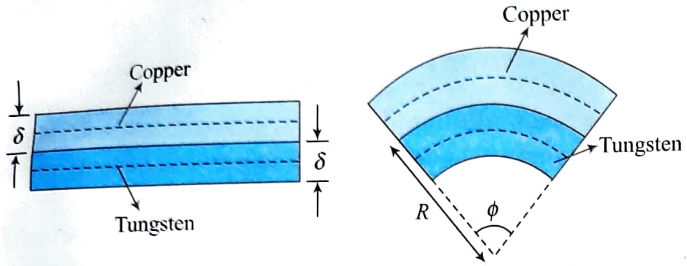


ILLUSTRATION 1.8

A copper plate and a tungsten plate having a thickness δ each are riveted together so that at 0°C they form a flat bimetallic plate. Find the average radius of the curvature of this plate at temperature T. The coefficients of linear expansion for copper and tungsten are α_C and α_t.

Sol. The average length of copper plate at temperature T is $l_C = l_0(1 + \alpha_C T)$, where l_0 is the length of copper plate at 0°C. The length of the tungsten plate is $l_t = l_0(1 + \alpha_t T)$



We shall assume that the edges of plates are not displaced during deformation and that an increase in the plate thickness due to heating can be neglected.

From figure, we have

Length of copper plates, $l_C = \phi(R + \delta/2)$

and length of tungsten plate, $l_t = \phi(R - \delta/2)$

Consequently, $\phi(R + \delta/2) = l_0(1 + \alpha_C T)$ (i)

$\phi(R - \delta/2) = l_0(1 + \alpha_t T)$ (ii)

To eliminate the unknown quantities, ϕ and l_0, we divide the equation (i) by (ii) term wise and get

$$\frac{(R+\delta/2)}{(R-\delta/2)} = \frac{(1+\alpha_C T)}{(1+\alpha_t T)}$$

$$\Rightarrow \quad R = \delta\frac{\left[2+(\alpha_c+\alpha_t)T\right]}{\left[2(\alpha_c-\alpha_t)T\right]}$$

$$\Rightarrow \quad R = \frac{\delta}{(\alpha_C-\alpha_t)T} \qquad \text{[Neglecting } (\alpha_c + \alpha_t) \text{ in numerator]}$$

ILLUSTRATION 1.9

We would like to prepare a scale whose length does not change with temperature. It is proposed to prepare a unit scale of this type whose length remains, say 10 cm. We can use a bimetallic strip made of brass and iron strips of different lengths whose length (both components) would change in such a way that difference between their lengths remain constant. If $\alpha_{iron} = 1.2 \times 10^{-5}$/K and $\alpha_{brass} = 1.8 \times 10^{-5}$/K, what should we take as lengths of each strip?

Sol. Let $l_{0,iron}$ and $l_{0,brass}$ be the initial lengths of iron and brass strips, respectively. According to question the difference between their lengths remains constant. Now, after change in the temperature by Δt, the difference in the lengths of the strips will be

$$l_{0,\text{iron}} - l_{0,\text{brass}} = l_{\text{iron}} - l_{\text{brass}} = 10 \text{ cm} \qquad \ldots(i)$$

$$\Rightarrow \quad l_{0,\text{iron}}(1 + \alpha_{\text{iron}}\Delta t) - l_{0,\text{brass}}(1 + \alpha_{\text{brass}}\Delta t) = l_{0,\text{iron}} - l_{0,\text{brass}}$$

$$\Rightarrow \quad l_{0,\text{iron}}\,\alpha_{\text{iron}} = l_{0,\text{brass}}\,\alpha_{\text{brass}}$$

$$\Rightarrow \quad \frac{l_{0,\text{iron}}}{l_{0,\text{brass}}} = \frac{1.8}{1.2} = \frac{3}{2} \qquad \ldots(ii)$$

$$l_{0,\text{iron}} = \left(\frac{3}{2}\right) l_{0,\text{brass}}$$

From (i) and (ii), we get

$$\left(\frac{3}{2}\right) l_{0,\text{brass}} - l_{0,\text{brass}} = 10 \text{ cm}$$

$$\Rightarrow \quad \frac{1}{2} l_{0,\text{brass}} = 10 \text{ cm}$$

$$\Rightarrow \quad l_{0,\text{brass}} = 20 \text{ cm}$$

$$\therefore \quad l_{0,\text{iron}} = 30 \text{ cm}$$

EXPANSION OF CAVITY

If a solid object has a hole in it, what happens to the size of the hole when the temperature of the object increases? A common misconception is that if the object expands, the hole will shrink because material expands into the hole. But the truth of the matter is that if the object expands, the hole will expand too. When an object undergoes thermal expansion, any holes in the object expand as well.

Thermal expansion of an isotropic object may be imagined as a photographic enlargement. So if there is a hole A in a plate C (or cavity A inside a body C), the area of hole (or volume of cavity) will increase when body expands on heating, just as if the hole (or cavity) were solid plate of the same material.

$l' = l_0(1 + \alpha\Delta T)$ and $R' = R_0(1 + \alpha\Delta T)$

While applying the concept of linear expansion in such cases actually the length is that of air section but we use the coefficient of linear expansion for the material enclosing this air section. Now we take some example for understanding the linear expansion in detail.

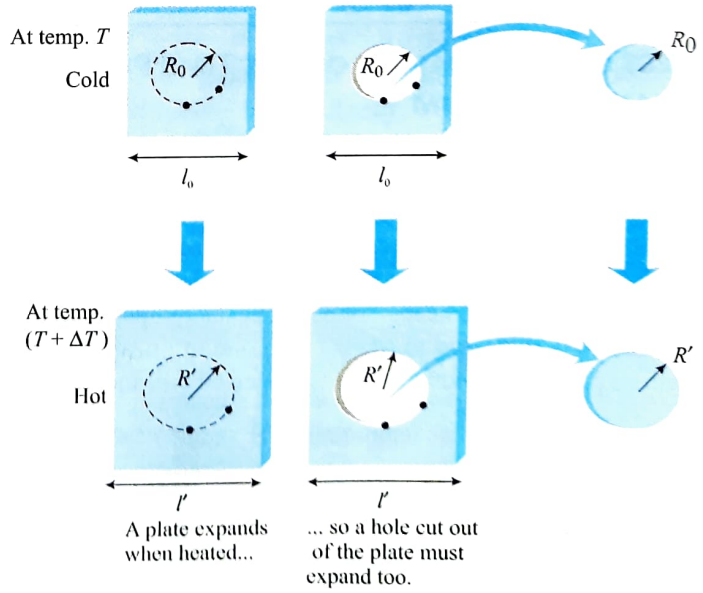
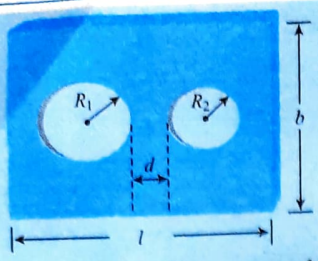


ILLUSTRATION 1.10

The figure shows a rectangular plate of dimension $(l \times b)$ from which two circular holes of radii R_1 and R_2 $(R_1 > R_2)$ have been cut. The distance between the two holes is d $(< R_2)$.



(a) What happens to all these distances and dimensions when the plate is heated up?

(b) If l', b', R_1', R_2' and d' are the respective lengths at a higher temperature, then determine the relations between the ratios $\left(\frac{l'}{l}\right), \left(\frac{b'}{b}\right), \left(\frac{R_1'}{R_1}\right), \left(\frac{R_2'}{R_2}\right)$ and $\left(\frac{d'}{d}\right)$.

Sol.

(a) Thermal expansion of an isotropic object may be imagined as a photographic enlargement. Hence, all the dimensions increases as the temperature is increased.

(b) Let the temperature increase be ΔT.

Then $l' = l(1+\alpha\Delta T) \Rightarrow \frac{l'}{l} = (1+\alpha\Delta T)$

$b' = b(1+\alpha\Delta T) \Rightarrow \frac{b'}{b} = (1+\alpha\Delta T)$

$R_1' = R_1(1+\alpha\Delta T) \Rightarrow \frac{R_1'}{R_1} = (1+\alpha\Delta T)$

$R_2' = R_2(1+\alpha\Delta T) \Rightarrow \frac{R_2'}{R_2} = (1+\alpha\Delta T)$

$d' = d(1+\alpha\Delta T) \Rightarrow \frac{d'}{d} = (1+\alpha\Delta T)$

Hence all these ratios are equal.

$$\frac{l'}{l} = \frac{b'}{b} = \frac{R_1'}{R_1} = \frac{R_2'}{R_2} = \frac{d'}{d} = 1+\alpha\Delta T = \text{constant}$$

Effect of Temperature on the Time Period of a Simple Pendulum

The time represented by the clock hands of a pendulum clock depends on the number of oscillations performed by pendulum. Every time it reaches to its extreme position. The second hand of the clock advances by one second that means second hand moves by two seconds when one oscillation is complete.

A pendulum clock keeps proper time at temperature θ_0. If temperature is increased to $\theta(>\theta_0)$ then due to linear expansion, length of pendulum and hence its time period will increase.

Let $T = 2\pi\sqrt{\frac{L_0}{g}}$ at temperature θ_0 and $T' = 2\pi\sqrt{\frac{L}{g}}$ at temperature θ.

$$\frac{T'}{T} = \sqrt{\frac{L'}{L}} = \sqrt{\frac{L[1+\alpha\Delta\theta]}{L}} = 1+\frac{1}{2}\alpha\Delta\theta$$

Therefore, change (loss or gain) in time per unit time lapsed is $\frac{T'-T}{T} = \frac{1}{2}\alpha\Delta\theta$; gain or loss in time in duration of 'T' is $\Delta T = \frac{1}{2}\alpha\Delta\theta T$. If T is the correct time then

(a) $\theta < \theta_0$, $T' < T$; clock becomes fast and gain time

(b) $\theta > \theta_0$, $T' > T$; clock becomes slow and lose time

Fractional change in time period, $\frac{\Delta T}{T} = \frac{1}{2}\alpha\Delta\theta$

Important Points:

- Due to increment in its time period, a pendulum clock becomes slow in summers and will lose time.
- Loss of time in a time period, $\Delta t = \frac{1}{2}\alpha\Delta\theta t$
- Time lost by the clock in a day ($t = 86400$ sec),
$$\Delta t = \frac{1}{2}\alpha\Delta\theta t = \frac{1}{2}\alpha\Delta\theta\,(86400)\text{ sec}$$
- The clock will lose time i.e., will become slow if $\theta' > \theta$ (in summers) and will gain time i.e., will become fast if $\theta' < \theta$ (in winters).
- The gain or loss in time is independent of time period T and depends on the time interval t.
- Since coefficient of linear expansion (α) is very small for invar, hence pendulums are made of invar to show the correct time in all seasons.

Illustration 1.11

A clock with a brass pendulum shaft keeps correct time at a certain temperature.

(a) How closely must the temperature be controlled if the clock is not to gain or lose more than 1 s a day? Does the answer depend on the period of the pendulum?

(b) Will an increase of temperature cause the clock to gain or lose? ($\alpha_{brass} = 2\times10^{-5}/°C$)

Sol.

(a) Number of seconds lost or gained per day $= \frac{1}{2}\alpha\theta\times86400$

where θ = rise or drop in temperature; α = coefficient of linear expansion of shaft.

We want that $\left|\frac{1}{2}\alpha\theta\times86400\right| < 1$

$$\Rightarrow \quad |\theta| < \frac{2}{2\times10^{-5}\times86400} \quad \Rightarrow |\theta| < 1.1574°C$$

Hence temperature should not increase or decrease by more than 1.1574°C. It does not depend upon time period.

(b) An increase in temperature makes the pendulum slow and hence clock loses time.

Illustration 1.12

A pendulum clock loses 12 s a day if the temperature is 40°C and goes fast by 4 s a day if the temperature is 20°C. Find the temperature at which the clock will show correct time and the coefficient of linear expansion of the metal of the pendulum shaft.

Sol. Let T be the temperature at which the clock shows correct time.

Time lost per day = $1/2\alpha$ (rise in temperature) × 86400

$$\Rightarrow \quad 12 = 1/2\alpha(40-T)\times86400 \qquad ...(i)$$

Time gained per day = 1/2 α (drop in temperature) × 86400

$\Rightarrow \quad 4 = 1/2\alpha(T-20)\times 86400$ ...(ii)

Adding Eqs. (i) and (ii), we get $32 = 86400\alpha(40-20)$

$\Rightarrow \quad \alpha = 1.85\times10^{-5}/°C$

Dividing Eq. (i) by Eq. (ii), we get $12(T-20) = 4\,(40-T)$

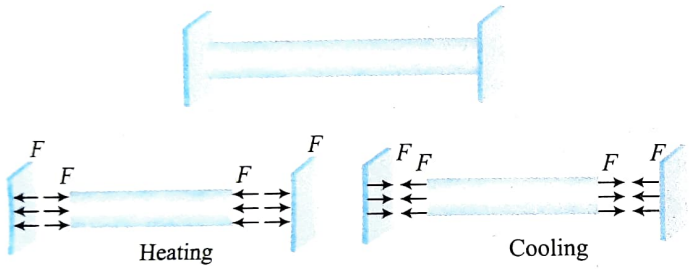
$\Rightarrow \quad T = 25°C$

$\Rightarrow$ Clock shows correct time at 25°C.

THERMAL STRESS IN A RIGIDLY FIXED ROD

If the rod is free to expand then there will be no stress and strain. Stress and strain are produced only when an object is restricted to expand or contract according to change in temperature. When the temperature of the rod is decreased or increased under constrained condition, compressive or tensile stresses are developed in the rod. These stresses are known as thermal stresses. Due to this thermal stress the rod will exert a large force on the supports.

Consider a rod of length L which is fixed between to rigid end separated at a distance L now if the temperature of the rod is increased by $\Delta\theta$ then the strain produced in the rod will be



Thermal strain, $\varepsilon = \dfrac{\Delta L}{L} = \dfrac{\text{Final length} - \text{Original length}}{\text{Original length}} = \alpha\Delta\theta$

$$\left[\text{As } \alpha = \frac{\Delta L}{L}\times\frac{1}{\Delta\theta}\right]$$

So, Thermal stress = $Y\alpha\Delta\theta$ $\quad\left[\text{As } Y = \dfrac{\text{stress}}{\text{strain}}\right]$

Force on the supports, $F = YA\alpha\Delta\theta$

Thermal stresses may become very large so as to stress the rod beyond its elastic limit or even beyond its ultimate strength. Hence, in the design of any structure which is subject to changes of temperature, some provision must be made for expansion.

We can also easily calculate thermal stress (i.e., pressure ΔP) exerted by an expanding or contracting body.

We know that

$$B = \frac{\Delta P}{\Delta V/V} \quad \text{or} \quad \Delta P = B\left(\frac{\Delta V}{V}\right) \qquad ...(i)$$

Also, if the temperature of the body rises by an amount ΔT, then

$$\gamma = \frac{\Delta V}{V\Delta T} \quad \text{or} \quad \frac{\Delta V}{V} = \gamma\,\Delta T \qquad ...(ii)$$

From Eqs. (i) and (ii), thermal stress,

$$\Delta P = B\gamma\Delta T \qquad ...(iii)$$

Thus, a solid expanding due to rise in temperature ΔT exerts a pressure $B\gamma\Delta T$ in all directions.

ILLUSTRATION 1.13

A brass wire of length 1.8 m is held taut at 27°C with little tension between two rigid supports. If the wire is cooled to a temperature of –39°C, what is the tension developed in the wire, if its diameter is 2.0 mm? Coefficient of linear expansion of brass = $2.0\times10^{-5}/°C$, Young's modulus of brass = 0.91×10^{11} Pa.

Sol. As the wire is not free to contract, the thermal stress is developed at the wire.

The change in temperature, $\Delta T = 27°C - (-39°C) = 66\ °C$

Let ΔL be the change in length of the wire.

$$\Delta L = \alpha L\,\Delta T = (2\times10^{-5})\times1.8\times66 = 2.376\times10^{-3}\text{ m}$$

As $\quad Y = \dfrac{FL}{A\,\Delta L} = \dfrac{FL}{(\pi D^2/4)\Delta L} = \dfrac{4FL}{\pi D^2\Delta L}$

$$F\text{ (tension in the wire)} = \frac{Y\pi D^2\,\Delta L}{4L}$$

$$= \frac{(0.91\times10^{11})\times3.142\times(2\times10^{-3})^2\times(2.376\times10^{-3})}{4\times1.8} = 3.8\times10^{2}\text{N}$$

ILLUSTRATION 1.14

A rod has length 2 m at a temperature of 20°C. Find the free expansion of the rod, if the temperature is increased to 50°C, then find stress produced when the rod is **(a)** fully prevented to expand, **(b)** permitted to expand by 0.4 mm. $Y = 2\times10^{11}\ N/m^2$; $\alpha = 15\times10^{-6}/°C$.

Sol. Free expansion of the rod = $\alpha L\Delta\theta$

$= 15\times10^{-6}/°C \times 2\text{ m} \times (50-20)°C = 9\times10^{-4}\text{ m} = 0.9\text{ mm}$

(a) If the expansion is fully prevented, then

$$\text{Strain} = \frac{9\times10^{-4}}{2} = 4.5\times10^{-4}$$

$\therefore$ Temperature stress = Strain × $Y = 4.5\times10^{-4}\times2\times10^{11}$

$= 9\times10^{7}\ N/m^2$

(b) If 0.4 mm expansion is allowed, then length restricted to expand = 0.9 – 0.4 = 0.5 mm

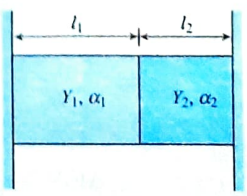
$$\therefore \text{ Strain} = \frac{5\times10^{-4}}{2} = 2.5\times10^{-4}$$

$\therefore$ Temperature stress = Strain × $Y = 2.5\times10^{-4}\times2\times10^{11}$

$= 5\times10^{7}\ N/m^2$

ILLUSTRATION 1.15

Two rods of different metals having the same area of cross section A are placed between the two massive walls as shown in the given figure. The first rod has a length l_1, coefficient of linear expansion α_1 and Young's modulus Y_1. The corresponding quantities for second rod are l_2, α_2 and Y_2. The temperature of both the rods is now raised by t°C.



(a) Find the force with which the rods act on each other (at higher temperature) in terms of given quantities.
(b) Also find the length of the rods at higher temperature.

Sol.

(a) Let t°C = increase in the temperature.
Increase in length of first rod = $l_1\alpha_1 t$
Increase in length of second rod = $l_2\alpha_2 t$
$\therefore$ Total extension in length due to rise in temperature

$$= l_1\alpha_1 t + l_2\alpha_2 t = (l_1\alpha_1 + l_2\alpha_2)t \quad \ldots(i)$$

Since the walls are rigid, this increase in length will not happen. This will be compensated by equal and opposite forces F, F producing decrease in the lengths of the rods due to elasticity.

$\therefore$ Decrease in length of first rod = $\dfrac{F \times l_1}{Y_1 \times A}$

And decrease in length of second rod = $\dfrac{F \times l_2}{Y_2 \times A}$

$\therefore$ Total decrease in length due to elastic force

$$= \frac{F}{A}\left(\frac{l_1}{Y_1} + \frac{l_2}{Y_2}\right) \quad \ldots(ii)$$

From Eqs. (i) and (ii), we have

$$\frac{F}{A}\left(\frac{l_1}{Y_1} + \frac{l_2}{Y_2}\right) = (l_1\alpha_1 + l_2\alpha_2)t$$

or $$F = \frac{A(l_1\alpha_1 + l_2\alpha_2)t}{\left(\frac{l_1}{Y_1} + \frac{l_2}{Y_2}\right)} \quad \ldots(iii)$$

(b) Length of the first rod = original length + increase in length due to temperature − decrease in length due to force

$$= \left(l_1 + l_1\alpha_1 t - \frac{F}{A}\frac{l_1}{Y_1}\right)$$

and length of second rod = $l_2 + l_2\alpha_2 t - \dfrac{F}{A}\dfrac{l_2}{Y_2}$

$\therefore$ The total length is same = $l_1 + l_2$ at all temperatures.

ILLUSTRATION 1.16

A steel ball initially at a pressure of 10^5 Pa is heated from 20°C to 120°C keeping its volume constant. Find the final pressure inside the ball. Given that coefficient of linear expansion of steel is 1.1×10^{-5}/°C and bulk modulus of steel is 1.6×10^{11} N/m^2.

Sol. On increasing temperature of ball by 100°C (from 20°C to 120°C), the thermal expansion in its volume can be given as

$$\Delta V = \gamma_{st} V\Delta T = 3\alpha_{st} V\Delta T \quad \ldots(i)$$

Here it is given that no change of volume is allowed. This implies that the volume increment by thermal expansion is compressed elastically by external pressure. Thus elastic compression in the sphere must be equal to that given in Eq. (i). Bulk modulus of a material is defined as

$$B = \frac{\text{increase in pressure}}{\text{volume strain}} = \frac{\Delta P}{\Delta V/V}$$

Here the externally applied pressure to keep the volume of ball constant is given as

$$\Delta P = B \times \frac{\Delta V}{V} = B(3\alpha_{st}\Delta T) \qquad \text{[From Eq. (i)]}$$

$$= 1.6 \times 10^{11} \times 3 \times 1.1 \times 10^{-5} \times 100$$

$$= 5.28 \times 10^8 \text{ N/m}^2 = 5.28 \times 10^8 \text{ Pa}$$

Thus this must be the excess pressure inside the ball at 120°C to keep its volume constant during heating.

ILLUSTRATION 1.17

A composite rod is made by joining a copper rod, end to end, with a second rod of different material but of the same area of cross section. At 25°C, the composite rod is 1 m long and the copper rod is 30 cm long. At 125°C the length of the composite rod increases by 1.91 mm. When the composite rod is prevented from expanding by holding it between two rigid walls, it is found that the constituent rods have remained unchanged in length in spite of rise of temperature. Find Young's modulus and the coefficient of linear expansion of the second rod (Y of copper = 1.3×10^{10} N/m^2 and α of copper = 17×10^{-6}/K).

Sol. Since total expansion = expansion of the constituent rods

$$1.91 \times 10^{-3} = 0.3 \times 17 \times 10^{-6} \times (125 - 25) + 0.7 \times \alpha \times (125 - 25)$$

or $$1.91 \times 10^{-3} = 5.1 \times 10^{-4} + 70\alpha$$

or $$70\alpha = (1.91 - 0.51) \times 10^{-3} = 1.4 \times 10^{-3}$$

or $$\alpha = 2 \times 10^{-5}/\text{K}$$

When prevented from expanding thermal expansion is accompanied by elastic contraction. Since the lengths remain unchanged, thermal expansion is equal to elastic contraction.

$\therefore$ $$\text{Strain} = \frac{l\alpha\Delta t}{l} = \alpha\Delta t = 17 \times 10^{-6} \times 100 = 17 \times 10^{-4}$$

and $$\text{stress} = Y \times \text{strain} = 1.3 \times 10^{10} \times 17 \times 10^{-4} = 22.1 \times 10^6$$

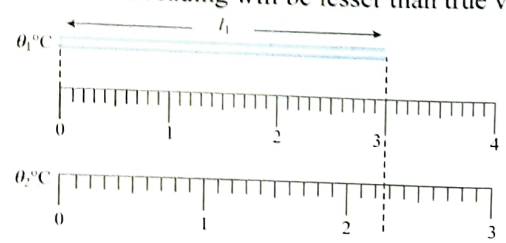
For the second rod,
Strain = $\alpha\Delta t = 2 \times 10^{-5} \times 100 = 2 \times 10^{-3}$
Stress = $Y \times$ strain = $Y \times 2 \times 10^{-3}$
But the same stress is effective throughout the composite rod.

$\therefore$ $$Y \times 2 \times 10^{-3} = 22.1 \times 10^6$$

or $$Y = 11.1 \times 10^9 \text{ N/m}^2$$

ERROR IN SCALE READING DUE TO EXPANSION OR CONTRACTION

If a scale gives correct reading at temperature θ, then at temperature θ' ($>\theta$) due to linear expansion of scale, the scale will expand and scale reading will be lesser than true value.



So, true value = Scale reading (l) + Expansion of length L of the scale ($\alpha_s l\Delta\theta$) where $\Delta\theta = (\theta' - \theta)$.

$\Rightarrow$ True value = Scale reading $\left[1 + \alpha_s(\theta' - \theta)\right]$

α_s = Linear expansion coefficient of measuring scale

i.e., TV = SR[1 + $\alpha_s \Delta\theta$], where $\Delta\theta = (\theta' - \theta)$

However, if $\theta' < \theta$, due to contraction of scale, scale reading will be more than true value, so true value will be lesser than scale reading and will still be given by above equation with $\Delta\theta = (\theta' - \theta)$ negative.

ILLUSTRATION 1.18

At room temperature (25°C) the length of a steel rod is measured using a brass centimetre scale. The measured length is 20 cm. If the scale is calibrated to read accurately at temperature 0°C, find the actual length of steel rod at room temperature.

Sol. The brass scale is calibrated to read accurately at 0°C. This means at 0°C, each division of scale has exact 1 cm length. Thus at higher temperature the division length of scale will be more than 1 cm due to thermal expansion. Thus at higher temperature the scale reading for length measurement is not appropriate and as at higher temperature the division length is more, the length this scale reads will be lesser than the actual length to be measured. For illustration in this case the length of each division on brass scale at 25°C is

$$l_{1\text{div}} = (1\,\text{cm})[1+\alpha_{\text{br}}(25-0)] = 1 + \alpha_{\text{br}}(25)$$

where α_{br} is coefficient of linear expansion of brass scale.

It is given that at 25°C the length of steel rod measured is 20 cm. Actually it is not 20 cm, it is 20 divisions on the brass scale. Now, we can find the actual length of the steel rod at 25°C as

$$l_{25^\circ\text{C}} = (20\,\text{cm}) \times l_{1\text{div}}$$

or $$l_{\text{act}} = 20[1+\alpha_{\text{br}}(25)] \quad \ldots(\text{i})$$

The above expression is a general relation using which you can find the actual lengths of the objects of which lengths are measured by a metallic scale at some temperature other than the graduation temperature of the scale.

ILLUSTRATION 1.19

A steel tape of length 1 m is correctly calibrated for a temperature of 27.0°C. The length of a steel rod measured by this tape is found to be 63.0 cm on a hot day when the temperature is 45°C. What is the actual length of the steel rod on that day? What is the length of the same steel rod on a day when the temperature is 27.0°C? Coefficient of linear expansion of steel = 1.20×10^{-5} /K.

Sol. The steel tape gives correct reading only at the temperature 27°C at which it has been calibrated. At any other temperature 45°C the scale will expand and give less reading than the true value.

Hence length of the steel rod at 27°C, L = 63 cm

Let ΔL be the increase in the length of the steel tape when temperature rises from 27°C to 45°C, i.e.,,

ΔT = 45°C – 27°C = 18°C = 18 K

Clearly, $\Delta L = \alpha_s L \Delta T = (1.20 \times 10^{-5}/\text{K}) \times (63\ \text{cm}) \times 18\ \text{K}$

$= 0.0136$ cm

Actual length of the rod at 45°C = 63 cm + 0.0136 cm

= 63.0136 cm

Alternatively:

We can use direct formula, TV = SR[1 + $\alpha_s \Delta T$]

TV = true value

SR = scale reading

$\therefore$ TV = $63.0[1 + 1.20 \times 10^{-5} \times 18] = 63.0136$ cm

ILLUSTRATION 1.20

A steel scale measures the length of a copper rod as 80 cm when both are at 20°C, which is the calibration temperature for the scale. What would the scale read for the length of the rod when both are at 40°C? α for steel = 1.1×10^{-6}°C and α for copper = 1.7×10^{-6}°C.

Sol.

1 cm length of steel scale at 40°C

$$= 1 + 1 \times (11 \times 10^{-6}) \times (40 - 20) = 1.00022\ \text{cm}$$

Length of copper rod at 40°C

$$= 80 + (80 \times 17 \times 10^{-6}) \times (40 - 20) = 80.0272\ \text{cm}$$

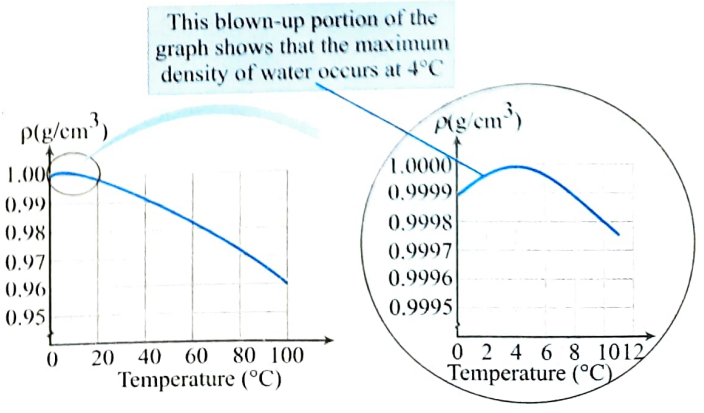
Number of division on the steel scale = $\dfrac{80.0272}{1.00022} = 80.0096$

Scale reading for length of the rod = 80.0096 cm

ANOMALOUS EXPANSION OF WATER

We know, matter expands on heating and contracts on cooling. However, water is special. It contracts when cooled, down to a temperature of 4°C but thereafter begins to expand as it reaches 0°C and turns into ice. Anomalous expansion of water is an abnormal property of water whereby it expands instead of contracting when the temperature goes from 4°C to 0°C, and it becomes less dense. In the range 0°C to 4°C, water contracts on heating and expands on cooling, i.e. γ is negative.

This anomalous behavior of water causes ice to form first at the surface of a lake in cold weather. As winter approaches, the water temperature increases initially at the surface. The water there sinks because of its increased density. Consequently, the surface reaches 0°C first and the lake becomes covered with ice. . Further, water and ice are not good conductors of heat. All this help in the maintenance of temperature of the water at the bottom at 4°C. It is in this layer that marine life is able to sustain itself. Had water been like any other liquid, the whole depth of the water would have been frozen and aquatic life would have been destroyed completely.



THERMAL EXPANSION IN LIQUIDS

Liquids do not have linear and superficial expansion but these only have volume expansion. Since liquids are always to be heated along with the vessel that contains them so initially on heating the system (liquid + vessel), the level of liquid in vessel falls (as vessel expands more since it absorbs heat and liquid expands less), but later on, it starts rising due to faster expansion of the liquid.

The actual increase in the volume of the liquid = The apparent increase in the volume of liquid + the increase in the volume of the vessel.

Liquids have two coefficients of volume expansion.

(i) Coefficient of apparent expansion (γ_{app}) It is due to apparent (that appears to be, but is not) increase in the volume of liquid if expansion of vessel containing the liquid is not taken into account.

$$\gamma_{app} = \frac{\text{Apparent expansion in volume}}{\text{Initial volume} \times \Delta T} = \frac{(\Delta V)_{app}}{V_0 \times \Delta T}$$

(ii) Coefficient of real expansion (γ_{real}) It is due to the actual increase in volume of liquid due to heating.

$$\gamma_{real} = \frac{\text{Real increase in volume}}{\text{Initial volume} \times \Delta T} = \frac{(\Delta V)_{real}}{V_0 \times \Delta T}$$

The coefficient of expansion of vessel, $\gamma_{vessel} = \dfrac{(\Delta V)_{vessel}}{V_0 \times \Delta T}$

Hence, $\gamma_{real} = \gamma_{app} + \gamma_{vessel}$

The change (apparent change) in volume in liquid relative to vessel is:

$$\Delta V_{app} = V_0 \gamma_{app} \Delta T = V_0(\gamma_{real} - \gamma_{vessel})\Delta T = V_0(\gamma_{real} - 3\alpha)\Delta T$$

α = Coefficient of linear expansion of the vessel

Different levels of liquid in vessel

γ_{app}	ΔV_{app}	Level
$\gamma_{real} > 3\alpha$ $\gamma_{app} > 0$	ΔV_{app} is positive	Level of liquid in vessel will rise on heating.
$\gamma_{real} < 3\alpha$ $\gamma_{app} < 0$	ΔV_{app} is negative	Level of liquid in vessel will fall on heating.
$\gamma_{real} = 3\alpha$ $\gamma_{app} = 0$	$\Delta V_{app} = 0$	Level of liquid in vessel will remain unchanged.

ILLUSTRATION 1.21

A 1L flask contains some mercury. It is found that at different temperatures, the volume of air inside the flask remains the same. What is the volume of mercury in the flask, given that the coefficient of linear expansion of glass = 9×10^{-6} /°C and the coefficient of volume expansion of Hg = 1.8×10^{-4} /°C?

Sol. Since the volume above mercury remains the same at all temperatures, the expansion of the glass vessel must be the same as that of mercury in the vessel.

Also, $\gamma_g = 3 \times \alpha_g = 27 \times 10^{-6}$/°C. Let V be the volume of the mercury. Then, from

$$\Delta V = V\gamma\Delta T = V \times (1.8 \times 10^{-4}) \times \Delta T = 10^{-3} \times 27 \times 10^{-6} \Delta T$$

($\because$ 1 L = 10^{-3} m^3 and $\gamma = 3\alpha$)

or $V = 150 \times 10^{-6}$ m^3

ILLUSTRATION 1.22

A 250 cm^3 glass bottle is completely filled with water at 50°C. The bottle and water are heated to 60°C. How much water runs over if:

(a) the expansion of the bottle is neglected;

(b) the expansion of the bottle is included? Given the coefficient of areal expansion of glass $\beta = 1.2 \times 10^{-5}$/K and $\gamma_{water} = 60 \times 10^{-5}$/°C.

Sol. Water overflow = (final volume of water) – (final volume of bottle)

(a) If the expansion of bottle is neglected:

Water overflow = $250\,(1 + \gamma_{water}\theta) - 250$

$= 250 \times 60 \times 10^{-5} \times 10 = 1.5$ cm^3

(b) If the bottle (glass) expands:

Water overflow

$= 250(1 + \gamma_{water}\theta) - 250(1 + \gamma_g\theta)$

$= 250\,(\gamma_l - \gamma_g)\theta$, where $\gamma_g = \dfrac{3\beta}{2} = 1.8 \times 10^{-5}$/°C

$= 250\,(58.2 \times 10^{-5}) \times (60 - 50)$

Water overflow = 1.455 cm^3

ILLUSTRATION 1.23

A hollow aluminium cylinder 20.0 cm deep has an internal capacity of 2.000 L at 20.0°C. It is completely filled with turpentine and then slowly warmed to 80.0°C.

(a) How much turpentine overflows?

(b) If the cylinder is then cooled back to 20.0°C, how far below the cylinder's rim does the turpentine's surface recede?

Sol. When the temperature is increased from 20.0°C to 80.0°C both the cylinder and the turpentine increase in volume by $\Delta V = \gamma V \Delta T$:

(a) The overflow is $V_{over} = \Delta V_{turp} - \Delta V_{Al}$

$V_{over} = (\gamma V \Delta T)_{turp} - (\gamma V \Delta T)_{Al} = V_i \Delta T(\gamma_{turp} - 3\alpha_{Al})$

$V_{over} = (2.000\text{ L})\,(60.0°\text{C})\,(9.00 \times 10^{-4}/°\text{C} - 0.720 \times 10^{-4}/°\text{C})$

$= 0.099\,4$ L

(b) After warming, the whole volume of the turpentine,

$V' = 2000\,\text{cm}^3 + (9.00 \times 10^{-4}/°\text{C})(2000\,\text{cm}^3)(60.0°\text{C})$

$= 2108\,\text{cm}^3$

The fraction lost = $\dfrac{99.4\,\text{cm}^3}{2108\,\text{cm}^3} = 4.71 \times 10^{-2}$

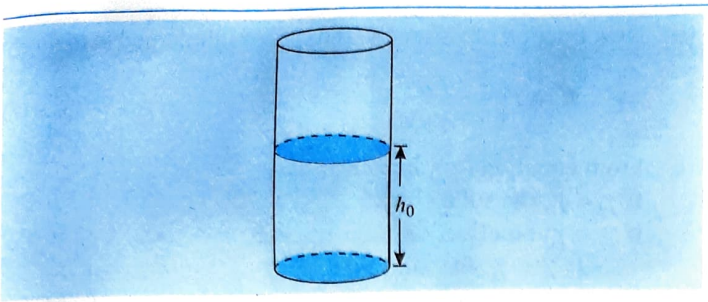
This also is the fraction of the cylinder that will be empty after cooling. Therefore, change in level,

$\Delta h = (4.71 \times 10^{-2})(20.0\,\text{cm}) = 0.943$ cm

The change in volume of the container is not negligible, but is 8% of the change in volume of the turpentine.

ILLUSTRATION 1.24

The height of liquid column in the cylindrical container is h_0, at 0°C. If the temperature of liquid rises to θ°C, assuming the expression of both solid (container) and liquid, find the height of the liquid column. Assume γ_l and γ_s as the coefficient cubical expression of liquid and solid.



Sol. The mass of water at θ°C is equal to mass of water 0°C because mass does not change with temperature

Hence, $m_\theta = m_0$

or $V_\theta \rho_\theta = V_0 \rho_0$

or $A_\theta h_\theta \rho_\theta = A_0 h_0 \rho_0$

or $h_\theta = \left(\frac{A_0}{A_\theta}\right)\left(\frac{h_0}{h_\theta}\right) h_0 = (1-\beta_s\theta)(1+\gamma_l\theta)h_0$

or $h_\theta \simeq [1+(\gamma_l-\beta_s)\theta]h_0 = \left[1+\left(\gamma_l-\frac{2}{3}\gamma_s\right)\theta\right]h_0$

VARIATION OF DENSITY WITH TEMPERATURE

Most substances (solid and liquid) expand when they are heated, i.e., volume of a given mass of a substance increases on heating, so the density should decrease $\left(\text{as } \rho \propto \frac{1}{V}\right)$. For a given mass $\rho \propto \frac{1}{V}$

$$\Rightarrow \quad \frac{\rho'}{\rho} = \frac{V}{V'} = \frac{V}{V+\Delta V} = \frac{V}{V+\gamma V \Delta\theta} = \frac{1}{1+\gamma\Delta\theta}$$

$$\Rightarrow \quad \rho' = \frac{\rho}{1+\gamma\Delta\theta} \qquad \ldots(i)$$

For solids we can write $\gamma = 3\alpha$, hence equation (i) can be written as:

$$\rho' = \frac{\rho}{1+3\alpha\Delta\theta}$$

For small value of γ, we can approximate, it as:

$$\rho' = \rho(1+\gamma\Delta\theta)^{-1} = \rho(1-\gamma\Delta\theta)$$

EFFECT OF TEMPERATURE ON UPTHRUST

The thrust (F_{th}) on volume V of a body in a liquid of density σ is given by $F_{th} = V\sigma g$

Now with rise in temperature by $\Delta\theta$ °C, due to expansion, volume of the body will increase while density of liquid will decrease according to the relations

$$V' = V(1+\gamma_S\Delta\theta) \text{ and } \sigma' = \frac{\sigma}{(1+\gamma_L\Delta\theta)}$$

So the thrust $F'_{th} = V'\sigma' g$

$$\Rightarrow \quad \frac{F'_{th}}{F_{th}} = \frac{V'\sigma' g}{V\sigma g} = \frac{(1+\gamma_S\Delta\theta)}{(1+\gamma_L\Delta\theta)}$$

The apparent weight of the body, W_{app} = Actual weight – Thrust

As $\gamma_S < \gamma_L$ Therefore, $F'_{th} < F_{th}$

With rise in temperature, thrust also decreases and apparent weight of body increases.

ILLUSTRATION 1.25

The coefficient of volume expansion of glycerine is 49×10^{-5}/°C. What is the fractional change in its density for 30°C rise in temperature?

Sol. Coefficient of volume expansion of glycerine.

$\gamma = 49 \times 10^{-5}$/°C

Rise in temperature, $\Delta T = 30$°C

As $\gamma = \frac{1}{\Delta T}\left(\frac{\Delta V}{V}\right)$

$$\Rightarrow \quad \frac{\Delta V}{V} = \gamma \times \Delta T$$

or $\frac{\Delta V}{V} = (49\times10^{-5})\times 30 = 0.0147$

Since mass remains constant, Fractional change in density = Fractional change in volume = 0.0147 = 1.5×10^{-2}

ILLUSTRATION 1.26

A solid floats in a liquid at 20°C with 75% of it immersed. When the liquid is heated to 100°C, the same solid floats with 80% of it immersed in the liquid. Calculate the coefficient of expansion of the liquid. Assume the volume of the solid to be constant.

Sol. Let m be the mass of the solid and V its volume. By the law of flotation

Weight of floating object = Buoyant force

In Case I: $mg = \left(\frac{3}{4}V\right)\rho_{20}g$

where ρ_{20} = density of liquid at 20°C

In Case II: $mg = \left(\frac{80}{100}V\right)\rho_{100}g$

where ρ_{100} = density of liquid at 100°C

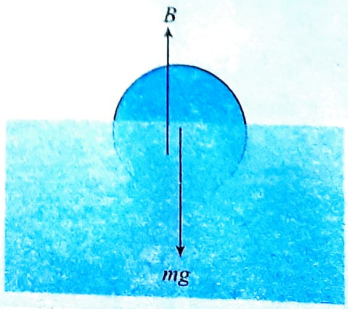
Considering both the cases, we get

$$\frac{3}{4}\rho_{20} = \frac{4}{5}\rho_{100} \quad \Rightarrow \frac{3}{4}\left(\frac{\rho_0}{1+\gamma\times 20}\right) = \frac{4}{5}\left(\frac{\rho_0}{1+\gamma\times 100}\right)$$

After solving we get $\gamma = \frac{1}{1180} = 8.47\times10^{-4}$/°C

ILLUSTRATION 1.27

A body is floating inside a liquid. If we increases temperature then what changes occur in buoyancy force (Assume body is always in floating condition)



Sol. Body is in equilibrium, buoyant force will balance the weight of the body.

$\therefore \quad mg = B$

The gravitational force does not change with change in temperature. So Buoyancy force remains constant.

With increase in temperature, density of liquid decreases. So volume of body inside the liquid increases to keep the buoyance force constant to be equal to gravitational force.

ILLUSTRATION 1.28

A metal sphere floats on mercury. The fraction of volume of the sphere submerged in mercury is f. The coefficients of expansion of the metal and mercury are γ_1 and γ_2, the temperature of system is increased by ΔT. Then

(a) find the fraction of the volume of the sphere submerged in mercury at this situation.

(b) under what conditions will the sphere sink or lift up considering expansion of solid?

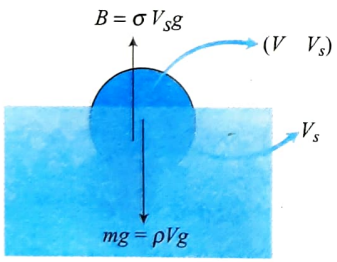
(c) if we neglect the expansion of solid, determine the temperature rise at which the solid is completely immersed?

Sol.

(a) Let total volume of the sphere in air and mercury be V and V_S, respectively. Also, let ρ and σ be the densities of the sphere and mercury, respectively. When the sphere floats in equilibrium,

Weight of the sphere = Buoyant force of the mercury

$$\rho V g = \sigma V_s g \qquad ...(i)$$



Fraction of the volume submerged, $f = \dfrac{V_S}{V} = \dfrac{\rho}{\sigma}$ ...(ii)

When the temperature changes, the fraction of volume submerged changes as densities of metal and mercury change, so

$$f' = \frac{\rho'}{\sigma'} \qquad ...(iii)$$

Density of liquid decreases with temperature increase. So

$$\rho' = \frac{\rho}{1+\gamma_1 \Delta T}, \quad \sigma' = \frac{\sigma}{1+\gamma_2 \Delta T}$$

On substituting these values in equation (iii), we get

$$f' = \frac{\rho'}{\sigma'} = \frac{\rho}{1+\gamma_1 \Delta T} \times \frac{1+\gamma_2 \Delta T}{\sigma}$$

$$\Rightarrow \quad f' = \frac{\rho}{\sigma}\left(\frac{1+\gamma_2 \Delta T}{1+\gamma_1 \Delta T}\right) = f\left(\frac{1+\gamma_2 \Delta T}{1+\gamma_1 \Delta T}\right)$$

(b) New fraction of volume of the sphere submerged in mercury

$$\Rightarrow \quad f' = f\left(\frac{1+\gamma_2 \Delta T}{1+\gamma_1 \Delta T}\right) \qquad ...(iv)$$

From equation (iv), it is clear that

If $\gamma_2 > \gamma_1$, the solid sinks.

If $\gamma_2 = \gamma_1$, no effect on submergence.

If $\gamma_2 < \gamma_1$, the solid lifts up.

(c) When the temperature changes, the fraction of volume submerged

$$f' = \frac{\rho'}{\sigma'} = \frac{\rho}{\sigma}(1+\gamma \Delta T) = f(1+\gamma \Delta T)$$

Expansion of solid has been ignored and $\gamma_2 = \gamma$

When the solid is completely submerged,

$$f' = 1 = f(1+\gamma \Delta T) \Rightarrow \Delta T = \frac{1-f}{\gamma f}$$

ILLUSTRATION 1.29

A sinker of weight W_0 has an apparent weight W_1 when placed in a liquid at a temperature T_1 and W_2 when weighed in the same liquid at a temperature T_2. The coefficient of cubical expansion of the material of the sinker is γ_s. What is the coefficient of volume expansion of the liquid?

Sol. Let $\theta = T_2 - T_1$ and γ = coefficient of volume expansion of liquid.

Let density of liquid at temperatures T_1 and T_2 be ρ_1 and ρ_2 respectively.

$$\Rightarrow \quad \rho_1 = \rho_2(1+\gamma\theta) \qquad ...(i)$$

Let V_1 and V_2 be the volumes of the sinker at temperatures T_1 and T_2, respectively.

$$\Rightarrow \quad V_2 = V_1(1+\gamma_s\theta) \qquad ...(ii)$$

The loss in weight at $T_1 = V_1\rho_1 g$

$$\Rightarrow \quad W_0 - W_1 = V_1\rho_1 g \qquad ...(iii)$$

The loss in weight at $T_2 = V_2\rho_2 g$

$$\Rightarrow \quad W_0 - W_2 = V_2\rho_2 g \qquad ...(iv)$$

Dividing Eq. (iii) by Eq. (iv), $\dfrac{W_0 - W_1}{W_0 - W_2} = \dfrac{V_1\rho_1}{V_2\rho_2}$

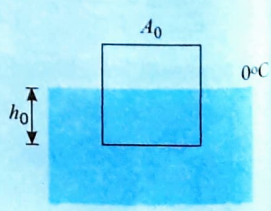
Using Eqs. (i) and (ii), $\dfrac{W_0 - W_1}{W_0 - W_2} = \dfrac{1+\gamma\theta}{1+\gamma_s\theta}$

$$\Rightarrow \quad 1+\gamma\theta = \frac{W_0 - W_1}{W_0 - W_2} + \gamma_s\left(\frac{W_0 - W_1}{W_0 - W_2}\right)\theta$$

$$\Rightarrow \quad \gamma = \left(\frac{W_2 - W_1}{W_0 - W_2}\right)\frac{1}{T_2 - T_1} + \left(\frac{W_0 - W_1}{W_0 - W_2}\right)\gamma_s$$

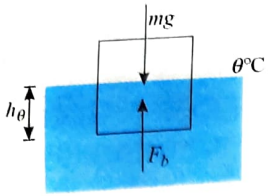
ILLUSTRATION 1.30

A cubical of coefficient of linear expansion α_s floats in the liquid of coefficient of volume expansion γ_l. If the depth of immersion of the cube at 0°C is h_0, find the depth of immersion of the cube at θ°C.



Sol. As the cube is floating. For the equilibrium of the body

$F_b = mg$



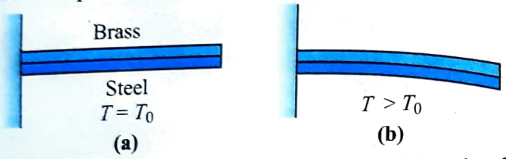
Or $V\rho g = mg \Rightarrow A_\theta \rho_\theta h_\theta = A_0 \rho_0 h_0$

Or $h_\theta = \dfrac{A_0\rho_0}{A_\theta\rho_\theta} h_0 = (1-\beta_s\theta)(1+\gamma_l\theta)h_0$

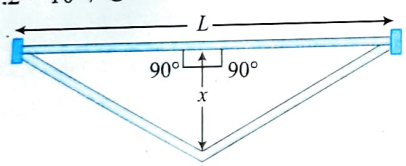
$= [1+(\gamma_l - \beta_s)\theta]h_0 = [1+(\gamma_l - 2\alpha_s)\theta]h_0$

CONCEPT APPLICATION EXERCISE 1.1

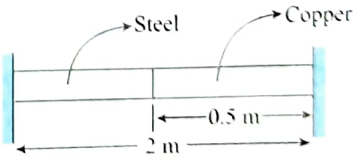
1. In figure which strip brass or steel have higher coefficient of linear expansion?



2. A rail track made of steel having length 10 m is clamped on a railway line at its two ends (see figure). On a summer day due to rise in temperature by 20° C, it is deformed as shown in figure. Find x (displacement of the centre) if $\alpha_{steel} = 1.2 \times 10^{-5}$/°C



3. Calculate the stress developed inside a tooth cavity filled with copper when hot tea at temperature of 57°C is drunk. You can take body (tooth) temperature to be 37°C and $\alpha_{Cu} = 1.7 \times 10^{-5}$/°C bulk modulus for copper $B_{Cu} = 140 \times 10^9$ N/m^2.
4. The design of some physical instrument requires that there be a constant difference in length of 10 cm between an iron rod and a copper cylinder laid side by side at all temperatures. Find their lengths.
($\alpha_{Fe} = 11 \times 10^{-6}$ °C^{-1}, $\alpha_{Cu} = 17 \times 10^{-6}$ °C^{-1})
5. A metal rod of 30 cm length expands by 0.075 cm when its temperature is raised from 0°C to 100°C. Another rod of a different metal of length 45 cm expands by 0.045 cm for the same rise in temperature. A composite rod C made by joining A and B end to end expands by 0.040 cm when its length is 45 cm and it is heated from 0°C to 50°C. Find the length of each portion of the composite rod.
6. A clock with a metallic pendulum gains 5 s each day at a temperature of 15°C and loses 10 s each day at a temperature of 30°C. Find the coefficient of thermal expansion of the pendulum metal.
7. A glass vessel measures exactly 10 cm × 10 cm × 10 cm at 0°C. It is filled completely with mercury at this temperature. When the temperature is raised to 10°C, 1.6 cm^3 of mercury overflows. Calculate the coefficient of volume expansion of mercury. Coefficient of linear expansion of glass = 6.5×10^{-6}/°C.
8. A metal ball immersed in alcohol weighs W_1 at 0°C and W_2 at 50°C. The coefficient of cubical expansion of the metal is less than alcohol. Assuming that density of the metal is large compared to that of the alcohol, find which of W_1 and W_2 is greater?
9. A brass scale is graduated at 10°C. What is the true length of a zinc rod which measures 60.00 cm on this scale at 30°C? Coefficient of linear expansion of brass = 18×10^{-6} K^{-1}.
10. A mercury-in-glass thermometer has a stem of internal diameter 0.06 cm and contains 43 g of mercury. The mercury thread expands by 10 cm when the temperature changes from 0°C to 50°C. Find the coefficient of cubical expansion of mercury. Relative density of mercury = 13.6 and $\alpha_{glass} = 9 \times 10^{-6}$/K.
11. A sphere of diameter 7 cm and mass 266.5 g floats in a bath of liquid. The temperature is raised, and the sphere begins to sink at 35°C. If the density of the liquid is 1.527 at 0°C, find the coefficient of cubical expansion of the liquid. Neglect the expansion of the sphere.
12. On a Celsius thermometer the distance between the readings 0°C and 100°C is 30 cm and the area of cross section of the narrow tube containing mercury is 15×10^{-4} cm^2. Find the total volume of mercury in the thermometer at 0°C. α of glass = 9×10^{-6}/K and the coefficient of real expansion of mercury = 18×10^{-5}/K.
13. The height of a mercury column measured with a brass scale, which is correct and equal to H_0 at 0°C, is H_1 at t°C? The coefficient of linear expansion of brass is α and the coefficient of volume expansion of mercury is γ. Relate H_0 and H_1.
14. A glass bulb contains air and mercury. What fraction of the bulb must be occupied by mercury if the volume of air in the bulb is to remain constant at all temperatures? The coefficient of linear expansion of glass is 9×10^{-6}/K and the coefficient of expansion of mercury is 1.8×10^{-4}/K.
15. When composite rod is free, composite length increases to 2.002 m when temperature increases from 20°C to 120°C. When composite rod is fixed between the support, there is no change in component length. Find Y and α of steel if $Y_{cu} = 1.5 \times 10^{13}$ N/m^2 $\alpha_{cu} = 1.6 \times 10^{-5}$/°C.



ANSWERS

1. Brass strip
2. $20\sqrt{30} \times 10^{-3}$ cm
3. 42×10^6 N/m^2
4. $l_1 = 28.3$ cm, $l_2 = 18.3$ cm
5. $l_1 = 23.3$ cm, $l_2 = 21.7$ cm
6. 23×10^{-6} K^{-1}
7. 1.795×10^{-4}/°C
8. $W_2 > W_1$
9. 60.02 cm
10. 20.7×10^{-5} K^{-1}
11. 8.3×10^{-4}/°C
12. 2.94 cc
13. $H_0 = H_1(1+\alpha)(1-\gamma t)$
14. $\dfrac{3}{20}$
15. 3×10^{13} N/m^2

CALORIMETRY

Heat is defined as a process of transferring energy across the boundary of a system because of a temperature difference between the system and its surroundings. It is also the amount of energy Q transferred by this process. When you heat a substance, you transfer energy into it by placing it in contact with surroundings that have a higher temperature. This is the case, for example, when you place a pan of cold water on a stove burner. The burner is at a higher temperature than the water, and so the water gains energy by heating. The process of heat exchange between bodies and system is known as calorimetry.

UNITS OF HEAT

Early studies of heat focused on the resultant increase in temperature of a substance, which was often water. Initial notions of heat were based on a fluid called caloric that flowed from one substance to another and caused changes in temperature. From the name of this mythical fluid came an energy unit related to thermal processes, the calorie (cal), which is defined as the amount of energy necessary to raise the temperature of 1 g of water from 14.5°C to 15.5°C.

ILLUSTRATION 1.31

What is wrong with the following statement? "Given any two bodies, the one with the higher temperature contains more heat".

Sol. The statement shows a misunderstanding of the concept of heat. Heat is a process by which energy is transferred, not a form of energy that is held or contained. If you wish to speak of energy that is contained, you speak of internal energy, not heat.

Further, even, if the statement used the term 'internal energy', it would still be incorrect, since the effects of specific heat and mass are both ignored. A 1 kg mass of water at 20°C has more internal energy than a 1 kg mass of air at 30°C.

Similarly, the earth has far more internal energy than a drop of molten titanium metal.

Correct statements would be: **1.** 'Given any two bodies in thermal contact, the one with the higher temperature will transfer energy to the other by heat'. **2.** 'Given any two bodies of equal mass, the one with the higher product of absolute temperature and specific heat contains more internal energy'. All to say is that internal energy depends not only on temperature but also on mass and nature of body.

SPECIFIC HEAT CAPACITY AND ABSORPTION OF HEAT

When heat is supplied to a substance, it increases the energy of molecules of the substance and hence, its temperature increases. The rise in temperature is proportional to heat supplied to it and also depends on the mass and material of the substance. If ΔQ is the amount of heat supplied to a body of mass m and due to this if its temperature increases by ΔT, then

(i) $\Delta Q \propto m$ (ii) $\Delta Q \propto \Delta T$

or $\Delta Q \propto m\,\Delta T$

or $\Delta Q = mc\,\Delta T$

or $\Delta Q = m\int_{T_1}^{T_2} c(T)\,dT$

where c is a constant for material of body known as specific heat capacity of the material of body. If $m = 1$ g and $\Delta T = 1$°C then $c = \Delta Q$. Thus specific heat capacity of a material is defined as **"The amount of heat required for unit mass of a body to raise its temperature by 1°C".**

The material with high specific capacity requires a lot of heat to change its temperature, while a material with low specific heat capacity requires little heat to change its temperature. The unit of specific heat capacity is cal/g°C or J/kg°C.

Important Points:

- As $\Delta Q = mc\Delta T \Rightarrow c = \dfrac{\Delta Q}{m\Delta T}$, if the substance undergoes the change of state which occurs at constant temperature $(\Delta T = 0)$, then $c = \dfrac{\Delta Q}{0} = \infty$. Thus the specific heat of a substance when it melts or boils at constant temperature is infinite are:
- The specific heats of some common materials are

Substance	Specific heat (in cal/g°C at 25°C)
Water	1.00
Ice	0.50
Steam	0.48

ILLUSTRATION 1.32

The temperature of a silver bar rises by 10.0°C when it absorbs 1.23 kJ of energy by heat. The mass of bar is 525 g. Determine the specific heat of silver.

Sol. We find its specific heat from the definition, which is contained in the equation $Q = mc_{silver}\,\Delta T$ for energy input by heat to produce a temperature change. Solving we have

$$c_{silver} = \frac{Q}{m\Delta T}$$

$$c_{silver} = \frac{1.23\times10^{3}\,\text{J}}{(0.525\,\text{kg})(10.0°\text{C})} = 234\ \text{J/kg°C}$$

ILLUSTRATION 1.33

A 10 kW drilling machine is used to drill a bore in a small aluminium block of mass 8.0 kg. What is the rise in temperature of the block in 2.5 minutes, assuming 50% of power is used up in heating the machine itself or lost to the surroundings? Specific heat of aluminium = 0.91 J/g°C.

Sol. We are given that power of the machine,

$P = 10\ \text{kW} = 10^4\ \text{W}$

Time for which the machine is used, $t = 2.5\ \text{min} = 150\ \text{s}$

Mass of the aluminium block, $m = 8\ \text{kg} = 8 \times 10^3\ \text{g}$

Specific heat of aluminium, $c = 0.91\ \text{J/g°C}$

Since 50% of the energy is used up in heating the machine itself or lost to the surroundings,

Energy absorbed by the block,

$Q = 0.5\ Pt = 0.5 \times 10^4 \times 150\ \text{J} = 7.5 \times 10^5\ \text{J}$

Let ΔT be the rise in temperature of the block.

As $Q = mc\,\Delta T$

$$\Rightarrow \quad \Delta T = \frac{Q}{mc} = \frac{7.5\times10^5\,(\text{J})}{8\times10^3\,(\text{g})\times0.91\,(\text{J/g°C})} = 103°\text{C}$$

THE MECHANICAL EQUIVALENT OF HEAT

In early days, heat was not recognised as a form of energy. Heat was supposed to be something needed to raise the temperature of a body or to change its phase. Calorie was defined as the unit of heat. A number of experiments were performed to show that the temperature may also be increased by doing mechanical work on the system. These experiments established that heat is equivalent to mechanical energy and measured how much mechanical energy is equivalent to a calorie. Joule demonstrated by his experiments that mechanical energy can be directly converted into thermal energy and he also measured the numerical factor relating mechanical units to thermal units.

Whenever heat is converted into mechanical work or mechanical work is converted into heat, the ratio of work done to heat produced always remains constant. i.e., $W \propto Q$ or $\frac{W}{Q} = J$;

This is Joule's law and J is called mechanical equivalent of heat. J is expressed in joule/calorie. The value of J gives how many joules of mechanical work is needed to raise the temperature of 1 g of water by 1°C.

From $W = JQ$ if $Q = 1$ then $J = W$. Hence the amount of work done necessary to produce unit amount of heat is defined as the mechanical equivalent of heat.

J is neither a constant, nor a physical quantity rather it is a conversion factor which is used to convert **joule** or **erg** into **calorie** or **kilo calories** and vice-versa.

$$\text{Value of } J = 4.2\frac{\text{joule}}{\text{cal}} = 4.2\times10^7\frac{\text{erg}}{\text{cal}} = 4.2\times10^3\frac{\text{joule}}{\text{k cal}}$$

Important Points:

- When water in a stream falls from height h, then its potential energy is converted into heat and temperature of water rises slightly.

  $W = JQ \Rightarrow mgh = J(mc\,\Delta\theta)$

  [where m = Mass of water, c = Specific heat of water, $\Delta\theta$ = rise in temperature]

  Rise in temperature, $\Delta\theta = \frac{gh}{Jc}$°C

- The kinetic energy of a bullet fired from a gun gets converted into heat on striking the target. By this heat, the temperature of bullet increases by $\Delta\theta$.

  $$W = JQ \quad \Rightarrow \quad \frac{1}{2}mv^2 = J(mc\,\Delta\theta)$$

  [where m = Mass of the bullet, v = Velocity of the bullet, c = Specific heat of the bullet]

  Rise in temperature, $\Delta t = \frac{v^2}{2Jc}$°C

ILLUSTRATION 1.34

A geyser heats water flowing at the rate of 3.0 litre per minute from 27°C to 77°C. If the geyser operates on a gas burner, what is the rate of combustion of fuel, considering its heat of combustion as 4.0×10^4 J/g.

Sol. Mass of water heated, $M = 3 \times 1000 \times 1 = 3000$ g/min

Rise in temperature, $\Delta T = 77 - 27 = 50°\text{C}$

Amount of heat spent, $Q = mc\Delta T$

or $Q = 3000 \times 1 \times 50$ cal $= 3000 \times 50 \times 4.2$ J/min

$= 6.3 \times 10^5$ J/min

$$\text{Rate of combustion of fuel} = \frac{6.3\times10^5\ \text{J/min}}{4.0\times10^4\ \text{J/g}}$$

$= 15.75$ g/min ≈ 16 g/min

ILLUSTRATION 1.35

What is the change in potential energy (in calories) of a 10 kg mass after 41.8 m fall?

Sol. Change in potential energy

$\Delta U = mgh = 10 \times 10 \times 41.8 = 4180$ J

If mechanical work/energy(W) in joule produces the same temperature change as heat(H), we write, $W = JH$

where $J = 4.18$ is called mechanical equivalent of heat. J is expressed in joule/calorie.

$$\text{Hence, equivalent calories will be } H = \frac{\Delta U}{J} = \frac{4180}{4.18} = 1000 \text{ cal}$$

ILLUSTRATION 1.36

A 60 kg boy running at 5.0 m/s while playing basketball falls down on the floor and skids along on his leg until he stops. How many calories of heat are generated between his leg and the floor?

Assume that all this heat energy is confined to a volume of 2.0 cm³ of his flesh. What will be temperature change of the flesh? Assume c = 1.0 cal/g °C and $\rho = 950$ kg/m³ for flesh.

Sol. Kinetic energy of boy $K = \frac{1}{2}mv^2 = \frac{1}{2}(60)(5)^2 = 750$ J

Changing this energy in joule into calorie

$$Q_1 = \frac{K}{J} = \frac{750}{4.2} = 179 \text{ cal} \qquad \text{...(i)}$$

As this kinetic energy changes into heat energy (Q_2) and this heat is confined to flesh of boy,

$$Q_2 = mc\Delta T = (\rho V)c\Delta T \qquad \text{...(ii)}$$

As $Q_1 = Q_2$, hence from Eqs. (i) and (ii)

$179 = 0.95 \times 2 \times \Delta T \Rightarrow \Delta T = 94°\text{C}$

ILLUSTRATION 1.37

An electric heater supplies 1.8 kW of power in the form of heat to a tank of water. How long will it take to heat the 200 kg of water in the tank from 10°C to 70°C? Assume heat losses to the surroundings to be negligible.

Sol. The heat added to tank ΔQ = power × time

$\therefore \quad \Delta Q = 1.8 \times 10^3 \times t$ (J) (i)

The heat absorbed in water

$$\Delta Q = mc\Delta T = 200\times10^3\times1\times60 = 12\times10^6 \text{ cal}$$

$$= 12\times10^6\times4.2 \text{ J}$$

Hence $t = \dfrac{12\times10^6\times4.2}{1.8\times10^3} = 2.78\times10^4 \text{ s} = 7.75 \text{ h}$

ILLUSTRATION 1.38

In the Joule's experiment, a mass of 20 kg falls through 1.5 m at a constant velocity to stir the water in a calorimeter. If the calorimeter has a water equivalent of 2 g and contains 12 g of water, what is J, the mechanical equivalent of heat, for a temperature rise of 5.0°C?

Sol. Change in potential energy is $\Delta U = mgh = 20\times10\times1.5 = 300$ J. This change in potential energy is utilized to increase the temperature of calorimeter and water $Q = W\Delta T + mc\Delta T$ (W is the water equivalent of calorimeter)

$\therefore \quad Q = 2\times(5.0)+(12)(1)(5.0) = 70$ cal

Hence $Q\cdot J = \Delta U \Rightarrow J = \dfrac{\Delta U}{Q} = \dfrac{300}{70} = 4.2$ J/cal

HEAT CAPACITY

It is defined as the amount of heat required to raise the temperature of the whole body (mass m) through 0°C or 1 K.

Heat capacity $C = mc \times 1 = mc$

The value of heat capacity of a body depends upon the nature of the body and its mass. The unit of heat capacity is cal/°C or J/°C.

ILLUSTRATION 1.39

Two bodies have the same heat capacity. If they are combined to form a single composite body, show that the equivalent specific heat of this composite body is independent of the masses of the individual bodies.

Sol. Let the two bodies have masses m_1, m_2 and specific heats s_1 and s_2. Then

$$m_1s_1 = m_2s_2 \quad \text{or} \quad m_1/m_2 = s_2/s_1$$

Let s = specific heat of the composite body.

Then $(m_1 + m_2)\, s = m_1s_1 + m_2s_2 = 2\, m_1s_1$

$$s = \frac{2m_1s_1}{m_1+m_2} = \frac{2m_1s_1}{m_1+m_1(s_1/s_2)} = \frac{2s_1s_2}{s_2+s_1}$$

Let s = specific heat of the composite body.

Here we can observe that the equivalent specific heat of this composite body is independent of the masses of the individual bodies.

WATER EQUIVALENT

Water equivalent of a body is defined as the mass of water which would absorb or evolve the same amount of heat as done by the body in raising or dropping through the same range of temperature. It is represented by W.

Let m = Mass of the body, c = Specific heat of body, $\Delta\theta$ = Rise in temperature. Then,

Heat given to body, $\Delta Q = mc\Delta\theta$ ...(i)

If same amount of heat is given to W gram of water and its temperature also rises by $\Delta\theta$, then,

Heat given to water, $Q = W \times 1 \times \Delta\theta$ ...(ii)

[As $c_{water} = 1$]

From equation (i) and (ii), $\Delta Q = mc\Delta\theta = W \times 1 \times \Delta\theta$

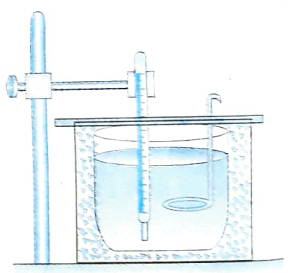
$\Rightarrow$ Water equivalent (W) = mc gram

Important Points:

- Thermal capacity of the body and its water equivalent are numerically equal.
- If thermal capacity of a body is expressed in terms of mass of water, it is called water equivalent of the body.
- If a solid and a hollow sphere of same radius and material are heated to the same temperature then expansion of both will be equal because thermal expansion of isotropic solids is similar to true photographic enlargement. It means that the expansion of cavity is same as if it has been a solid body of the same material. But if same heat is given to the two spheres, due to lesser mass, rise in temperature of hollow sphere will be more $\left(\text{As } \Delta\theta = \dfrac{\Delta Q}{mc}\right)$. Hence, its expansion will be more.

Determination of Water Equivalent of a Calorimeter

A calorimeter is a simple apparatus employed to determine the heat changes occurring in various cases of heat loss and gain. It consists of a cylindrical copper vessel containing a copper stirrer and a sensitive thermometer. It is placed in a wooden box, on two wooden wedges. The space between the outer wooden box and the calorimeter is occupied by some heat insulator, such as cotton, wool, sawdust, etc., to minimize the heat losses from the copper vessel by conduction and convection, Further, to minimize, the heat losses by radiation, the outer surface of the copper vessel is polished.



Calorimeter

The calorimeter is cleaned with water and dried. A known mass of pure water (say m_1) is taken in the calorimeter at room temperature (say T_1). Next, another known mass of pure water (say) m_2 at a temperature (say T_2) about 10°C to 20°C above the room temperature is transferred into the calorimeter, maintaining all the precautions to check any sort of undue heat loss or gain from outside the system. After thorough, but gentle stirring the equilibrium temperature (say T) is noted from the thermometer.

Calculations: Let s_w and W be the heat capacity of water and the water equivalent of the calorimeter, respectively. Then

Heat lost by the hot water = $m_2s_w(T_2 - T)$

Heat gained by the cold water = $m_1s_w(T - T_1)$

and, heat gained by the calorimeter = $Ws_w(T - T_1)$

From, principle of calorimetry,

Heat gained by (cold water + calorimeter) = Heat lost by hot water

i.e., $m_1s_w(T - T_1) + Ws_w(T - T_1) = m_2s_w(T_2 - T)$

or $W = m_2\left(\dfrac{T_2 - T}{T - T_1}\right) - m_1$

Note that, temperatures can be in any unit. The unit of mass (m_1 and m_2) employed will be the unit of water equivalent of the calorimeter.

ILLUSTRATION 1.40

When 0.400 kg of water at 30°C is mixed with 0.150 kg of water at 25°C contained in a calorimeter, the final temperature is found to be 27°C. Find the water equivalent of the calorimeter.

Sol. Water of mass $m_1 = 0.150$ kg is taken in the calorimeter at temperature $T_1 = 25°C$ and mixed with another known mass of pure water $m_2 = 0.400$ kg at a temperature $T_2 = 30°C$ and final temperature is found to be $T = 27°C$. Let s_w and W be the heat capacity of water and the water equivalent of the calorimeter, respectively.

Heat gained by (water + calorimeter) at temperature T_1 = Heat lost by water at temperature T_2

i.e., $$m_1 s_w (T - T_1) + W s_w (T - T_1) = m_2 s_w (T_2 - T)$$

or $$W = m_2 \left(\frac{T_2 - T}{T - T_1} \right) - m_1$$

Hence, $$W = 0.400 \text{ kg} \left[\frac{30°C - 27°C}{27°C - 25°C} \right] - 0.150 \text{ kg} = 0.450 \text{ kg}$$

PHASE CHANGE AND LATENT HEAT

Phase Change

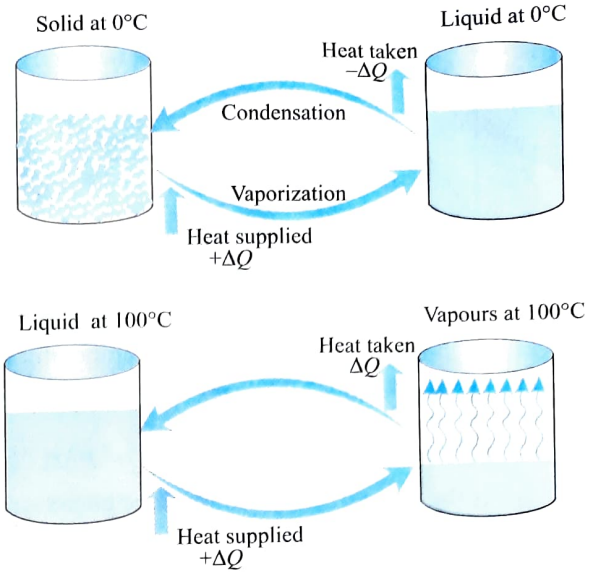
We use the term phase to describe a specific state of matter, such as solid, liquid or gas. A substance can undergo a change in temperature when energy is transferred between it and its surroundings. In some situations, however, the transfer of energy does not result in a change in temperature. That is the case whenever the physical characteristics of the substance change from one form to another; such a change is commonly referred to as a **phase change**.

Two common phase changes are from solid to liquid (melting) and from liquid to gas (boiling); another is a change in the crystalline structure of a solid. All such phase changes involve a change in the system's internal energy but no change in its temperature.

In solids, the forces between the molecules are large and the molecules are almost fixed in their positions inside the solid. In a liquid, the forces between the molecules are weaker and the molecules may move freely inside the volume of the liquid. However, they are not able to come out of the surface. In vapours or gases, the intermolecular forces are almost negligible and the molecules may move freely anywhere in the container. When a solid melts, its molecules move apart against the strong molecular attraction. This needs energy which must be supplied from outside. Thus, the internal energy of a given body is larger in liquid phase than in solid phase. Similarly, the internal energy of a given body in vapour phase is larger than that in liquid phase.

For any given pressure a phase change takes place at a definite temperature, usually accompanied by absorption or emission of heat and a change of volume and density.

In phase change ice at 0°C melts into water at 0°C. Water at 100°C boils to form steam at 100°C.



Latent Heat

The amount of heat required to change the state of the mass m of the substance is written as $Q = mL$, where L is called latent heat (literally, the "hidden" heat) because this added or removed energy does not result in a temperature change. Latent heat is also called heat of transformation. Its unit is cal/g or J/kg and dimension is $[L^2T^{-2}]$.

The value of L for a substance depends on the nature of the phase change as well as on the properties of the substance.

Latent heat of fusion: The latent heat of fusion is the heat energy required to change 1 kg of the material in its solid state at its melting point to 1 kg of the material in its liquid state. It is also the amount of heat energy released when at melting point 1 kg of liquid changes to 1 kg of solid. For water at its normal freezing temperature or melting point (0°C), the latent heat of fusion (or latent heat of ice) is

$$L_F = L_{ice} \approx 80 \text{ cal/g} \approx 336 \text{ kJ/kg}$$

Latent heat of vaporization: The latent heat of vaporization is the heat energy required to change 1 kg of the material in its liquid state at its boiling point to 1 kg of the material in its gaseous state. It is also the amount of heat energy released when 1 kg of vapour changes into 1 kg of liquid. For water at its normal boiling point or condensation temperature (100°C), the latent heat of vaporization (latent heat of steam) is

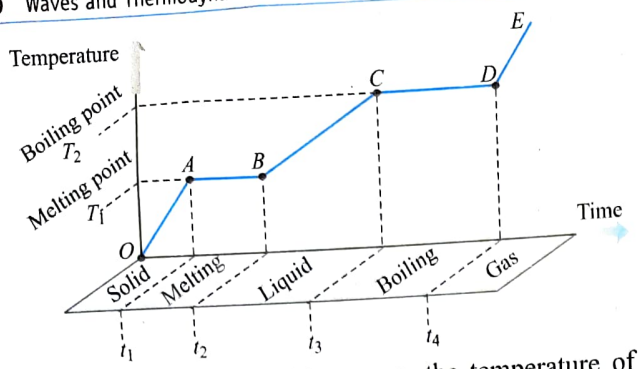
$$L_V = L_{steam} \approx 540 \text{ cal/g} \approx 40.8 \text{ kJ/mol} \approx 2260 \text{ kJ/kg}$$

Note:

Latent heat of vaporization is more than the latent heat of fusion. This is because when a substance gets converted from liquid to vapour, there is a large increase in volume. Hence, more amount of heat is required. But when a solid gets converted to a liquid, then the increase in volume is negligible. Hence, very less amount of heat is required. So, latent heat of vaporization is more than the latent heat of fusion.

HEATING CURVE

If to a given mass (m) of a solid, heat is supplied at constant rate P and a graph is plotted between temperature and time, the obtained graph is as shown in figure and is called **heating curve**.



Part A: On this portion of the curve, the temperature of the system varies linearly with the energy added, so the experimental result is a straight line on the graph.

$$Q = mc_s\Delta T \Rightarrow P\Delta t = mc_s\Delta T \quad [\text{as } Q = P\Delta t]$$

But as $(\Delta T/\Delta t)$ is the slope of temperature–time curve

$$c_s \propto \frac{1}{\text{Slope of line } OA}$$

i.e., specific heat (or thermal capacity) is inversely proportional to the slope of temperature–time curve.

Part B: When the temperature of the system reaches T_1, at A melting starts and at B all solid is converted into liquid. So between A and B substance is partly solid and partly liquid. The solid–liquid mixture remains at this temperature—even though energy is being added until all the solid melts i.e., melting of solid with melting point T_1. If L_F is the latent heat of fusion.

$$Q = mL_F \quad \text{or} \quad L_F = \frac{P(t_2 - t_1)}{m} \quad [\text{as } Q = P(t_2 - t_1)]$$

or $L_F \propto$ length of line AB

i.e., latent heat of fusion is proportional to the length of line of zero slope.

In this region, specific heat $\propto \frac{1}{\tan 0} = \infty$

Part C: In the region BC, temperature of liquid increases, so specific heat (or thermal capacity) of liquid will be inversely proportional to the slope of line BC

i.e., $c_L \propto \frac{1}{\text{Slope of line } BC}$

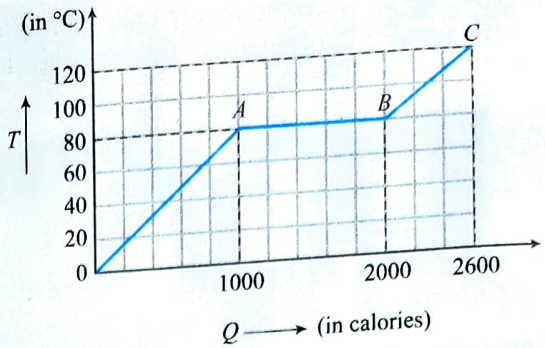
Part D: In the region CD, temperature is constant, so it represents the change of state, i.e., boiling with boiling point T_2. At C, all substance is in liquid state while at D, in vapour state and between C and D, partly liquid and partly gas. The length of line CD is proportional to latent heat of vaporization i.e., $L_v \propto$ Length of line CD

In this region, specific heat $\propto \frac{1}{\tan 0} = \infty$

Part E: The line DE represents gaseous state of substance with its temperature increasing linearly with time. The reciprocal of slope of line will be proportional to specific heat or thermal capacity of substance in vapour state.

ILLUSTRATION 1.41

A substance is in the solid form at 0°C. The amount of heat added to this substance and its temperature are plotted in the following graph.



If the relative specific heat capacity of the solid substance is 0.5, find from the graph **(i)** the mass of the substance; **(ii)** the specific latent heat of the melting process and **(iii)** the specific heat of the substance in the liquid state.

Specific heat capacity of water = 1000 cal/kg/K

Sol.

(i) 1000 cal of heat raises the temperature of the substance from 0°C to 80°C.

$\therefore$ $1000 = m\,(1000 \times 0.5) \times 80$

($\because$ specific heat = relative sp. heat × sp. heat of water)

or $m = 0.025$ kg

(ii) Latent heat = $200 \times 5 = 1000$ cal ($\because$ 1 div reads 200 cal)

$= 0.025 \times L_F$

$\therefore$ $L_F = 40000$ cal/kg

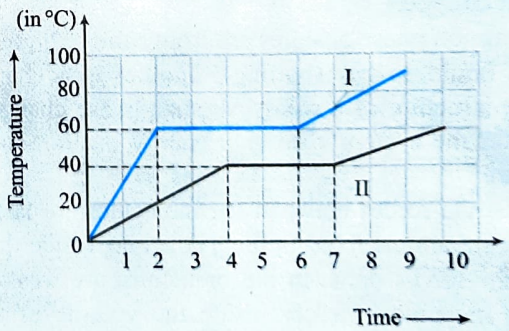
(iii) In the liquid state temperature rises from 80°C to 120°C that is, by 40°C after absorbing 600 cal.

$\therefore$ $0.025\ s \times 40 = 600$ or $s = 600$ cal/kg/K

ILLUSTRATION 1.42

Two bodies of equal masses are heated at a uniform rate under identical conditions. The change in temperature in the two cases is shown graphically. What are their melting points?

Also, find the ratio of their specific heats and latent heats.



Sol. The melting points of liquids I and II are 60°C and 40°C, respectively. Let R be the rate of supply of heat. We note from the graph that liquid I is heated through 60°C in 2 units of time and that liquid II is heated through 40°C in 4 units of time.

$\therefore$ $2R = m \times c_1 \times 60$ and $4R = m \times c_2 \times 40$

Hence, $\frac{c_1}{c_2} = \frac{1}{3}$

We note further that the temperature of I remains constant for 4 units of time and that of II for 3 units of time.

$\therefore$ $4R = mL_1$ and $3R = mL_2 \Rightarrow \frac{L_1}{L_2} = \frac{4}{3}$

ILLUSTRATION 1.43

(a) 10 g of water at –10°C heated to becomes steam at 100°C. Find the total heat absorbed it.

(b) Draw the graph temperature (θ)-heat (Q), showing the heat absorbed in each stage.

Sol.

(a) The heat required to increase the temperature from –10°C to 0°C is

$$Q_1 = mc_{ice}\Delta\theta = 10\times\frac{1}{2}[0-(-10)] = 50 \text{ cal}$$

The heat required to melt is $Q_2 = mL_f = 10\times 82 = 820$ cal

The heat required to rise the temperature of water from 0°C to 100°C is

$$Q_3 = mc_w\Delta\theta = 10\times 1\times(100-0) = 1000 \text{ cal}$$

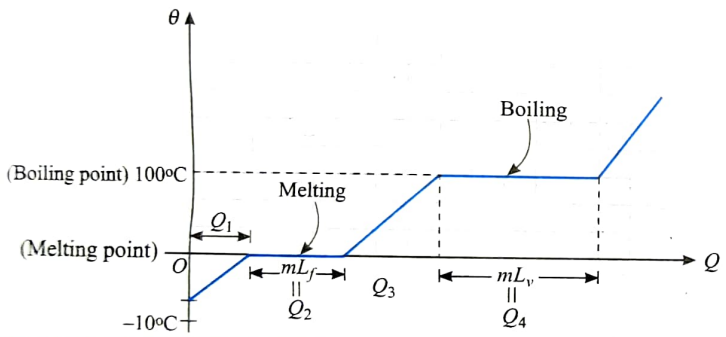
The heat required to vaporize the water is

$$Q_4 = mL_v = 10\times 539 = 5390 \text{ cal}$$

Then, the total heat required is $Q = Q_1 + Q_2 + Q_3 + Q_4$

$$= 50 + 820 + 1000 + 5390 = 7260 \text{ cal}$$

(b) θ-Q graph is given.



PRINCIPLE OF CALORIMETRY

Calorimetry means 'measuring heat'. When two bodies (one being solid and other being liquid or both being liquid) at different temperatures are mixed, heat will be transferred from the body at higher temperature to the body at lower temperature till both acquire same temperature. The body at higher temperature releases heat while the body at lower temperature absorbs it, so that

Heat lost = Heat gained

i.e., principle of calorimetry represents the law of conservation of heat energy.

Temperature of mixture (θ_{mix}) is always ≥ lower temperature (θ_L) and ≤ higher temperature (θ_H), i.e., $\theta_L \le \theta_{mix} \le \theta_H$.

It means that the temperature of mixture can never be lesser than lower temperature (as a body cannot be cooled below the temperature of cooling body) and greater than higher temperature (as a body cannot be heated above the temperature of heating body). Furthermore, usually rise in temperature of one body is not equal to the fall in temperature of the other body. Though, heat gained by one body is equal to the heat lost by the other.

Mixing of two substances when temperature changes only: It means that there is no phase change. Suppose that two substances having masses m_1 and m_2 gram specific heats c_1 and c_2, temperatures θ_1 and θ_2 ($\theta_1 > \theta_2$) are mixed together such that temperature of mixture at equilibrium is θ_{mix}.

Hence, Heat lost = Heat gained

$$\Rightarrow \quad m_1c_1(\theta_1 - \theta_{mix}) = m_2c_2(\theta_{mix} - \theta_2)$$

$$\Rightarrow \quad \theta_{mix} = \frac{m_1c_1\theta_1 + m_2c_2\theta_2}{m_1c_1 + m_2c_2}$$

ILLUSTRATION 1.44

A copper calorimeter of mass 1.5 kg has 200 g of water at 25°C. How much water at 50°C is required to be poured in the calorimeter, so that the equilibrium temperature attained is 40°C? (Sp. heat capacity of copper = 390 J/kg-°C)

Sol. Let 'm' g be the required mass of water.

Then heat lost by hot water,

$$Q_{lost} = (mg)\left(1\frac{\text{cal}}{\text{g-°C}}\right)(50-40)°\text{C} \qquad ...(i)$$

Heat gained by the calorimeter,

$$Q_1 = \left[(1500 \text{ g})\left(\frac{390}{4.2\times10^3}\frac{\text{cal}}{\text{g-°C}}\right)\right](40-25)°\text{C}$$

Heat gained by the water present in it,

$$Q_2 = \left[(200 \text{ g})\left(1\frac{\text{cal}}{\text{g-°C}}\right)\right](40-25)°\text{C}$$

Total heat gained by the calorimeter and the water present in it,

$$Q_{gain} = Q_1 + Q_2 \qquad ...(ii)$$

From the principle of calorimetry, Heat lost by hot body = Heat gained by cold body

So, equating Eqs. (i) and (ii), and solving for m, we get

$$m = 508.93 \text{ g}$$

ILLUSTRATION 1.45

The temperatures of equal masses of three different liquids A, B and C are 12°C, 19°C and 28°C, respectively. The temperature when A and B are mixed is 16°C, while when B and C are mixed, it is 23°C. What would be the temperature when A and C are mixed?

Sol. Let m = mass of each liquid, when A and B are mixed,

Heat lost by B = heat gained by A

$$\Rightarrow \quad ms_B(19-16) = ms_A(16-12) \quad \Rightarrow 3s_B = 4s_A \qquad ...(i)$$

When B and C are mixed,

Heat lost by C = heat gained by B

$$\Rightarrow \quad ms_C(28-23) = ms_B(23-19) \Rightarrow 5s_C = 4s_B \qquad ...(ii)$$

From Eqs. (i) and (ii), we get

$$16s_A = 12s_B = 15s_C \qquad ...(iii)$$

When A and C are mixed, Let θ = final temperature

Heat lost by C = heat gained by A

$$\Rightarrow \quad ms_C(28-\theta) = ms_A(\theta-12)$$

Using Eq. (iii), we get

$$\Rightarrow \quad 15s_C(28-\theta) = 15s_A(\theta-12)$$

$$\Rightarrow \quad 16s_A(28-\theta) = 15s_A(\theta-12)$$

On solving for θ, we get $\theta = \dfrac{16\times28+12\times15}{16+15}$

$$\Rightarrow \quad \theta = 20.26°\text{C}$$

ILLUSTRATION 1.46

When a block of metal of specific heat 0.1 cal/g°C and weighing 110 g is heated to 100°C and then quickly transferred to a calorimeter containing 200 g of a liquid at 10°C, the resulting temperature is 18°C. On repeating the experiment with 400 g of same liquid in the same calorimeter at same initial temperature, the resulting temperature is 14.5°C. Find:

(a) specific heat of the liquid,

(b) the water equivalent of calorimeter.

Sol. Let s be the specific heat of the liquid and W be the water equivalent of the calorimeter.

Heat lost by the block = heat gained by (liquid + calorimeter)

For the first case: $\Rightarrow 110 \times 0.1 \times (100 - 18)$

$= 200 \times s \times (18 - 10) + W \times 1 \times (18 - 10)$

$\Rightarrow \quad 1600s + 8W = 902$ ...(i)

For the second case: $\Rightarrow 110 \times 0.1 \times (100 - 14.5)$

$= 400 \times s \times (14.5 - 10) + W \times 1 \times (14.5 - 10)$

$\Rightarrow \quad 1800s + 4.5W = 940.5$ ...(ii)

On solving Eqs. (i) and (ii), we get $s = 0.48$ cal/g/°C and $W = 16.6$ g.

Mixing of two substances when temperature and phase both change or only phase changes: A very common example for this category is ice-water mixing.

Suppose water at temperature θ_W°C is mixed with ice at 0°C, first ice will melt and then its temperature rises to attain thermal equilibrium.

Hence, heat given = Heat taken

$$\Rightarrow \quad m_W c_W (\theta_W - \theta_{mix}) = m_i L_i + m_i c_W (\theta_{mix} - 0)$$

$$\Rightarrow \quad \theta_{mix} = \frac{m_W \theta_W - \frac{m_i L_i}{c_W}}{m_W + m_i}$$

ILLUSTRATION 1.47

Calculate the heat of fusion of ice from the following data for ice at 0°C added to water. Mass of calorimeter = 60 g, mass of calorimeter + water = 460 g, mass of calorimeter + water + ice = 618 g, initial temperature of water = 38°C, final temperature of the mixture = 5°C. The specific heat of calorimeter = 0.10 cal/g°C. Assume that the calorimeter was also at 0°C initially.

Sol. Mass of water = 460 – 60 = 400 g

Mass of ice = 618 – 460 = 158 g

Heat lost by water = heat gained by ice to melt + heat gained by (water obtained from melting of ice + calorimeter) to reach 5°C

$\Rightarrow \quad 400 \times 1 \times (38 - 5) = 158 \times L + 158 \times 1 \times 5 + 60 \times 0.1 \times 5$

(where L is the latent heat of fusion of ice)

$\Rightarrow \quad L = 78.35$ cal/g

ILLUSTRATION 1.48

A copper block of mass 2.5 kg is heated in a furnace to a temperature of 500°C and then placed on a large ice block. What is the maximum amount of ice that can melt? (Specific heat of copper = 0.39 J/g-°C; heat of fusion of water = 335 J/g).

Sol. Here, mass of copper block, $M = 2.5$ kg $= 2.5 \times 10^3$ g

Change in temperature of the copper block,

$\Delta T = 500°C - 0°C = 500°C$

Specific heat of copper, $c = 0.39$ J/g-°C

Latent heat of fusion of water, $L = 335$ J/g

If m is the mass of ice melted, then on applying the principle of calorimetry, we get

Heat gained by ice = Heat lost by copper block

i.e., $mL = Mc\,\Delta T$

or $$m = \frac{Mc\,\Delta T}{L} = \frac{2.5 \times 10^3 (g) \times 0.39 (J/g\text{-}°C) \times 500 (°C)}{335 (J/g)}$$

$$= 1455 \text{ g} = 1.455 \approx 1.5 \text{ kg}$$

ILLUSTRATION 1.49

Two 50 g ice cubes are dropped into 200 g of water in a thermally insulated container. If the water is initially at 25°C, and the ice comes directly from a freezer at –15°C.

(a) What is the final temperature at thermal equilibrium?

(b) What is the final temperature if only one ice cube is used?

[Given: specific heat of water = 4190 J/kg°C,
specific heat of ice = 2220 J/kg°C,
latent heat of fusion of ice = 333×10^3 J/kg]

Sol.

(a) There are three possibilities:

- No ice melts, and the water-ice system reaches thermal equilibrium at a temperature that is at or below the melting point of ice.
- The system reaches thermal equilibrium at the melting point of ice, with some of the ice melted.
- All of the ice melts, and the system reaches thermal equilibrium at a temperature at or above the melting point of ice.

First, suppose that no ice melts. The temperature of the water decreases from $T_{Wi} = 25°C$ to some final temperature T_f, and the temperature of the ice increases from $T_{Ii} = -15°C$ to T_f. If m_W is the mass of the water and c_W is its specific heat, then the heat rejected by water,

$$|Q| = c_W m_W (T_{Wi} - T_f)$$

If m_I is the mass of the ice and c_I is its specific heat, then the heat absorbed by ice,

$$Q = c_I m_I (T_f - T_{Ii})$$

Since no energy is lost to the environment, these two heats (in absolute value) must be the same. Consequently,

$$c_W m_W (T_{Wi} - T_f) = c_I m_I (T_f - T_{Ii})$$

Thus, the equilibrium temperature,

$$T_f = \frac{c_W m_W T_{Wi} + c_I m_I T_{Ii}}{c_W m_W + c_I m_I}$$

$$= \frac{4190 \times 0.200 \times 25 + 2220 \times 0.100 \times (-15)}{4190 \times 0.200 + 2220 \times 0.100} = 16.6°C$$

This value is above the melting point of ice, which invalidates our assumption that no ice has melted. That is, the calculation just completed does not take into account the melting of the ice and is in error. Consequently, we start with a new assumption: that the water and ice reach

thermal equilibrium at $T_f = 0°C$, with mass m ($< m_I$) of the ice melted.

The magnitude of the heat rejected by the water,

$$|Q| = c_W m_W T_{Wi},$$

And the heat absorbed by the ice, $Q = c_I m_I (0 - T_{Ii}) + mL_F$,

where L_F is the heat of fusion for water. The first term is the energy required to warm all the ice from its initial temperature to 0°C and the second term is the energy required to melt mass m of the ice. The two heats are equal, so

$$c_W m_W T_{Wi} = -c_I m_I T_{Ii} + mL_F$$

This equation can be solved for the mass m of ice melted:

$$m = \frac{c_W m_W T_{Wi} + c_I m_I T_{Ii}}{L_F}$$

$$= \frac{4190 \times 0.200 \times 25 + 2220 \times 0.100 \times (-15)}{333 \times 10^3}$$

$$= 5.3 \times 10^{-2} \text{ kg} = 53 \text{g}$$

Since the total mass of ice present initially was 100 g, there is enough ice to bring the water temperature down to 0°C. This is when the solution, the ice and water reach thermal equilibrium at a temperature of 0°C with 53 g of ice melted.

(b) Now, there is less than 53 g of ice present initially. All the ice melts, and the final temperature is above the melting point of ice.

The heat rejected by the water, $|Q| = c_W m_W (T_{Wi} - T_f)$ and the heat absorbed by the ice and the water it becomes when it melts,

$$|Q| = c_I m_I (0 - T_{Ii}) + c_W m_I (T_f - 0) + m_I L_F$$

The first term is the energy required to raise the temperature of the ice to 0°C, the second term is the energy required to raise the temperature of the melted ice from 0°C to T_f, and the third term is the energy required to melt all the ice. Since the two heats are equal,

$$c_W m_W (T_{Wi} - T_f) = c_I m_I (-T_{Ii}) + c_W m_I T_f + m_I L_F$$

The solution for T_f is: $T_f = \frac{c_W m_W T_{Wi} + c_I m_I T_{Ii} - m_I L_F}{c_W (m_W + m_I)}$.

Using the given values, we obtain $T_f = 2.5°C$.

ILLUSTRATION 1.50

A calorimeter contains 400 g of water at a temperature of 5°C. Then, 200 g of water at a temperature of +10°C and 400 g of ice at a temperature of –60°C are added. What is the final temperature of the contents of calorimeter?
Specific heat capacity of water = 1000 cal/kg K
Specific latent heat of fusion of ice = 80 × 1000 cal/kg
Relative specific heat of ice = 0.5

Sol. Let t be the common temperature above zero degree celsius.

Heat lost by calorimeter and water added

$$= 400 \times 10^{-3} \times 1000 (5 - t) + 200 \times 10^{-3} \times 1000 (10 - t)$$

Heat gained by ice $= 400 \times 10^{-3} \times 500 (60) + 400 \times 10^{-3} \times 80000 + 400 \times 10^{-3} \times 1000t$

$\because$ Heat lost = Heat gained

$\therefore$ $400 (5 - t) + 200 (10 - t) = 200 (60) + 400 \times 80 + 400t$

or $t = -40°C$

This is contradictory to the assumption that t is above zero.

Hence, the final temperature is either 0°C or less than 0°C.

Let the temperature be t°C (less than 0°C).

Then $400 \times 10^{-3} \times 1000 \times 5 + 200 \times 10^{-3} \times 1000 \times 10 + 600 \times 10^{-3} \times 80000 + 600 \times 10^{-3} \times 500 \times t = 400 \times 10^{-3} \times 500 (60 - t)$

or $400 \times 5 + 200 \times 10 + 600 \times 80 + 300t = 200 (60 - t)$

or $20 + 20 + 480 + 3t = 120 - 2t$

or $t = -80°C$

This is absurd because the lowest temperature is –60°C and we discard this second assumption also.

So the final temperature is 0°C.

Let x grams of water be converted into ice.

Heat lost $= 400 \times 10^{-3} \times 1000 \times 5 + 200 \times 10^{-3} \times 1000 \times 10 + x \times 10^{-3} \times 8000 = 4000 + 80x$

Heat gained $= 400 \times 10^{-3} \times 500 \times 60$

or $4000 + 80x = 12000$

or $x = 100$ g

Hence, the final result is 500 g of ice and 500 g of water at 0°C.

ILLUSTRATION 1.51

Determine the final result when 200 g of water and 20 g of ice at 0°C are in a calorimeter having a water equivalent of 30 g and 50 g of steam is passed into it at 100°C.

Sol. When steam is passed, the final temperature can be 0°C, between 0°C and 100°C, or 100°C.

We will consider all three possibilities.

Case I:

Final temperature = 0°C

In this case, all the steam condenses and then cools down to 0°C.

Heat given out by steam $= 50 \times 540 + 50 \times 1 \times (100 - 0) = 32000$ cal

Mass of ice which will melt by this heat $= \frac{32000}{80} = 400$ g

But there is only 20 g of ice in the calorimeter.

Hence final temperature cannot be 0°C.

Case II:

Final temperature = θ and $0 < \theta < 100°C$

Heat lost by steam = heat gained by (ice + water + calorimeter)

$\Rightarrow$ $50 \times 540 + 50 \times 1 \times (100 - \theta)$

$= 20 \times 80 + (20 + 200 + 30) \times 1 \times (\theta - 0)$

$\Rightarrow$ $\theta = 101.3°C$

The assumption ($0 < \theta < 100°C$) is proved to be wrong. Hence, the final temperature cannot be between 0°C and 100°C

$\Rightarrow$ Thus, the final temperature will be 100°C.

Case III:

Let m = mass of steam condensed.

Heat lost by steam = heat gained by ice to melt + heat gained by (water + water + calorimeter) to reach 100°C

$\Rightarrow$ $m (540) = 20 \times 80 + (20 + 200 + 30) \times (100 - 0)$

$\Rightarrow$ $m = 26600/540 \approx 49$ g

$\Rightarrow$ 49 g of steam gets condensed and the final temperature is 100°C.

ILLUSTRATION 1.52

1 kg ice at –20°C is mixed with 1 kg steam at 200°C. Then find equilibrium temperature and mixture content.

Sol. Let equilibrium temperature be 100°C, heat required to convert 1 kg ice at –20°C to 1 kg water at 100°C is equal to

$$H_1 = 1 \times \frac{1}{2} \times 20 + 1 \times 80 + 1 \times 1 \times 100 = 190 \text{ kcal}$$

Heat released by steam to convert 1 kg steam at 200°C to 1 kg water at 100°C is equal to

$$H_2 = 1 \times \frac{1}{2} \times 100 + 1 \times 540 = 590 \text{ kcal}$$

1 kg ice at –20°C = H_1 + 1 kg water at 100°C ...(i)

1 kg steam at 200°C = H_2 + 1 kg water at 100°C ...(ii)

By adding Eqs. (i) and (ii)

1 kg ice at –20°C + 1 kg steam at 200°C = $H_1 + H_2$ + 2 kg water at 100°C.

Here heat required to ice is less than heat supplied by steam, so mixture equilibrium temperature is 100°C. Then steam is not completely converted into water.

So mixture has water and steam, which is possible only at 100°C. Mass of steam which converted into water is equal to

$$m = \frac{190 - 1 \times \frac{1}{2} \times 100}{540} = \frac{7}{27} \text{ kg}$$

So mixture content

$$\text{Mass of steam} = 1 - \frac{7}{27} = \frac{20}{27} \text{ kg}$$

$$\text{Mass of water} = 1 + \frac{7}{27} = \frac{34}{27} \text{ kg}$$

CONCEPT APPLICATION EXERCISE 1.2

1. A child running a temperature of 101° F is given an antipyrin (i.e., a medicine that lowers fever) which causes an increase in the rate of evaporation of sweat from his body. If the fever is brought down to 98°F in 20 min, what is the average rate of extra evaporation caused by the drug? Assume the evaporation mechanism to be the only way by which the heat is lost. The mass of the child is 30 kg. The specific heat of human body is approximately the same as that of water, and latent heat of water at that temperature is about 580 cal/g.
2. A piece of ice of mass 50 g exists at a temperature of –20°C. Determine the total heat required to convert it completely to steam at 100°C. (Specific heat capacity of ice = 0.5 cal/g-°C; specific latent heat of fusion for ice = 80 cal/g and specific latent heat of vaporisation for water = 540 cal/g).
3. Ice of mass 600 g and at a temperature of –10°C is placed in a copper vessel heated to 350°C. The resultant mixture is 550 g of ice and water. Find the mass of the vessel. The specific heat capacity of copper (c) = 100 cal/kg-K.
4. When a small ice crystal is placed in overcooled water it begins to freeze instantaneously.
   i. What amount of ice is formed from 1 kg of water overcooled to –8°C? L of water = 336×10^3 J/kg and s of water = 4200 J/kg/K.
   ii. What should be the temperature of the overcooled water in order that all of it be converted into ice at 0°C?
5. An electric heater whose power is 54 W is immersed in 650 cm^3 water in a calorimeter. In 3 min the water is heated by 3.4°C. What part of the energy of the heater passes out of the calorimeter in the form of radiant energy?
6. An ice cube whose mass is 50 g is taken from a refrigerator where its temperature was –10°C. If no heat is gained or lost from outside, how much water will freeze onto the cube if it is dropped into a beaker containing water at 0°C? Latent heat of fusion = 80 kcal/kg, specific heat capacity of ice = 500 cal/kg/K.
7. Equal volumes of three liquids of densities ρ_1, ρ_2 and ρ_3, specific heat capacities c_1, c_2 and c_3 and temperatures t_1, t_2 and t_3, respectively, are mixed together. What is the temperature of the mixture? Assume no changes in volume on mixing.
8. Victoria Falls in Africa is 122 m in height. Calculate the rise in temperature of the water if all the potential energy lost in the fall is converted into heat.
9. Equal masses of three liquids A, B and C are taken. Their initial temperatures are 10°C, 25°C and 40°C, respectively. When A and B are mixed the temperature of the mixture is 19°C. When B and C are mixed, the temperature of the mixture is 35°C. Find the temperature if all three are mixed.
10. An earthenware vessel loses 1 g of water per second due to evaporation. The water equivalent of the vessel is 0.5 kg and the vessel contains 9.5 kg of water. Find the time required for the water in the vessel to cool to 20°C from 30°C. Neglect radiation losses. Latent heat of vapourization of water in this range of temperature is 540 cal/g.
11. A certain amount of ice is supplied heat at a constant rate for 7 min. For the first 1 min, the temperature rises uniformly with time; then it remains constant for the next 4 min and again rises at a uniform rate for the last 2 min. Explain physically these observations and calculate the final temperature. L of ice = 336×10^3 J/kg and c_{water} = 4200 J/kg/K.
12. 1 g steam at 100°C is passed in an insulated vessel having 1 g ice at 0°C. Find the equilibrium temperature of the mixture. Neglect heat capacity of the vessel.
13. The ratio of the densities of the two bodies is 3 : 4 and the ratio of specific heats is 4 : 3. Find the ratio of their thermal capacities for unit volume?

ANSWERS

1. 4.3 g/min **2.** 36500 cal **3.** 200 g

4. (i) 100 g (ii) 80°C **5.** 438 J **6.** 3.125 g

7. $\frac{\rho_1 c_1 t_1 + \rho_2 c_2 t_2 + \rho_3 c_3 t_3}{\rho_1 c_1 + \rho_2 c_2 + \rho_3 c_3}$ **8.** 0.29 K **9.** 30.5°C

10. 3 min 5 s **11.** 40°C **12.** 100°C **13.** 1:1

Solved Examples

EXAMPLE 1.1

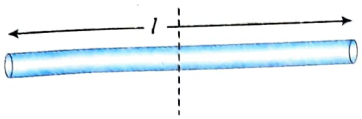
Find out the increase in moment of inertia I of a uniform rod (coefficient of linear expansion α) about its perpendicular bisector when its temperature is slightly increased by ΔT.

Sol. **Method 1:** As temperature increases, length of the rod also increases and hence, moment of inertia of the rod also increases.

Let the mass and length of the uniform rod be M and l, respectively.

Moment of inertia of the rod about its perpendicular bisector,

$$I = \frac{Ml^2}{12}$$



Increase in length of the rod when temperature is increased by Δt, is given by

$$\Delta l = l\alpha\Delta T \quad ...(i)$$

$\therefore$ Thus, new moment of inertia of the rod,

$$I' = \frac{M}{12}(l + \Delta l)^2 = \frac{M}{12}(l^2 + \Delta l^2 + 2l\,\Delta l)$$

As change in length Δl is very small, neglecting $(\Delta l)^2$, we get

$$I' = \frac{M}{12}(l^2 + 2l\,\Delta l)$$

$$\Rightarrow \quad I' = \frac{Ml^2}{12} + \frac{Ml\Delta l}{6} = I + \frac{Ml\Delta l}{6}$$

Hence, increase in moment of inertia,

$$\Delta I = I' - I = \frac{Ml\Delta l}{6} = 2 \times \left(\frac{Ml^2}{12}\right)\frac{\Delta l}{l}$$

$$\Rightarrow \quad \Delta I = 2I\alpha\,\Delta T \quad \text{[Using Eq. (i)]}$$

Method 2: We have $I = \frac{Ml^2}{12}$

We can write $dI = d\left(\frac{Ml^2}{12}\right) = \frac{M}{12}d(l^2)$

$$\Rightarrow \quad dI = \frac{M}{12}\left(\frac{d(l^2)}{dl}\right)\cdot dl = \frac{M}{12}\cdot 2l\cdot dl \quad ...(i)$$

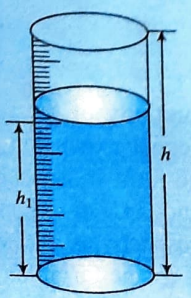
We know that $\Delta l = l\alpha\Delta T$ ...(ii)

From (i) and (ii), we get

$$dI = \frac{M}{12}2l\cdot l\alpha dT = 2\left(\frac{Ml^2}{12}\right)\alpha\,dT = 2I\alpha\,dT$$

EXAMPLE 1.2

Consider a cylindrical container of cross-section area A, length h and having coefficient of linear expansion α_c. The container is filled by liquid of real expansion coefficient γ_L up to height h_1. When temperature of the system is increased by $\Delta\theta$ then



(a) Find out new height, area and volume of cylindrical container and new volume of liquid.

(b) Find the height of liquid level when expansion of container is neglected.

(c) Find the relation between γ_L and α_c for which volume of container above the liquid level (i) increases, (ii) decreases and (iii) remains constant.

(d) On the surface of a cylindrical container a scale is attached for the measurement of level of liquid filled inside it. If we increase the temperature of the system by $\Delta\theta$, then (i) Find the height of liquid level as shown by the scale on the vessel. Neglect expansion of liquid. (ii) Find the height of liquid level as shown by the scale on the vessel. Neglect expansion of container.

Sol. On increasing the temperature, the height, area of cross section and volume of the cylinder will increase.

(a) New height $h_f = h\{1 + \alpha_c\Delta\theta\}$

New area of cross section $A_f = A\{1 + 2\alpha_c\,\Delta\theta\}$

New volume of container $V_f = Ah\,\{1 + 3\,\alpha_c\,\Delta\theta\}$

New volume of liquid $V_w = Ah_1(1 + \gamma_L\,\Delta\theta)$

(b) The height of liquid level when expansion of container is neglected

$$h_f = \frac{V_w}{A} = \frac{Ah_1(1 + \gamma_L\Delta\theta)}{A}$$

$$\Rightarrow \quad h_f = h_1\,\{1 + \gamma_L\,\Delta\theta\}$$

(c) The initial volume of container above the liquid

$$\Delta V_1 = Ah - Ah_1$$

Final volume of container above the liquid

$$\Delta V_2 = Ah(1 + 3\alpha_c\Delta\theta) - Ah_1(1 + \gamma_L\Delta\theta)$$

If volume of above container increases $\Delta V_2 > \Delta V_1$

$$[Ah(1 + 3\alpha_c\Delta\theta) - Ah_1(1 + \gamma_L\Delta\theta)] > [Ah - Ah_1]$$

Which gives $3h\,\alpha_c > h_1\,\gamma_L$

Similarly, we can prove if the volume above container decreases, then $3\,h\alpha_c < h_1\,\gamma_L$ and for no change in volume $3\,h\,\alpha_c = h_1\,\gamma_L$

(d) (i) The area of container will increase, the area of container at this temperature will be

$$A_f = A\{1 + 2\alpha_c\,\Delta\theta\}$$

As liquid does not expand the volume of the liquid will be same as initial volume $= Ah_1$.

Hence height of the liquid column will be

$$h_1' = \frac{Ah_1}{A_f} = \frac{Ah_1}{A(1 + 2\alpha_c\Delta\theta)}$$

$$\Rightarrow \quad h'_1 = h_1\,(1 - 2\alpha_c\,\Delta\theta)$$

(ii) In this case we are neglecting the expansion of container.

The volume of liquid at this temperature

$$V_w = Ah_1\,(1 + \gamma_L\Delta\theta)$$

Hence height of the liquid in container is

$$h_f = \frac{V_w}{A} = \frac{Ah_1(1 + \gamma_L\Delta\theta)}{A}$$

$$\Rightarrow \quad h_f = h_1\,(1 + \gamma_L\,\Delta\theta)$$

EXAMPLE 1.3

An ice cube of mass 0.1 kg at 0°C is placed in an isolated container which is at 227°C. The specific heat S of the container varies with temperature T according to the empirical relation $S = A + BT$, where $A = 100$ cal/kg-K and $B = 2 \times 10^{-2}$ cal/kg-K^2. If the final temperature of the container is 27°C, determine the mass of the container.

(Latent heat of fusion for water = 8×10^4 cal/kg-K, Specific heat of water = 10^3 cal/kg-K)

Sol. Heat received by ice, $Q_1 = mL + mc\Delta T$

$$= 0.1\times8\times10^4 + 0.1\times10^3\times(27-0) = 10700 \text{ cal}$$

Heat received by the container,

$$Q_2 = \int_{T_1}^{T_2} m_c S\,dT = m_c \int_{T_1}^{T_2} (A+BT)\,dT$$

$$\Rightarrow \quad Q_2 = m_c\left[AT + \frac{BT^2}{2}\right]_{T_1}^{T_2} = m_c\left[A(T_2-T_1) + \frac{B}{2}(T_2^2 - T_1^2)\right]$$

$$= m_c(T_2-T_1)\left[A + \frac{B}{2}(T_2+T_1)\right]$$

Here, $T_1 = 273 + 227°C = 500$ K

and $T_2 = 273 + 27°C = 300$ K

$$\therefore \quad Q_2 = m_c(300-500)\left[100 + \frac{2\times10^{-2}}{2}(300+500)\right]$$

$$= -m_c \times 200 \times 108 = -21600\, m_c \text{ cal}$$

Hence, heat given by container $= -Q_2 = 21600\, m_c$ cal

From principle of calorimetry,

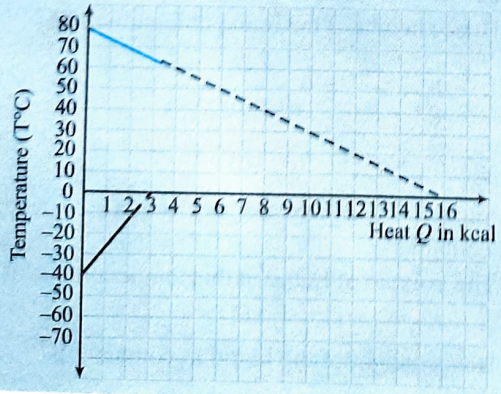
Heat received by ice = Heat given by container

$\therefore \quad 10700 = 21600 m_c$

$$\Rightarrow \quad m_c = \frac{107}{216} = 0.495 \text{ kg}$$

EXAMPLE 1.4

Figure shows the temperature versus heat evolved/absorbed by a mixture of ice and water. Sketch the remaining portion of the graph. Determine the equilibrium temperature, and the total heat exchanged.



Sol. From the graph, it is clear the initial temperature of water is 80°C and that of ice is –40°C. Let the mass of water and ice be m_1 and m_2, respectively.

From the graph, if water finally comes to the temperature 0°C, the heat released by water,

$$Q_1 = m_1 s_w \Delta T$$

$$\Rightarrow \quad 16000 = m_1 \times 1 \times (80-0) \Rightarrow m_1 = 200 \text{ g}$$

$$Q_2 = m_1 s_l \Delta T$$

$$\Rightarrow \quad 3\times1000 = m_2 \times \frac{1}{2} \times 40 \Rightarrow m_2 = 150 \text{ g}$$

Again following the graph, it can be observed that the he required to melt the ice completely is less than the heat release by water. Hence, final temperature of the mixture should b greater than 0°C. Let us assume final temperature of the mixtu is θ°C.

Heat given by water, $Q_1 = m_1 s_w (80 - \theta)$

$$= 200\times1(80-\theta) = 200(80-\theta) \quad \ldots(i$$

Heat received by ice,

$$Q_2 = m_2 s_{ice}[0-(-40)] + m_2 L + m_2 s_w(\theta - 0)$$

$$= 150\times\frac{1}{2}\times40 + 150\times80 + 150\times1\times\theta$$

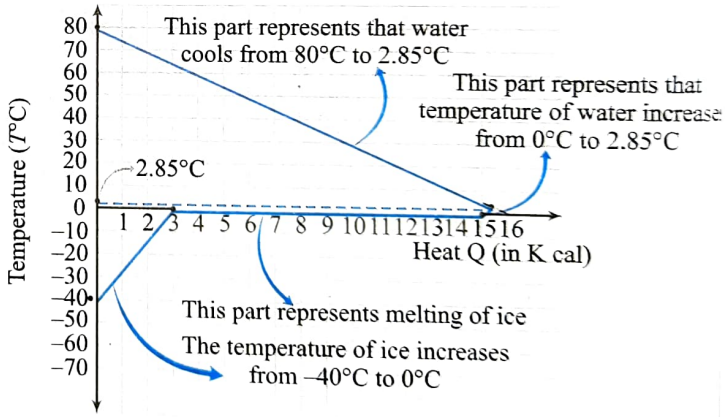
$$= 15000 + 150\theta \quad \ldots(ii$$

Using principle of calorimetry, Heat given by water = Hea received by ice

$$\Rightarrow \quad 200(80-\theta) = 15000 + 150\theta$$

$$\Rightarrow \quad 350\theta = 1000 \Rightarrow \theta = \frac{20}{7} = 2.85° \text{ C}$$

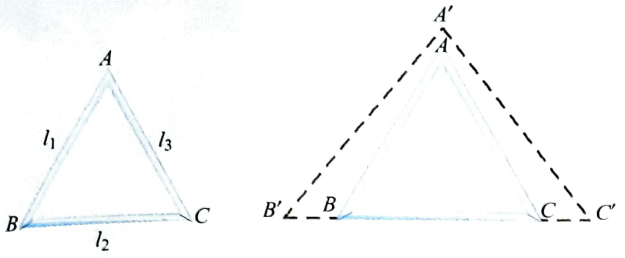
Hence, final temperature of the mixture should be 2.85°C. Th complete graph for the situation is as follows:



EXAMPLE 1.5

Three rods AB, BC and CA, identical in shape and size, are hinged together to form an equilateral triangle. Rods AB and CA have same coefficient of linear expansion i.e., $\alpha_1 = 12 \times 10^{-6}$/°C, while that of rod BC is $\alpha_2 = 18 \times 10^{-6}$/°C. What should be the rise in temperature of the system of rods, so that the angle opposite to the rod BC increases by an amount 0.02°?

Sol. The rods are initially of the same length. As they ar heated, due to the unequal expansion of the rods BC and A and CA ($\alpha_2 > \alpha_1$) would form an isosceles triangle, with side B greater than sides AB and CA. Let l_1, l_2 and l_3 be the lengths of th sides AB, BC and CA respectively and θ be the angle opposite t the side BC, then using cosine formula,



At temperature T At temperature $T + dT$

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos\theta$$

or $$l_2^2 = l_1^2 + l_3^2 - 2l_1 l_3 \cos\theta$$

or $$2l_1 l_3 \cos\theta = l_1^2 + l_3^2 - l_2^2 \quad ...(i)$$

Differentiating (i) with respect to temperature T, we get,

$$2\left[l_1 l_3(-\sin\theta)\frac{d\theta}{dT} + l_3\cos\theta\frac{dl_1}{dT} + l_1\cos\theta\frac{dl_3}{dT}\right]$$

$$= 2\left[l_1\frac{dl_1}{dT} + l_3\frac{dl_3}{dT} - l_2\frac{dl_2}{dT}\right]$$

When $l_1 = l_2 = l_3 = l$ (say) and $\theta = 60°$,

$$-l\left(\frac{\sqrt{3}}{2}\right)\frac{d\theta}{dT} + \frac{1}{2}\frac{dl_1}{dT} + \frac{1}{2}\frac{dl_3}{dT} = \frac{dl_1}{dT} + \frac{dl_3}{dT} - \frac{dl_2}{dT}$$

Let $\frac{dl_1}{dT} = \frac{dl_3}{dT} = l\alpha_1$ and $\frac{dl_2}{dT} = l\alpha_2$

$$\Rightarrow \quad -\left(\frac{\sqrt{3}}{2}\right)\frac{d\theta}{dT} + \alpha_1 = 2\alpha_1 - \alpha_2$$

or $$dT = \frac{\sqrt{3}}{2}\frac{d\theta}{(\alpha_2 - \alpha_1)}$$

Now, $d\theta = 0.02° = 0.02 \times \frac{\pi}{180}$ radian

and $\alpha_2 - \alpha_1 = 6 \times 10^{-6}$ /°C

$$\therefore \quad dT = \frac{\sqrt{3}}{2} \times \frac{\pi \times 10^{-2} \times 10^6}{90 \times 6} = \frac{250\pi\sqrt{3}}{27}\,°\text{C}$$

EXAMPLE 1.6

An iron cube of edge 40 cm floats on mercury, at a temperature of 0°C. What is the depth of its immersion into the mercury at 0°C. When temperature is increased to 100°C, how much further will the block sink?
Take density of Hg and Fe at 0°C as 13.6 g/cc and 7.6 g/cc, respectively, while the coefficient of volume and linear expansions of Hg and Fe are 1.8×10^{-4}/°C and 1.1×10^{-5}/°C, respectively.

Sol. Let L_0, h_0 and ρ_0 be edge of the iron cube, depth of its immersion into the mercury and density of mercury, respectively, all referred to 0°C. Further, let L_1, h_1 and ρ_1 be the corresponding values at a temperature of 100°C.
As the cube is floating, using Archimedes' principle, Weight of iron cube = Weight of Hg displaced

i.e., $$(L_0^3\sigma_0)g = (L_0^2 h_0 \rho_0)g \quad ...(i)$$

$$\Rightarrow \quad h_0 = \frac{L_0\sigma_0}{\rho_0} = \frac{40\text{ cm} \times 7.6\text{ g/cc}}{13.6\text{ g/cc}} = 22.35\text{ cm}$$

Similarly at 100°C, $L_1^3\sigma_1 = L_1^2 h_1 \rho_1$ ...(ii)

Since, the mass of iron remains the same, hence equating equations (i) and (ii), we get

$$L_0^2 h_0 \rho_0 = L_1^2 h_1 \rho_1$$

or, $$\frac{h_1}{h_0} = \left(\frac{L_0}{L_1}\right)^2\left(\frac{\rho_0}{\rho_1}\right) \quad ...(iii)$$

But $$\rho_0 = \rho_1(1 + \gamma_m \times 100)$$

$$\Rightarrow \quad \frac{\rho_0}{\rho_1} = (1 + 100\ \gamma_m) \quad ...(iv)$$

and $$L_1 = L_0(1 + \alpha_s \times 100)$$

$$\Rightarrow \quad \frac{L_0}{L_1} = \frac{1}{(1 + 100\alpha_s)} = (1 + 100\alpha_s)^{-1}$$

$$\Rightarrow \quad \left(\frac{L_0}{L_1}\right)^2 = (1 + 100\alpha_s)^{-2} = (1 - 200\,\alpha_s) \quad ...(v)$$

Using equations (ii), (iv) and (v), we can write

$$\frac{h_1}{h_0} = (1 - 200\,\alpha_s)(1 + 100\,\gamma_m)$$

$$= (1 - 200\,\alpha_s + 100\,\gamma_m - 200\alpha_s \times 100\gamma_m)$$

$$= 1 + 100(\gamma_m - 2\alpha_s)$$

(Ignoring the product of α_s and γ_m; two or more small terms)

or $$\frac{h_1 - h_0}{h_0} = 100(\gamma_m - 2\alpha_s)$$

or $$h_1 - h_0 = 100\ h_0(\gamma_m - 2\alpha_s)$$

Substituting h_0, γ_m and α_s, we get $h_1 - h_0 = 35.31 \times 10^{-2}$ cm

EXAMPLE 1.7

A beaker contains 1 litre of water at room temperature (i.e., 30°C). A piece of ice weighing 400 g at –10°C is dropped gently into the beaker. Find the final temperature and contents of the mixture. (Specific heat capacity of ice = 0.5 cal/g-°C; specific latent heat of fusion for ice = 80 cal/g).

Sol. Heat required to raise the temperature of the ice from –10°C to its melting point (i.e., 0°C) is given by

$$\Delta Q_1 = ms\Delta T = (400\text{ g})\left(\frac{0.5\text{ cal}}{\text{g°C}}\right)(0 + 10°\text{C}) = 2000\text{ cal}$$

Heat required for the complete melting of ice will be

$$\Delta Q_2 = mL = (400\text{ g})(80\text{ cal/g}) = 32000\text{ cal}$$

Heat released when the water at 30°C, cools down to 0°C,

$$\Delta Q_3 = ms\Delta T = (1000\text{ g})\left(1\frac{\text{cal}}{\text{g°C}}\right)(30 - 0)°\text{C} = 30000\text{ cal}$$

Now, let us understand the process in detail. Initially, the temperature of 1 litre water (i.e., 30°C) is more than that of ice (i.e., –10°C); so the former loses heat to the latter, owing to a temperature difference between them. By the time the heat released by water is 2000 cal, the ice reaches its melting point ($\Delta Q_1 = 2000$ cal).

If T is the temperature of 1 litre of water at that instant, then

$$2000\,\text{cal} = (1000\,\text{g})\left(1\,\frac{\text{cal}}{\text{g°C}}\right)(30-T)°\text{C}$$

$\therefore \quad T = 28°\text{C}$

Now, the temperature of 1 litre of water is 28°C and that of ice is 0°C. Due to a temperature difference between the two bodies, the former still continues losing heat to the latter.

Now, if the temperature of 1 litre water were to come down to 0°C, the heat evolved would be 30000 – 2000 = 28000 cal

However, the complete fusion of ice requires a total heat of 32000 cal, so only a fraction of ice melts, and the entire mixture comes to a final temperature of 0°C.

If $m'g$ be the mass of ice melted, then $(m'g)$ (80 cal/g) = 28000 cal

or, $\quad m' = 350$ g

Thus, mass of water in the mixture = Mass of water initially + Mass of melted ice = 1000 + 350 = 1350 g and, the mass of ice (which remained unmelted) = 400 – 350 = 50 g

EXAMPLE 1.8

Two steel rods and an aluminium rod of equal length l_0 and equal cross-section are joined rigidly at their ends as shown in the figure below.

All the rods are in a state of zero tension at 0°C. The temperature is raised to θ. Coefficient of linear expansion of aluminium and steel are α_a and α_s respectively ($\alpha_a > \alpha_s$). Young's modulus of aluminium is Y_a and that of steel is Y_s. Find the stress developed in aluminium rod and final length of the system.

Sol. As the coefficient of linear expansion of aluminium is greater than that of steel. The aluminium has tendency to expand more than the steel. But system will reach to common length and hence, compression stress will be developed in the aluminium rod and tensile stress in the steel rods.

Net change in the length in aluminium rod,

$$\Delta l = \Delta l_{\text{thermal}} - \Delta l_{\text{stress}}$$

$$\Rightarrow \quad \Delta l = l_0\alpha_a\theta - \frac{\sigma l_0}{Y_a} \qquad \ldots(\text{i})$$

σ is the stress developed in aluminium rod.

Net external force on the system is zero and hence, force applied by aluminium rod will be equal to force experienced by steel rods. The force applied by aluminium rod is equally shared by both steel rods as area or cross section of rods are same hence, stress developed in the steel rod should be equal to half the stress developed in aluminium rod.

Net change in the length of a steel rod, $\Delta l = \Delta l_{\text{thermal}} + \Delta l_{\text{stress}}$

$$\Rightarrow \quad \Delta l = l_0\alpha_s\theta + \frac{\sigma l_0}{2Y_s} \qquad \ldots(\text{ii})$$

Hence, from (i) and (ii), we get $l_0\alpha_a\theta - \dfrac{\sigma l_0}{Y_a} = l_0\alpha_s\theta + \dfrac{\sigma l_0}{2Y_s}$

$$\sigma l_0\left[\frac{1}{Y_a} + \frac{1}{2Y_s}\right] = -l_0\alpha_s\theta + l_0\alpha_a\theta$$

$$\sigma = \frac{(\alpha_a - \alpha_s)\theta \cdot Y_a \cdot 2Y_s}{(2Y_s + Y_a)}$$

Substituting the value of σ in (i),

$$\Delta l = l_0\alpha_a\theta - \frac{l_0}{Y_a}\left[\frac{(\alpha_a - \alpha_s)\theta \cdot 2Y_a \cdot Y_s}{(2Y_s + Y_a)}\right]$$

$$= l_0\theta\left[\alpha_a - \frac{2Y_s(\alpha_a - \alpha_s)}{(2Y_s + Y_a)}\right]$$

$$= l_0\theta\left[\frac{\alpha_a(2Y_s + Y_a) - 2Y_s(\alpha_a - \alpha_s)}{(2Y_s + Y_a)}\right]$$

$$= l_0\theta\left[\frac{\alpha_a Y_a + 2Y_s\alpha_s}{2Y_s + Y_a}\right]$$

or $\quad \Delta l = l_0\theta\left[\dfrac{\alpha_a Y_a + 2\alpha_s Y_s}{Y_a + 2Y_s}\right]$

Hence, final length $l' = l_0 + \Delta l$

$$l' = l_0\left[1 + \theta\left(\frac{\alpha_a Y_a + 2\alpha_s Y_s}{Y_a + 2Y_s}\right)\right]$$

EXAMPLE 1.9

A cube of coefficient of linear expansion α_s is floating in a bath containing a liquid of coefficient of volume expansion γ_l. When the temperature is raised by ΔT, the depth up to which the cube is submerged in the liquid remains the same. Find the relation between α_s and γ_l showing all the steps.

Sol. Let l be side of cube at initial temperature and d be the depth of cube submerged. Then according law of floatation

Weight of solid = weight of liquid displaced

$$Mg = l^2 d\rho_l g \qquad \ldots(\text{i})$$

When temperature is increased, the weight remains same. Length of side of cube increases, density of liquid decrease and depth remains unchanged (as given)

$\therefore \quad Mg = (l')^2\, d\rho_l g$

But $\quad l' = l(1 + \alpha_s\Delta T),\ \rho_l = \dfrac{\rho_l}{1 + \gamma_l\Delta T}$

Substituting these values in Eq. (iii), we get

$$l^2 d\rho_l g = l^2(1 + \alpha_s\Delta T)^2\,\frac{\rho_l}{1 + \gamma_l\Delta T}\,g$$

$$\Rightarrow \quad 1 + \gamma_l\Delta T = (1 + \alpha_s\Delta T)^2$$

As $\alpha_s\Delta T << 1$, using binomial theorem

$$1 + \gamma_l\Delta T = 1 + 2\alpha_s\Delta T \quad \Rightarrow \gamma_l = 2\alpha_s$$

EXAMPLE 1.10

In an insulated vessel, 0.05 kg steam at 373 K and 0.45 kg of ice at 253 K are mixed. Find the final temperature of the mixture (in Kelvin).

Given, $L_{fusion} = 80$ cal/g = 336 J/g

$L_{vapourization} = 540$ cal/g = 2268 J/g

$S_{ice} = 2100$ J/kg-K = 0.5 cal/g-K

and $S_{water} = 4200$ J/kg-K = 1 cal/g-K

Sol. 0.05 kg steam at 373 K $\xrightarrow{Q_1}$ 0.05 kg water at 373 K

0.05 kg water at 373 K $\xrightarrow{Q_2}$ 0.05 kg water at 273 K 0.45 kg ice at 253 K $\xrightarrow{Q_3}$ 0.45 kg ice at 273 K 0.45 kg ice at 273 K $\xrightarrow{Q_4}$ 0.45 kg water at 273 K

$Q_1 = (50)(540) = 27000$ cal = 27 kcal

$Q_2 = (50)(1)(100) = 5000$ cal = 5 kcal

$Q_3 = (450)(0.5)(20) = 4500$ cal = 4.5 kcal

$Q_4 = (450)(80) = 36000$ cal = 36 kcal

Now since $Q_1 + Q_2 > Q_3$ but $Q_1 + Q_2 < Q_4$ ice will come to 273 K from 253 K, but whole ice will not melt. Therefore, temperature of the mixture is 273 K.

EXAMPLE 1.11

A metal block of density 5000 kg/m^3 and mass 2kg is suspended by a spring of force constant 200 Nt/m. The spring block system is submerged in a water vessel. Total mass of water in it is 300 g and in equilibrium the block is at a height 40cm above the bottom of vessel. If the support is broken. Find the rise in temperature of water. Specific heat of the initial of block is 250 J/kg K and that of water is 4200 J/kg K. Neglect the heat capacities of the vessel and the spring.

Sol. When the block is in equilibrium in the water and spring is stretched by a distance x and the spring force balances the effective weight of block i.e. weight of block minus the buoyant force on block. Thus for equilibrium of block in water, we have

kx + weight of liquid displaced = weight of block

$$200x + \frac{2}{5000} \times 1000 \times 10 = 2 \times 10$$

or $$x = \frac{20-4}{200} = \frac{16}{200} = 0.08\,\text{m} = 8\,\text{cm}$$

Thus, in equilibrium, energy stored in spring is

$$U = \frac{1}{2}kx^2 = \frac{1}{2} \times 200 \times (0.08)^2 = 0.64 \text{ Joule}$$

When the support is broken, the mass fall down to the bottom of vessel and the potential energy stored in spring and the gravitational potential energy of block is released and used in heating the water and block. Thus, we have

$$0.64 + mg(0.4) = m_w \times s_w \times \Delta T + m_B \times s_B \times \Delta T$$

or $$0.64 + 2 \times 10 \times 0.4 = [0.3 \times 4200 + 2 \times 234]\Delta T$$

or $$\Delta T = \Delta T = \frac{8.64}{1768} = 0.005°\text{C}$$

EXAMPLE 1.12

A copper cube of mass 200 g slides down on a rough inclined plane of inclination 37° at a constant speed. Assume that any loss in mechanical energy goes into the block as thermal energy. Find the increase in temperature of block as it slides down through 60 cm. Given that specific heat of copper is 420 J/kg/K.

Sol. As block slides at constant speed, friction on block exactly balances the gravitational pull on it, $mg \sin\theta$. Thus

$$f = mg \sin\theta = 0.2 \times 10 \times \sin 37° = 2 \times \frac{3}{5} = 1.2 \text{ N}$$

As block slides 60 cm, work done by it against friction is

$$W = fl = 1.2 \times 0.6 = 0.72 \text{ J}$$

This energy is only used in increasing the temperature of copper block. Thus

$$0.72 = ms\,\Delta T$$

or $$\Delta T = \frac{0.72}{ms} = \frac{0.72}{0.2 \times 420} = 0.00857°\text{C}$$

EXAMPLE 1.13

A rod AB of length l is pivoted at an end A and freely rotated in a horizontal plane at an angular speed ω about a vertical axis passing through A. If coefficient of linear expansion of material of rod is α, find the percentage change in its angular velocity if temperature of system is increased by ΔT.

Sol. If temperature of surrounding increases by ΔT, the new length of rod becomes

$$l' = l(1 + \alpha\Delta T)$$

Due to change in length, moment of inertia of rod about an end A also changes and is given as

$$I'_A = \frac{Ml'^2}{3}$$

As no external force or torque is acting on rod thus its angular momentum remains constant during heating thus we have

$$I_A\omega = I'_A\omega'$$

(If ω' is the final angular velocity of rod after heating)

or $$\frac{Ml^2}{3}\omega = \frac{Ml^2(1+\alpha\Delta T)^2}{3}\omega'$$

or $$\omega' = \omega(1 - 2\alpha\Delta T)$$ (using binomial expansion for small α)

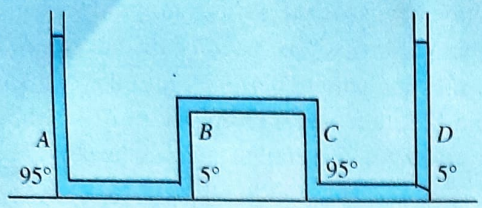
Thus percentage change in angular velocity of rod due to heating can given as

$$= \frac{\omega - \omega'}{\omega} \times 100\% = 2\alpha\Delta T \times 100\%$$

EXAMPLE 1.14

The apparatus shown in the figure consists of four glass columns connected by horizontal sections . The height of two central columns B and C are 49 cm each . The two outer columns A and D are open to the atmosphere . A and C are maintained at a temperature of 95°C while the columns B and D are maintained at 5°C. The heights of the liquid in A and D measured from the base line are 52.8 cm and 51 cm

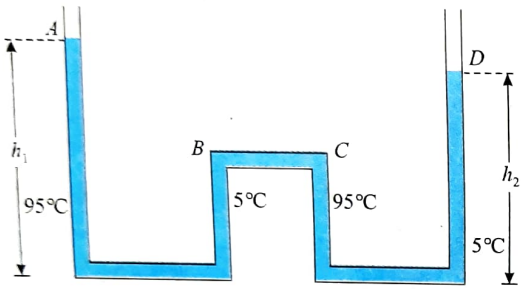
respectively. Determine the coefficient of thermal expansion of the liquid.



Sol. Density of a liquid varies with temperature as

$$\rho_{t°C} = \left(\frac{\rho_{0°C}}{1+\gamma t}\right)$$

Here γ is the coefficient of volume expansion of temperature.



In the figure, $h_1 = 52.8$ cm, h_2 cm and $h = 49$ cm

Now pressure at B = pressure at C. Therefore

$$P_0 + h_1 \rho_{95°} g - h\rho_{5°} g = P_0 + h_2 \rho_{5°} g - h\rho_{95°}$$

$$g\rho_{95°}(h_1 + h) = \rho_{5°}(h_2 + h)$$

$$\Rightarrow \quad \frac{\rho_{95°}}{\rho_{5°}} = \frac{h_2 + h}{h_1 + h} \Rightarrow \frac{\frac{\rho_{0°}}{(1+95\gamma)}}{\frac{\rho_{0°}}{(1+5\gamma)}} = \frac{h_2 + h}{h_1 + h}$$

$$\Rightarrow \quad \frac{1+5\gamma}{1+95\gamma} = \frac{51+49}{52.8+49} = \frac{100}{101.8}$$

Solving this equation, we get $\gamma = 2 \times 10^{-4}$ per °C

EXAMPLE 1.15

In a pitcher when water is filled some water comes to outer surface slowly through its porous walls and gets evaporated. Most of the latent heat needed for evaporation is taken from water inside and hence this water is cooled down. If 10 kg water is taken in the pitcher and 12 g water comes out and evaporated per minute. Neglect heat transfer by convection and radiation t surrounding, find the time in which the temperature of water i pitcher decrease by 5°C.

Sol. It is given that 12 g water is evaporated per minute, thu heat required per minute for it is

$$Q = mL_v = 12 \times 540 = 6480 \text{ cal/min}$$

After time t, mass of inside water is $m = 10000 - 12t$

If in further time dt, $dm = 12dt$, mass is vapourized, and th temperature of inside water falls by dT, we have

$$(10000 - 12t) \times 1 \times dT = 12\,dT = 12\,dt\,540$$

or $$dT = 12 \times 540 \frac{dt}{10000 - 12t}$$

Integrating the above expression in proper limits, we get

$$\int_{T_0}^{T_0-5} dT = 12 \times 540 \times \int_0^t \frac{dt}{10000 - 12t}$$

$$5 = 540\ \ell n\left(\frac{10000}{10000 - 12t}\right)$$

or $$e^{5/540} = \frac{10000}{10000 - 12t}$$

or $$t = \frac{10000}{12}[e^{1/108} - 1] = 7.737 \text{ min}$$

Alternative method

As here the amount of water vaporized is very small, we can assume that the quantity of water inside remains constant. Thus. to reduce the temperature of inside water, amount of heat to be rejected,

$$Q = ms\Delta T = 10000 \times 1 \times 5 = 50{,}000 \text{ calories}$$

As obtained from the given data that 6480 calories of heat is required per minute to vaporize 12 g water thus if in time t minutes, temperature of inside water falls by 5°C, we have

$$50000 = 64580\ t$$

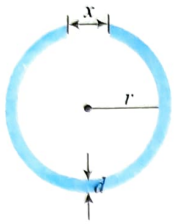
or $$t = \frac{50000}{6480} = 7.716 \text{ min}$$

Which is approximately same as obtained earlier. But student should note that this method will give the correct answer if the mass of inside water actually remains constant i.e., when the rate of evaporation is very very low.

Exercises

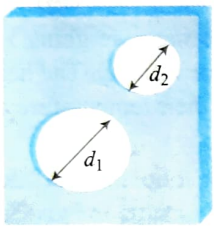
Single Correct Answer Type

1. A cylindrical metal rod of length L_0 is shaped into a ring with a small gap as shown. On heating the system



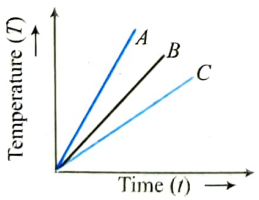
(1) x decreases, r and d increase
(2) x and r increase, d decreases
(3) x, r and d all increase
(4) Data is insufficient to arrive at a conclusion

2. Two holes of unequal diameters d_1 and d_2 ($d_1 > d_2$) are cut in a metal sheet.



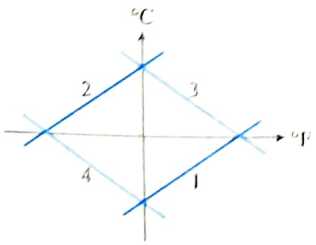
If the sheet is heated
(1) both d_1 and d_2 will decrease
(2) both d_1 and d_2 will increase
(3) d_1 will increase, d_2 will decrease
(4) d_1 will decrease, d_2 will increase

3. Which of the substances A, B and C has the highest specific heat? The temperature–time graph is shown below.



(1) A
(2) B
(3) C
(4) All have equal specific heat

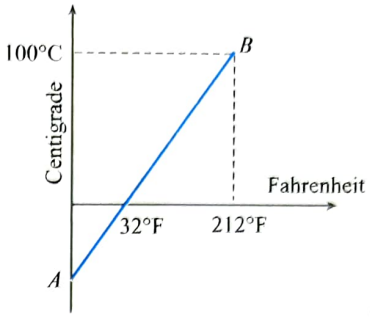
4. Which of the curves in figure represents the relation between Celsius and Fahrenheit temperatures?



(1) 1 (2) 2
(3) 3 (4) 4

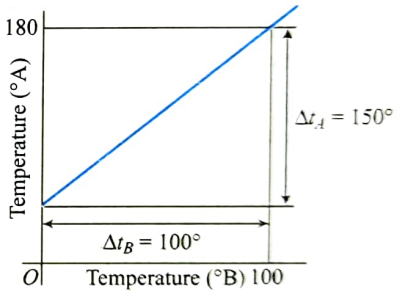
5. If a graph is plotted taking the temperature in Fahrenheit along Y-axis and the corresponding temperature in Celsius along the X-axis, it will be a straight line
(1) having a +ve intercept on Y-axis
(2) having a +ve intercept on X-axis
(3) passing through the origin
(4) having a –ve intercept on both the axes

6. The graph AB shown in figure is a plot of temperature of a body in degree Celsius and degree Fahrenheit. Then



(1) Slope of line AB is 9/5 (2) Slope of line AB is 5/9
(3) Slope of line AB is 1/9 (4) Slope of line AB is 3/9

7. The graph between two temperature scales A and B is shown in figure. Between upper fixed point and lower fixed point, there are 150 equal divisions on scale A and 100 on scale B. The relationship for conversion between the two scales is given by



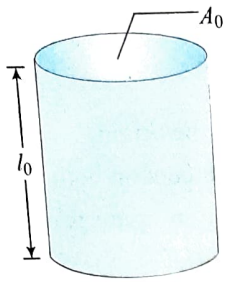
(1) $\frac{t_A - 180}{100} = \frac{t_B}{150}$ (2) $\frac{t_A - 30}{150} = \frac{t_B}{100}$
(3) $\frac{t_B - 180}{150} = \frac{t_A}{100}$ (4) $\frac{t_B - 40}{100} = \frac{t_A}{180}$

8. If a thermometer reads freezing point of water as 20°C and boiling point as 150°C, how much thermometer would read when the actual temperature is 60°C?
(1) 98°C (2) 110°C
(3) 40°C (4) 60°C

9. The apparent coefficient of expansion of a liquid when heated in a copper vessel is C and that when heated in a silver vessel is S. If A is the linear coefficient of expansion of copper, then the linear coefficient of expansion of silver is

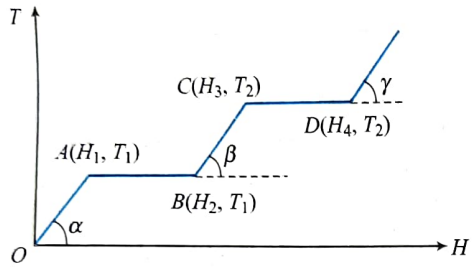
(1) $\frac{C + S - 3A}{3}$ (2) $\frac{C + 3A - S}{3}$
(3) $\frac{S + 3A - C}{3}$ (4) $\frac{C + S + 3A}{3}$

10. The figure shows a glass tube (linear coefficient of expansion is α) completely filled with a liquid of volume expansion coefficient γ. On heating, length of the liquid column does not change. Choose the correct relation between γ and α.



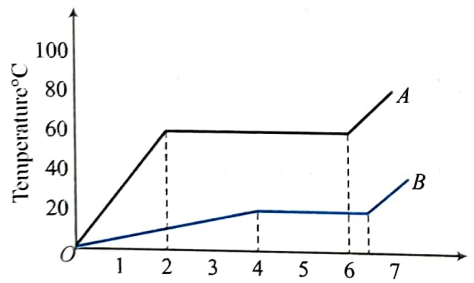
(1) $\gamma = \alpha$ (2) $\gamma = 2\alpha$
(3) $\gamma = 3\alpha$ (4) $\gamma = \alpha/3$

11. The graph shows the variation of temperature (T) of one kilogram of a material with the heat (H) supplied to it. At O, the substance is in the solid state. From the graph, we can conclude that



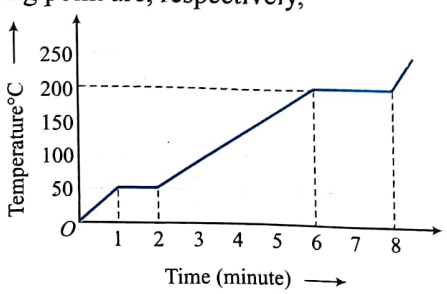
(1) T_2 is the melting point of the solid
(2) BC represents the change of state from solid to liquid
(3) $(H_2 - H_1)$ represents the latent heat of fusion of the substance
(4) $(H_3 - H_1)$ represents the latent heat of vaporization of the liquid

12. Two substances A and B of equal mass m are heated at uniform rate of 6 cal-s^{-1} under similar conditions. A graph between temperature and time is shown in figure. Ratio of heats absorbed (H_A/H_B) by them for complete fusion is



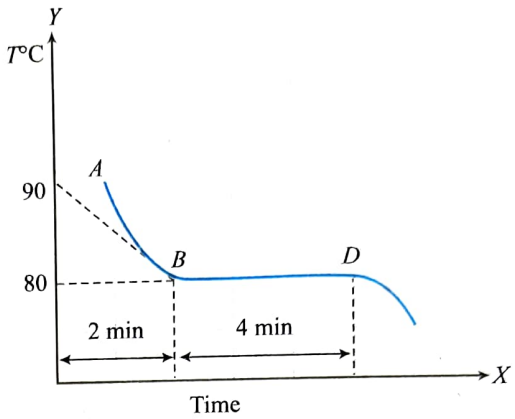
(1) $\frac{9}{4}$ (2) $\frac{4}{9}$
(3) $\frac{8}{5}$ (4) $\frac{5}{8}$

13. A student takes 50 g wax (specific heat = 0.6 k cal/kg-°C) and heats it till it boils. The graph between temperature and time is shown below. Heat supplied to the wax per minute and boiling point are, respectively,



(1) 500 cal, 50°C (2) 1000 cal, 100°C
(3) 1500 cal, 200°C (4) 200°C

14. The figure given below shows the cooling curve of pure wax material after heating. It cools from A to B and solidifies along BD. If L and C are respective values of latent heat and the specific heat of the liquid wax, the ratio L/C is



(1) 40 (2) 80
(3) 100 (4) 20

15. If two balls of same metal weighing 5 g and 10 g strike with a target with the same velocity. The heat energy so developed is used for raising their temperature alone. The temperature will be higher
(1) for bigger ball (2) for smaller ball
(3) equal for both the balls (4) None of these

16. A rod of length 20 cm is made of metal A. It expands by 0.075 cm when its temperature is raised from 0°C to 100°C. Another rod of a different metal B having the same length expands by 0.045 cm for the same change in temperature. A third rod of the same length is composed of two parts, one of metal A and the other of metal B. This rod expands by 0.060 cm for the same change in temperature. The portion made of metal A has the length
(1) 20 cm (2) 10 cm
(3) 15 cm (4) 18 cm

17. Density of a substance at 0°C is 10 g/cc and at 100°C, its density is 9.7 g/cc. The coefficient of linear expansion of the substance will be
(1) 10^2 (2) 10^{-2}
(3) 10^{-3} (4) 10^{-4}

18. A vertical column of length 50 cm at 50°C balances another column of length 60 cm having same liquid at 100°C. The coefficient of absolute expansion of the liquid is
(1) 0.005/°C (2) 0.0005/°C
(3) 0.002/°C (4) 0.0002/°C

19. A one litre glass flask contains some mercury. It is found that at different temperatures, the volume of air inside the flask remains the same. What is the volume of mercury in the flask if coefficient of linear expansion of glass is 9×10^{-6}/°C while that of volume expansion of mercury is 1.8×10^{-4}/°C
(1) 50 cc (2) 100 cc
(3) 150 cc (4) 200 cc

20. A glass flask is filled up to a mark with 50 cc of mercury at 18°C. If the flask and contents are heated to 38°C, how much mercury will be above the mark? (α for glass

9×10^{-6}/°C and coefficient of real expansion of mercury is 180×10^{-6}/°C)

(1) 0.85 cc (2) 0.46 cc

(3) 0.153 cc (4) 0.05 cc

21. The coefficient of volumetric expansion of mercury is 18×10^{-5}/°C. A thermometer bulb has a volume 10^{-6} m^3 and cross section of stem is of 0.004 cm^2. Assuming that bulb is filled with mercury at 0°C, the length of the mercury column at 100°C is

(1) 18.8 mm (2) 9.2 mm

(3) 7.4 cm (4) 4.5 cm

22. The coefficient of apparent expansion of mercury in a glass vessel is 153×10^{-6}/°C and that in a steel vessel is 144×10^{-6}/°C. If α for steel is 12×10^{-6}/°C, then that of glass is

(1) 9×10^{-6}/°C (2) 6×10^{-6}/°C

(3) 36×10^{-6}/°C (4) 27×10^{-6}/°C

23. A glass flask of volume one litre at 0°C is filled, level full of mercury at this temperature. The flask and mercury are now heated to 100°C. How much mercury will spill out, if coefficient of volume expansion of mercury is 1.82×10^{-4}/°C and linear expansion of glass is 0.1×10^{-4}/°C respectively?

(1) 21.2 cc (2) 15.2 cc

(3) 1.52 cc (4) 2.12 cc

24. A metal ball immersed in alcohol weighs W_1 at 0°C and W_2 at 59°C. The coefficient of cubical expansion of the metal is less than that of alcohol. Assuming that the density of metal is large as compared to that of alcohol, it can be shown that

(1) $W_1 > W_2$ (2) $W_1 = W_2$

(3) $W_1 < W_2$ (4) $W_2 = (W_1/2)$

25. An iron tyre is to be fitted on to a wooden wheel 1 m in diameter. The diameter of tyre is 6 mm smaller than that of wheel. The tyre should be heated so that its temperature increases by a minimum of (the coefficient of cubical expansion of iron is 3.6×10^{-5}/°C)

(1) 167°C (2) 334°C

(3) 500°C (4) 1000°C

26. A steel scale measures the length of a copper wire as 80.0 cm, when both are at 20°C (the calibration temperature for scale). What would be the scale read for the length of the wire when both are at 40°C? (Given $\alpha_{steel} = 11 \times 10^{-6}$ per °C and $\alpha_{copper} = 17 \times 10^{-6}$ per °C)

(1) 80.0096 cm (2) 80.0272 cm

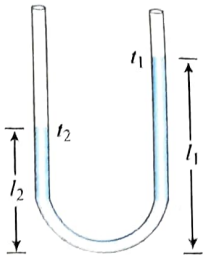
(3) 1 cm (4) 25.2 cm

27. A piece of metal weighs 46 g in the air. When it is immersed in the liquid of specific gravity 1.24 at 27°C, it weighs 30 g. When the temperature of liquid is raised to 42°C, the metal piece weighs 30.5 g. Specific gravity of the liquid at 42°C is 1.20. The linear expansion of the metal will be

(1) 3.316×10^{-5}/°C (2) 2.316×10^{-5}/°C

(3) 4.316×10^{-5}/°C (4) None of these

28. In a vertical U-tube containing a liquid, the two arms are maintained at different temperatures t_1 and t_2. The liquid columns in the two arms have heights l_1 and l_2, respectively. The coefficient of volume expansion of the liquid is equal to



(1) $\frac{l_1 - l_2}{l_2 t_1 - l_1 t_2}$ (2) $\frac{l_1 - l_2}{l_1 t_1 - l_2 t_2}$

(3) $\frac{l_1 + l_2}{l_2 t_1 + l_1 t_2}$ (4) $\frac{l_1 + l_2}{l_1 t_1 + l_2 t_2}$

29. A uniform metallic rod rotates about its perpendicular bisector with constant angular speed. If it is heated uniformly to raise its temperature slightly

(1) its speed of rotation increases

(2) its speed of rotation decreases

(3) its speed of rotation remains the same

(3) its speed increases because its moment of inertia increases

30. An aluminium sphere is dipped into water. Which of the following is true?

(1) Buoyancy will be less in water at 0°C than that in water at 4°C.

(2) Buoyancy will be more in water at 0°C than that in water at 4°C.

(3) Buoyancy in water at 0°C will be same as that in water at 4°C.

(4) Buoyancy may be more or less in water at 4°C depending on the radius of the sphere.

31. A lead bullet of 10 g travelling at 300 m/s strikes against a block of wood and comes to rest. Assuming 50% of heat is absorbed by the bullet, the increase in its temperature is (Specific heat of lead = 150 J/kg-K)

(1) 100°C (2) 125°C

(3) 150°C (4) 200°C

32. A water fall is 84 metres high. If half of the potential energy of the falling water gets converted to heat, the rise in temperature of water will be

(1) 0.098°C (2) 0.98°C

(3) 9.8°C (4) 0.0098°C

33. A steel ball of mass 0.1 kg falls freely from a height of 10 m and bounces to a height of 5.4 m from the ground. If the dissipated energy in this process is absorbed by the ball, the rise in its temperature is (Specific heat of steel = 460 Joule-kg^{-1}°C^{-1}, $g = 10$ ms^{-2})

(1) 0.01°C (2) 0.1°C

(3) 1°C (4) 1.1°C

34. 540 g of ice at 0°C is mixed with 540 g of water at 80°C. The final temperature of the mixture is

(1) 0°C (2) 40°C

(3) 80°C (4) Less than 0°C

35. A beaker contains 200 g of water. The heat capacity of the beaker is equal to that of 20 g of water. The initial temperature

of water in the beaker is 20°C. If 440 g of hot water at 92°C is poured in it, the final temperature (neglecting radiation loss) will be nearest to

(1) 58°C (2) 68°C
(3) 73°C (4) 78°C

36. If there are no heat losses, the heat released by the condensation of x gram of steam at 100°C into water at 100°C can be used to convert y gram of ice at 0°C into water at 100°C. Then the ratio $y : x$ is nearly

(1) 1 : 1 (2) 2.5 : 1
(3) 2 : 1 (4) 3 : 1

37. Calculate the amount of heat (in calories) required to convert 5 g of ice at 0°C to steam at 100°C.

(1) 3100 (2) 3200
(3) 3600 (4) 4200

38. Work done in converting one gram of ice at –10°C into steam at 100°C is

(1) 3045 J (2) 6056 J
(3) 721 J (4) 616 J

39. Steam is passed into 22 g of water at 20°C. The mass of water that will be present when the water acquires a temperature of 90°C (Latent heat of steam is 540 cal/g) is

(1) 24.8 g (2) 24 g
(3) 36.6 g (4) 30 g

40. 10 g of ice at –20°C is dropped into a calorimeter containing 10 g of water at 10°C; the specific heat of water is twice that of ice. When equilibrium is reached, the calorimeter will contain

(1) 20 g of water
(2) 20 g of ice
(3) 10 g ice and 10 g water
(4) 5 g ice and 15 g water

41. A lead bullet at 27°C just melts when stopped by an obstacle. Assuming that 25% of heat is absorbed by the obstacle, then the velocity of the bullet at the time of striking is (M.P. of lead = 327°C, specific heat of lead = 0.03 cal/g°C, latent heat of fusion of lead = 6 cal/g and $J = 4.2$ joule/cal)

(1) 410 m/sec (2) 1230 m/sec
(3) 307.5 m/sec (4) None of the above

42. The temperature of equal masses of three different liquids A, B and C are 12°C, 19°C and 28°C, respectively. The temperature when A and B are mixed is 16°C and that when B and C are mixed is 23°C. The temperature when A and C are mixed is

(1) 18.2°C (2) 22°C
(3) 20.2°C (4) 25.2°C

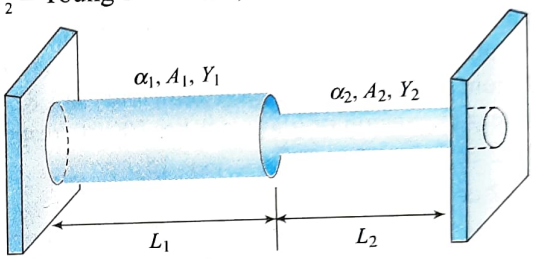
43. In an industrial process, 10 kg of water per hour is to be heated from 20°C to 80°C. To do this, steam at 150°C is passed from a boiler into a copper coil immersed in water. The steam condenses in the coil and is returned to the boiler as water at 90°C. How many kg of steam is required per hour?
(Specific heat of steam = 1 calorie per g°C, Latent heat of vaporization = 540 cal/g)

(1) 1 g (2) 1 kg
(3) 10 g (4) 10 kg

44. If earth suddenly stops rotating about its own axis, the increase in its temperature will be
(Here, R = radius of the Earth, ω = angular velocity of rotation of the Earth, J = mechanical equivalent of heat, and s = average specific heat capacity of the Earth)

(1) $\frac{R^2\omega^2}{5Js}$ (2) $\frac{R^2\omega^2}{Js}$
(3) $\frac{RM\omega^2}{5Js}$ (4) None of these

45. Two elastic rods are joined between fixed supports as shown in the figure. Condition for no change in the lengths of individual rods with the increase of temperature (α_1, α_2 = linear expansion coefficient, A_1, A_2 = area of rods, Y_1, Y_2 = Young's modulus) is



(1) $\frac{A_1}{A_2} = \frac{\alpha_1 Y_1}{\alpha_2 Y_2}$ (2) $\frac{A_1}{A_2} = \frac{L_1\alpha_1 Y_1}{L_2\alpha_2 Y_2}$
(3) $\frac{A_1}{A_2} = \frac{L_2\alpha_2 Y_2}{L_1\alpha_1 Y_1}$ (4) $\frac{A_1}{A_2} = \frac{\alpha_2 Y_2}{\alpha_1 Y_1}$

46. A wire is made by attaching two segments together end to end. One segment is made of aluminium and the other is steel. The effective coefficient of linear expansion of the two segment wire is 19×10^{-6}/(°C). The fraction length of aluminium is (linear coefficients of thermal expansion of aluminium and steel are 23×10^{-6}/(°C) and 12×10^{-6}/(°C). respectively)

(1) 5/11 (2) 6/11
(3) 7/11 (4) 8/11

47. Heat is required to change 1 kg of ice at –20°C into steam. Q_1 is the heat needed to warm the ice from –20°C to 0°C, Q_2 is the heat needed to melt the ice, Q_3 is the heat needed to warm the water from 0°C to 100°C and Q_4 is the heat needed to vapourize the water. Then

(1) $Q_4 > Q_3 > Q_2 > Q_1$ (2) $Q_4 > Q_3 > Q_1 > Q_2$
(3) $Q_4 > Q_2 > Q_3 > Q_1$ (4) $Q_4 > Q_2 > Q_1 > Q_3$

48. A block of wood is floating in water at 0°C. The temperature of water is slowly raised from 0°C to 10°C. With the rise in temperature the volume of block V above water level will

(1) increase
(2) decrease
(3) first increase and then decrease
(4) first decrease and then increase

49. An incandescent lamp consuming P = 54 W is immersed into a transparent calorimeter containing $V = 10^3$ cm^3 of

water. In 420 s the water is heated by 4°C. The percentage of the energy consumed by the lamp that passes out of the calorimeter in the form of radiant energy is

(1) 81.5% (2) 26%

(3) 40.5% (4) 51.5%

50. A brass wire 2 m long at 27°C is held taut with negligible tension between two rigid supports. If the wire is cooled to a temperature of –33°C, then the tension developed in the wire, its diameter being 2 mm, will be (coefficient of linear expansion of brass = 2.0×10^{-5}/°C and Young's modulus of brass = 0.91×10^{11} Pa)

(1) 3400 N (2) 34 kN

(3) 0.34 kN (4) 6800 N

51. A mass m of lead shot is placed at the bottom of a vertical cardboard cylinder that is 1.5 m long and closed at both ends. The cylinder is suddenly inverted so that the shot falls 1.5 m. If this process is repeated quickly 100 times, assuming no heat is dissipated or lost, the temperature of the shot will increase by (specific heat of lead = 0.03 cal/g°C)

(1) 0 (2) 5°C

(3) 7.3°C (4) 11.3°C

52. An iron rocket fragment initially at –100°C enters the earth's atmosphere almost horizontally and quickly fuses completely in atmospheric friction. Specific heat of iron is 0.11 kcal/kg °C its melting point is 1535°C and the latent heat of fusion is 30 kcal/kg. The minimum velocity with which the fragment must have entered the atmosphere is

(1) 0.45 km/s (2) 1.32 km/s

(3) 2.32 km/s (4) zero

53. A liquid of density 0.85 g/cm^3 flows through a calorimeter at the rate of 8.0 cm^3/s. Heat is added by means of a 250 W electric heating coil and a temperature difference of 15°C is established in steady-state conditions between the inflow and the outflow points of the liquid. The specific heat for the liquid will be

(1) 0.6 kcal/kgK (2) 0.3 kcal/kgK

(3) 0.5 kcal/kgK (4) 0.4 kcal/kgK

54. A flask of mercury is sealed off at 20°C and is completely filled with mercury. If the bulk modulus for mercury is 2500 MPa and the coefficient of volume expansion of mercury is 1.82×10^{-4}/°C and the expansion of glass is ignored, the pressure of mercury within the flask at 100°C will be

(1) 100 MPa (2) 72 MPa

(3) 36 MPa (4) 24 MPa

55. The densities of wood and benzene at 0°C are 880 kg/m^3 and 900 kg/m^3, respectively. The coefficients of volume expansion of wood is 1.2×10^{-3}/°C and of benzene 1.5×10^{-3}/°C. The temperature at which a piece of this wood would just sink in benzene at the same temperature is

(1) 53°C (2) 63°C

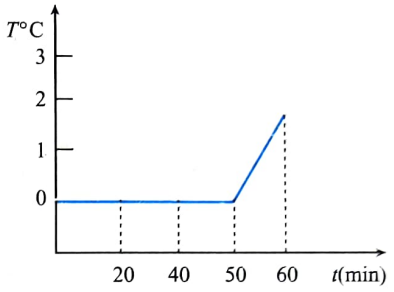
(3) 73°C (4) 83°C

56. A bullet of mass 5 g moving at a speed of 200 m/s strikes a rigidly fixed wooden plank of thickness 0.2 m normally and passes through it losing half of its kinetic energy. If it again strikes an identical rigidly fixed wooden plank and passes through it, assuming the same resistance in the two planks, the ratio of the thermal energies produced in the two planks is

(1) 1:1 (2) 1:2

(3) 2:1 (4) 4:1

57. A cooking vessel on a slow burner contains 5 kg of water and an unknown mass of ice in equilibrium at 0°C at time $t = 0$. The temperature of the mixture is measured at various times and the result is plotted as shown in given figure. During the first 50 min the mixture remains at 0°C . From 50 min to 60 min, the temperature increases to 2°C. Neglecting the heat capacity of the vessel, the initial mass of the ice is



(1) $\frac{10}{7}$ kg (2) $\frac{5}{7}$ kg

(3) $\frac{5}{4}$ kg (4) $\frac{5}{8}$ kg

58. The length of a steel rod exceeds that of a brass rod by 5 cm. If the difference in their lengths remains same at all temperatures, then the length of brass rod will be: (α for iron and brass are 12×10^{-6}/°C and 18×10^{-6}/°C, respectively)

(1) 15 cm (2) 5 cm

(3) 10 cm (4) 2 cm

59. A container of capacity 700 mL is filled with two immiscible liquids of volume 200 mL and 500 mL with respective volume expansivities as 1.4×10^{-5}/°C and 2.1×10^{-5}/°C. During the heating of the vessel, it is observed that neither any liquid overflows nor any empty space is created. The volume expansivity of the container is

(1) 1.9×10^{-5}/°C (2) 1.9×10^{-6}/°C

(3) 1.6×10^{-5}/°C (4) 1.6×10^{-6}/°C

60. The apparent coefficient of expansion of a liquid when heated, filled in vessel A and B of identical volumes, is found to be γ_1 and γ_2, respectively. If α_1 be the linear expansivity of A then that of B will be

(1) $\frac{(\gamma_1 - \gamma_2)}{3} - \alpha_1$ (2) $\frac{(\gamma_2 - \gamma_1)}{3} + \alpha_1$

(3) $\frac{(\gamma_2 - \gamma_1)}{3} - \alpha_1$ (4) $\frac{(\gamma_1 - \gamma_2)}{3} + \alpha_1$

61. A glass vessel is filled up to 3/5th of its volume by mercury. If the volume expansivities of glass and mercury be 9×10^{-6}/°C and 18×10^{-5}/°C, respectively, then the coefficient of apparent expansion of mercury is

(1) 17.1×10^{-5}/°C (2) 9.9×10^{-5}/°C

(3) 17.46×10^{-5}/°C (4) 16.5×10^{-5}/°C

62. A thin circular metal disc of radius 500.0 mm is set rotating about a central axis normal to its plane. Upon raising its temperature gradually, the radius increases to 507.5 mm. The percentage change in the rotational kinetic energy will be

(1) 1.5% (2) –1.5%

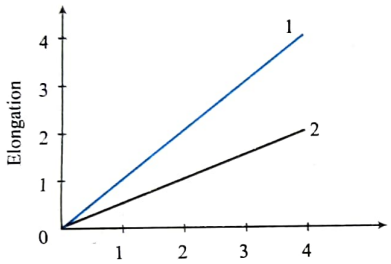
(3) 3% (4) –3%

63. A platinum sphere floats in mercury. Find the percentage change in the fraction of volume of sphere immersed in mercury when the temperature is raised by 80°C (volume expansivity of mercury is 182×10^{-6}/°C and linear expansivity of platinum is 9×10^{-6}/°C).

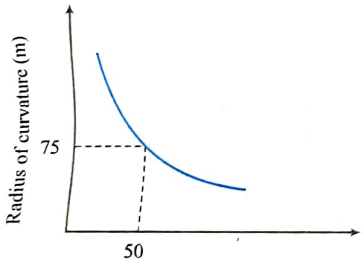
(1) 1.24% (2) 1.38%

(3) 2.48% (4) 2.76%

64. The given figure shows the graphs of elongation versus temperature for two different metals. If these metals are employed to form a straight bimetallic strip of thickness 6 cm and heated, it bends in the form of an arc, the radius of curvature changing with temperature approximately as shown in the figure. The linear expansivities of the two metals are



(a) Temperature (°C)



(b) Temperature (°C)

(1) 24×10^{-6}/°C and 12×10^{-6}/°C

(2) 20×10^{-6}/°C and 10×10^{-6}/°C

(3) 18×10^{-6}/°C and 9×10^{-6}/°C

(4) 16×10^{-6}/°C and 8×10^{-6}/°C

65. Water at 0°C, contained in a closed vessel, is abruptly opened in an evacuated chamber. If the specific latent heats of fusion and vapourization at 0°C are in the ratio λ:1, the fraction of water evaporated will be

(1) $\lambda/1$ (2) $\lambda/(\lambda+1)$

(3) $(1-\lambda)/\lambda$ (4) $(\lambda-1)/(\lambda+1)$

66. In a motor, the electrical power input is 500 W and the mechanical power output is 0.54 horse power. Heat developed in the motor in 1 h is (assuming that all the electric energy which is not converted to mechanical energy is converted to heat) is

(1) 4.18×10^4 cal (2) 3.6×10^5 cal

(3) 8.6×10^4 cal (4) 1.28×10^5 cal

67. The coefficient of linear expansion for a certain metal varies with temperature as $\alpha(T)$. If L_0 is the initial length of the metal and the temperature of metal is changed from T_0 to T ($T_0 > T$), then

(1) $L = L_0 \int_{T_0}^{T} \alpha(T)\,dT$

(2) $L = L_0 \left[1 + \int_{T_0}^{T} \alpha(T)\,dT\right]$

(3) $L = L_0 \left[1 - \int_{T_0}^{T} \alpha(T)\,dT\right]$

(4) $L > L_0$

68. A piece of metal floats in mercury. The coefficients of volume expansion of the metal and mercury are γ_1 and γ_2, respectively. If the temperatures of both mercury and the metal are increased by ΔT, the fraction of the volume of the metal submerged in mercury changes by the factor of

(1) $\dfrac{1+\gamma_2\Delta T}{1+\gamma_1\Delta T}$ (2) $1+\gamma_2\Delta T$

(3) $1+\gamma_1\Delta T$ (4) $\dfrac{1+\gamma_2\Delta T}{1-\gamma_1\Delta T}$

69. A uniform brass disc of radius a and mass m is set into spinning with angular speed ω_0 about an axis passing through centre of disc and perpendicular to the plane of disc. If its temperature increases from θ_1°C to θ_2°C without disturbing the disc, what will be its new angular speed? (The coefficient of linear expansion of brass is α.)

(1) $\omega_0\,[1 + 2a(\theta_2 - \theta_1)]$ (2) $\omega_0\,[1 + a(\theta_2 - \theta_1)]$

(3) $\dfrac{\omega_0}{[1+2\alpha(\theta_2-\theta_1)]}$ (4) None of these

70. Calculate the compressional force required to prevent the metallic rod of length l cm and cross-sectional area A cm^2 when heated through t°C, from expanding lengthwise. Young's modulus of elasticity of the metal is E and mean coefficient of linear expansion is α per degree celsius.

(1) $EA\alpha t$ (2) $EA\alpha t/(1+\alpha t)$

(3) $EA\alpha t/(1-at)$ (4) $El\alpha t$

71. The coefficient of linear expansion of glass is α_g per °C and the cubical expansion of mercury is γ_m per °C. The volume of the bulb of a mercury thermometer at 0°C is V_0 and cross section of the capillary is A_0. What is the length of mercury column in capillary at T°C, if the mercury just fills the bulb at 0°C?

(1) $\dfrac{V_0T(\gamma_m+3\alpha_g)}{A_0(1+2\alpha_gT)}$ (2) $\dfrac{V_0T(\gamma_m-3\alpha_g)}{A_0(1+2\alpha_gT)}$

(3) $\dfrac{V_0T(\gamma_m+2\alpha_g)}{A_0(1+3\alpha_gT)}$ (4) $\dfrac{V_0T(\gamma_m-2\alpha_g)}{A_0(1+3\alpha_gT)}$

72. A drilling machine of 10 kW power is used to drill a bore in a small aluminium block of mass 8 kg. If 50% of power is used up in heating the machine itself or lost to the surroundings then how much is the rise in temperature of the block in 2.5 min (given: specific heat of aluminium = 0.91 J/g°C)?

(1) 103°C (2) 130°C
(3) 105°C (4) 30°C

73. A ball of thermal capacity 10 cal/°C is heated to the temperature of furnace. It is then transferred into a vessel containing water. The water equivalent of vessel and the contents is 200 g. The temperature of the vessel and its contents rises from 10°C to 40°C. What is the temperature of furnace?
(1) 640°C (2) 64°C
(3) 600°C (4) 100°C

74. Two tanks A and B contain water at 30°C and 80°C, respectively; calculate the amount of water that must be taken from each tank respectively to prepare 40 kg of water at 50°C.
(1) 24 kg, 16 kg (2) 16 kg, 24 kg
(3) 20 kg, 20 kg (4) 30 kg, 10 kg

Multiple Correct Answers Type

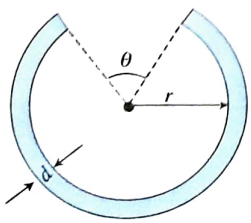
1. Due to thermal expansion with rise in temperature:
(1) metallic scale reading becomes lesser than true value
(2) pendulum clock becomes fast
(3) a floating body sinks a little more
(4) the weight of a body in a liquid increases

2. During heat exchange, temperature of a solid mass does not change. In this process, heat
(1) is not being supplied to the mass
(2) is not being taken out from the mass
(3) may have been supplied to the mass
(4) may have been taken out from the mass

3. During the melting of a slab of ice at 273 K at the atmospheric pressure
(1) positive work is done by the ice–water system on the atmosphere
(2) positive work is done on the ice–water system by the atmosphere
(3) the internal energy of the ice–water system increases
(4) the internal energy of the ice–water system decreases

4. The ends of a metal rod are kept at temperatures θ_1 and θ_2 with $\theta_2 > \theta_1$. The rate of flow of heat along the rod is directly proportional to
(1) the length of the rod
(2) the diameter of the rod
(3) the cross-sectional area of the rod
(4) the temperature difference $(\theta_2 - \theta_1)$ between the ends of the rod

5. A thin cylindrical metal rod is bent into a ring with a small gap as shown in figure. On heating the system,



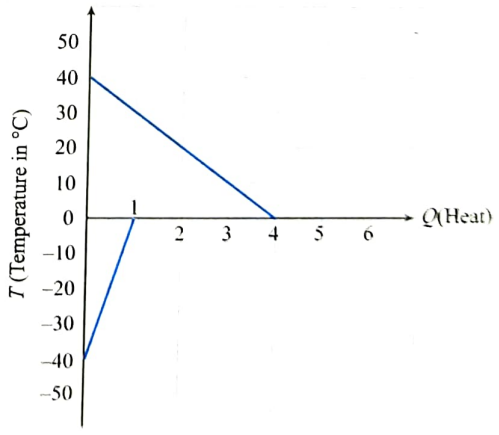
(1) θ decreases, r and d increase
(2) θ increases
(3) d and r increase
(4) θ is constant

6. When the temperature of a copper coin is raised by 80°C, its diameter increases by 0.2%.
(1) percentage rise in the area of a face is 0.4%
(2) percentage rise in the thickness is 0.4%
(3) percentage rise in the volume is 0.6%
(4) coefficient of linear expansion of copper is 0.25×10^{-4}/°C

7. A vessel is partly filled with liquid. When the vessel is cooled to a lower temperature, the space in the vessel unoccupied by the liquid remains constant. Then the volume of the liquid (V_L) volume of the vessel (V_V), the coefficient of cubical expansion of the material of the vessel (γ_V) and of the solid (γ_L) are related as
(1) $\gamma_L > \gamma_V$ (2) $\gamma_L < \gamma_V$
(3) $\frac{\gamma_V}{\gamma_L} = \frac{V_V}{V_L}$ (4) $\frac{\gamma_V}{\gamma_L} = \frac{V_L}{V_V}$

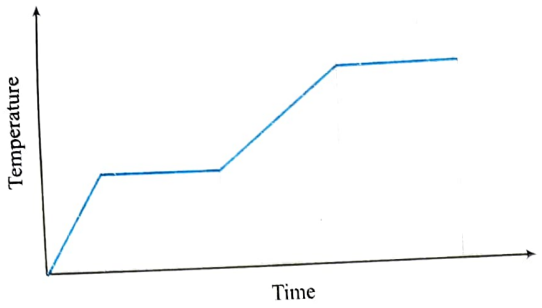
8. A clock is calibrated at a temperature of 20°C. Assume that the pendulum is a thin brass rod of negligible mass with a heavy bob attached to the end ($\alpha_{brass} = 19 \times 10^{-6}$/K)
(1) On a hot day at 30°C the clock gains 8.2 s
(2) On a hot day at 30°C the clock loses 8.2 s .
(3) On a cold day at 10°C the clock gains 8.2 s
(4) On a cold day at 10°C the clock loses 8.2 s

9. Given figure shows how the temperature of water and ice mixed with each other at different temperatures, changes with the amount of heat (Q) evolved by the water or absorbed by the ice. The scale corresponding to heat (Q) is arbitrary. Which of the following is/are conclusions drawn from the graph?



(1) The mass of water is double that of ice.
(2) The equilibrium temperature will be 0°C, with three-fourth of the ice melted.
(3) The equilibrium temperature will be 0°C with one-fourth of the water frozen.
(4) The equilibrium temperature will be 10°C.

10. Heat is supplied to a certain homogenous sample of matter, at a uniform rate. Its temperature is plotted against time, as shown.



Which of the following conclusions can be drawn?

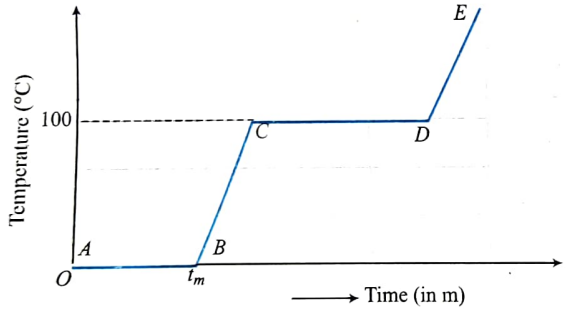
(1) Its specific heat capacity is greater in the solid state than in the liquid state.

(2) Its specific heat capacity is greater in the liquid state than in the solid state.

(3) Its latent heat of vaporization is greater than its latent heat of fusion.

(4) Its latent heat of vaporization is smaller than its latent heat of fusion.

11. Refer to the plot of temperature versus time (figure) showing the changes in the state if ice on heating (not to scale).



Which of the following is/are correct?

(1) The region *AB* represents ice and water in thermal equilibrium.

(2) At *B* water starts boiling.

(3) At *C* all the water gets converted into steam.

(4) *CD* represents water and steam in equilibrium at boiling point.

12. Mark the correct options

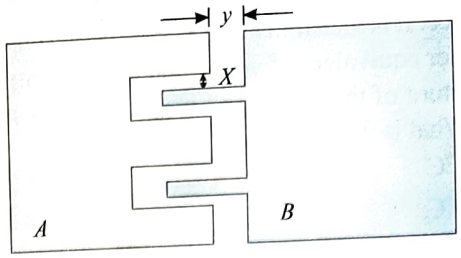
(1) A system X is in thermal equilibrium with Y but not with Z. The systems Y and Z may be in thermal equilibrium with each other.

(2) A system X is in thermal equilibrium with Y but not with Z. The systems Y and Z are not in thermal equilibrium with each other.

(3) A system X is neither in thermal equilibrium with Y nor with Z. The systems Y and Z must be in thermal equilibrium with each other.

(4) A system X is neither in thermal equilibrium with Y nor with Z. The systems Y and Z may be in thermal equilibrium with each other.

13. Two metal plates *A* and *B* made of same material are placed on a table as shown in the figure. If the plates are heated uniformly, the gaps indicated by *x* and *y* in the figure



(1) *x* increases,

(2) *y* decreases

(3) *x* decreases

(4) *y* increases,

14. A liquid having coefficient of volume expansion γ_0 is filled in a cylindrical glass vessel. Glass has a coefficient of linear expansion of α_g. The liquid along with the container is heated to raise their temperature by ΔT. Mass of the container is negligible.

(1) If the centre of mass of the system did not move due to heating then $\gamma_0 = 2\alpha_g$

(2) If the centre of mass of the system did not move due to heating then $\gamma_0 = 3\alpha_g$

(3) If the fraction of volume of the container occupied by the liquid does not change due to heating then $\gamma_0 = 3\alpha_g$

(4) If the fraction of volume of the container occupied by the liquid does not change due to heating then $\gamma_0 = 2\alpha_g$

15. Two rods of length L_1 and L_2 *are* made of materials of coefficients of linear expansions α_1 and α_2 respectively such that $L_1\alpha_1 = L_2\alpha_2$. The temperature of the rods is increased by ΔT and correspondingly the change in their respective lengths be ΔL_1 and ΔL_2

(1) $\Delta L_1 \neq \Delta L_2$

(2) $\Delta L_1 = \Delta L_2$

(3) Difference in length $(L_1 - L_2)$ is a constant and is independent of rise of temperature

(4) Data is insufficient to arrive at a conclusion

16. A bolt is passed through a pipe and a nut is just tightened. Coefficients of linear expansion for bolt and pipe material are α_b and α_p respectively. If the assembly is heated then:

(1) A tensile stress will be induced in the bolt if $\alpha_b < \alpha_p$

(2) A compressive stress will be induced in the bolt if $\alpha_b < \alpha$

(3) A compressive stress will be induced in the bolt is $\alpha_b = \alpha$

(4) No stress will be induced in the bolt if $\alpha_b > \alpha_p$

17. The temperature of an isotropic cubical solid of length l_0, density ρ_0 and coefficient of linear expansion α is increased by 20°C. Then at higher temperature, to a good approximation:

(1) Length is $l_0(1+20\alpha)$

(2) Total surface area is $l_0^2(1+40\alpha)$

(3) Total volume is $l_0^3(1+60\alpha)$

(4) Density is $\dfrac{\rho_0}{1+60\alpha}$

Linked Comprehension Type

For Problems 1–3

An immersion heater, in an insulated vessel of negligible heat capacity brings 100 g of water to the boiling point from 16°C in 7 min. Then

1. Power of heater is nearly
   (1) 8.4×10^3 (2) 84 W
   (3) 8.4×10^3 cal/s (4) 20 W
2. The water is replaced by 200 g of alcohol, which is heated from 16°C to the boiling point of 78°C in 6 min 12 s whereas 30 g are vapourized in 5 min 6 s. The specific heat of alcohol is
   (1) 0.6 J/kg°C (2) 0.6 cal/g°C
   (3) 0.6 cal/kg°C (4) 6 J/kg°C
3. The heat of vapourization of alcohol is
   (1) 854 J/kg (2) 854×10^3 J/kg
   (3) 204 cal/g (4) 204 cal/kg

For Problems 4–6

The internal energy of a solid also increases when heat is transferred to it from its surroundings. A 5 kg solid bar is heated at atmospheric pressure. Its temperature increases from 20°C to 70°C. The linear expansion coefficient of solid bar is 1×10^{-3}/C°. The density of solid bar is 50 kg/m^3. The specific heat capacity of solid bar is 200 J/kg C°. The atmospheric pressure is 1×10^5 N/m^2.

4. The work done by the solid bar due to thermal expansion, under atmospheric pressure, is
   (1) 500 J (2) 1000 J
   (3) 1500 J (4) 2000 J
5. The heat transferred to the solid bar is
   (1) 49000 J (2) 50000 J
   (3) 50500 J (4) 51000 J
6. The increase in the internal energy of the solid bar is
   (1) 49500 J (2) 48500 J
   (3) 49000 J (4) 50000 J

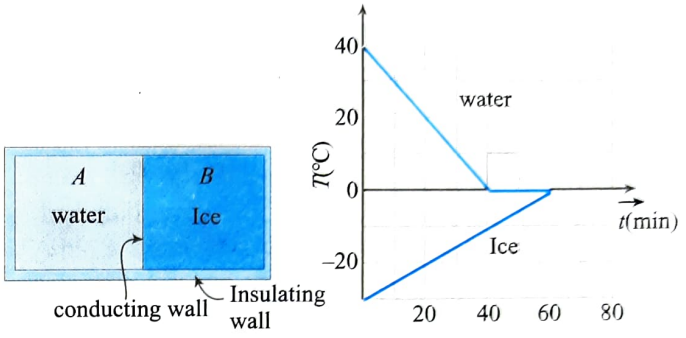
For Problems 7–9

A copper collar is to fit tightly about a steel shaft that has a diameter of 6 cm at 20°C. The inside diameter of the copper collar at that temperature is 5.98 cm.

7. To what temperature must the copper collar be raised so that it will just slip on the steel shaft, assuming the steel shaft remains at 20°C? ($\alpha_{copper} = 17 \times 10^{-6}$/K)
   (1) 324°C (2) 21.7°C
   (3) 217°C (4) 32.4°C
8. The tensile stress in the copper collar when its temperature returns to 20°C is ($Y = 11 \times 10^{10}$ N/m^2)
   (1) 1.34×10^5 N/m^2 (2) 3.68×10^{12} N/m^2
   (3) 3.68×10^8 N/m^2 (4) 1.34×10^{12} N/m^2
9. If the breaking stress of copper is 230 N/m^2, at what temperature will the copper collar break as it cools?
   (1) 20°C (2) 47°C
   (3) 94°C (4) 217°C

For Problems 10–12

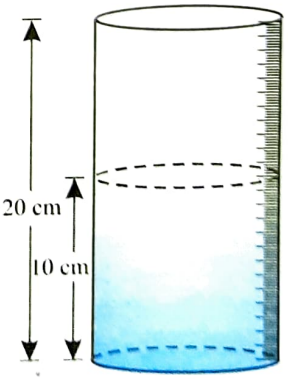
A 0.60 kg sample of water and a sample of ice are placed in two compartments A and B separated by a conducting wall, in a thermally insulated container. The rate of heat transfer from the water to the ice through the conducting wall is constant P, until thermal equilibrium is reached. The temperature T of the liquid water and the ice are given in graph as functions of time t. Temperature of the compartments remain homogeneous during whole heat transfer process. Given specific heat of ice = 2100 J/kg-K, specific heat of water = 4200 J/kg-K, and latent heat of fusion of ice = 3.3×10^5 J/kg.



10. The value of rate P is
    (1) 42.0 W (2) 36.0 W
    (3) 21.0 W (4) 18.0 W
11. The initial mass of the ice in the container is equal to
    (1) 0.36 kg (2) 1.2 kg
    (3) 2.4 kg (4) 3.6 kg
12. The mass of the ice formed due to conversion from the water till thermal equilibrium is reached is equal to
    (1) 0.12 kg (2) 0.15 kg
    (3) 0.25 kg (4) 0.40 kg

For Problems 13–15

At 20°C a liquid is filled upto 10 cm height in a container of glass of length 20 cm and cross-sectional area 100 cm^2 Scale is marked on the surface of container. This scale gives correct reading at 20°C. Given $\gamma_L = 5 \times 10^{-5} K^{-1}, \alpha_g = 1 \times 10^{-5} °C^{-1}$



13. The volume of liquid at 40°C is
    (1) 1002 cc (2) 1001 cc
    (3) 1003 cc (4) 1000.5 cc
14. The actual height of liquid at 40°C is
    (1) 10.01 cm (2) 10.006 cm
    (3) 10.6 cm (4) 10.1 cm

15. The reading of scale at 40°C is

(1) 10.01 cm (2) 10.004 cm

(3) 10.006cm (4) 10.04 cm

Matrix Match Type

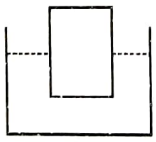
1. In a container of negligible mass 'm' grams of steam at 100°C is added to 100 g of water that has temperature 20°C. If no heat is lost to the surroundings at equilibrium, match the items given in Column I with that in Column II.

Column I	Column II
i. Mass of steam in the mixture, if $m = 20$ g (in g)	a. 114.8
ii. Mass of water in the mixture, if $m = 20$ g (in g)	b. 76.4
iii. If $m = 20$ g, final temperature of the mixture (in °C)	c. 5.2
iv. If $m = 10$ g, final temperature of the mixture (in °C)	d. 100

2. A piece of metal of density ρ_1 floats on mercury of density ρ_2. The coefficients of expansion of the metal and mercury are γ_1 and γ_2, respectively. The temperatures of both mercury and metal are increased by ΔT. Then match the following:

Column I	Column II
i. If $\gamma_2 > \gamma_1$	a. no effect on fraction of solid submerged in mercury
ii. $\gamma_2 = \gamma_1$	b. fraction of the volume of metal submerged in mercury increases
iii. If $\gamma_2 < \gamma_1$	c. the solid sinks
	d. the solid lifts up

3. A cylindrical isotropic solid of coefficient of thermal expansion α and density ρ (at STP) floats in a liquid of coefficient of volume expansion γ and density d (at STP) as shown in the diagram. Match the following:



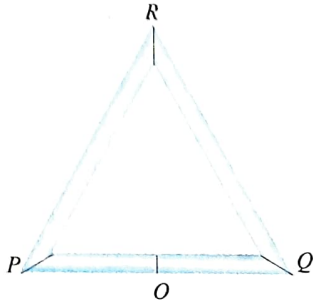
Column I	Column II
i. Volume of cylinder inside the liquid remains constant	a. $\gamma = 0$
ii. Volume of cylinder outside the liquid remains constant	b. $\gamma = 2\alpha$
iii. Height of cylinder outside the liquid remains constant	c. $\gamma = 3\alpha\frac{d}{\rho}$
iv. Height of cylinder inside the liquid remain constant	d. $\gamma = (2\alpha + \alpha\frac{d}{\rho})$

4. Three liquids A, B and C having same specific heat and mass m, 2m and 3m have temperature 20°C, 40°C and 60°C respectively. Temperature of the mixture when :

Column I	Column II
i. A and B are mixed	a. 35°C
ii. A and C are mixed	b. 52°C
iii. B and C are mixed	c. 50°C
iv. A, B and C all three are mixed	d. 45°C
	e. None

Numerical Value Type

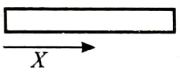
1. 1 g of steam at 100°C melts to how many grams of ice at 0°C? (Latent heat of ice = 80 cal/g and latent heat of steam = 540 cal/g)
2. 2 kg of ice at −20°C is mixed with 5 kg of water at 20°C in an insulating vessel having a negligible heat capacity. Calculate the final mass of water (in kg) remaining in the container.
3. 2 kg of ice at −15°C is mixed with 2.5 kg of water at 25°C in an insulating container. If the specific heat capacities of ice and water are 0.5 cal/g°C and 1 cal/g°C, find the amount of water present in the container? (in kg nearest integer)
4. In two experiments with a continuous flow calorimeter to determine the specific heat capacity of a liquid, an input power of 16 W produced a rise of 10 K in the liquid. When the power was doubled, the same temperature rise was achieved by making the rate of flow of liquid three times faster. Find the power lost (in W) to the surrounding in each case.
5. A clock with a metallic pendulum at 15°C runs faster by 5 s each day and at 30°C, runs slow by 10 s. Find the coefficient of linear expansion of the metal. (nearly in 10^{-5}/°C)
6. Two vessels connected at the bottom by a thin pipe with a sliding plug contain liquid at 20°C and 80°C respectively. The coefficient of cubic expansion of liquid is 10^{-3} K^{-1}. The ratio of heights of liquid columns in the vessel (h_{20}/h_{80}) is nearest to which integer?
7. Three rods of equal length l are joined to form an equilateral triangle PQR. O is the mid point of PQ. Distance OR remains the same for small change in temperature. Coefficient of linear expansion for PR and RQ is the same, i.e., α_2 but that for PQ is α_1. Then find the ratio $\alpha_1 : \alpha_2$.



8. A cylindrical beer can made up of aluminium contains 500 cm^3 of beer. The area of cross section of can is 125 cm^2 at 10°C. Find the rise in level of beer if temperature of can increases to 80°C. Given that coefficient of linear expansion of aluminium is $\alpha_{Al} = 2.3 \times 10^{-5}$/°C and that for cubical expansion of beer is $\gamma_{Beer} = 3.2 \times 10^{-4}$/°C.

9. We would like to increase the length of a 15 cm long copper rod of cross section 4 mm^2 by 1 mm. The energy absorbed by the rod if it is heated is E_1. The energy a bsorbed by the rod if it is stretched slowly is E_2. Then find E_1/E_2.
[Various parameters of copper are density = 9×10^3 kg/m^3, thermal coefficient of linear expansion = 16×10^{-6}/K, Young's modulus = 135×10^9 Pa, specific heat = 400 J/kg-K]

10. A rod has variable co-efficient of linear expansion $\alpha = \frac{x}{5000}$.
If length of the rod is 1m. Determine increase in length of the rod in (cm) on increasing temperature of the rod by 100°C.



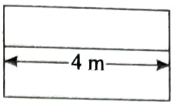
11. A clock pendulum made of invar has a period of 2 s at 20°C. If the clock is used in a climate where average temperature is 40°C, what correction (in seconds) may be necessary at the end of 10 days to the time given by clock ?

($\alpha_{invar} = 7 \times 10^{-7}$°$C^{-1}$, day = 8.64×10^4 s)

12. The temperature of a body rises by 44°C when a certain amount of heat is given to it. The same heat when supplied to 22g of ice at –8°C, raises its temperature by 16°C. Find the water equivalent of the body in grams [Given: $s_{water} = 1$ cal/g°C and $L_f = 80$ cal/g, $s_{ice} = 0.5$ cal/ g°C]

13. A fine steel wire of length 4m is fixed rigidly in a heavy brass frame as shown in figure. It is just taut at 20°C. Find the the tensile stress developed in steel wire(in $\times 10^6$ Nm^{-2}) if whole system is heated to 120°C.

(Given $\alpha_{brass} = 1.8 \times 10^{-5}$°$C^{-1}$, $\alpha_{steel} = 1.2 \times 10^{-5}$°$C^{-1}$, $Y_{steel} = 20 \times 10^{11}$ Nm^{-2}, $Y_{brass} = 1.7 \times 10^7$ Nm^{-2})



14. A lead ball at 25°C is dropped from a height of 2 km. It is heated due to air resistance and it is assumed that all of its kinetic energy is used in increasing the temperature of ball. If specific heat of lead is s = 125 Joule/kg °C, find the final temperature of ball in °C.

15. A certain mass of a solid exists at its melting temperature of 20°C. When a heat Q is added 3/4 of the material melts. When an additional Q amount of heat is added the material transforms to its liquid state at 50°C. Find the ratio of specific latent heat of fusion (in J/g) to the specific heat capacity of the liquid (in J g^{-1} $°C^{-1}$) for the material.

16. A block of ice of mass M = 13.44 kg slides on a horizontal surface. It starts with a speed v = 10 m/s and finally stops after moving a distance s = 30 m. What is the mass of ice (in grams) that has melted due to friction between block and surface? Latent heat of fusion of ice is $L_f = 80$ cal/g.

17. A calorimeter of water equivalent 10 g contains a liquid of mass 50 g at 40 °C. When m gram of ice at – 10°C is put into the calorimeter and the mixture is allowed to attain equilibrium, the final temperature was found to be 20°C. It is known that specific heat capacity of the liquid changes with temperature as $S = \left(1 + \frac{\theta}{500}\right)$ cal g^{-1} $°C^{-1}$ where θ is temperature in °C. The specific heat capacity of ice, water and the calorimeter remains constant and values are $S_{ice} = 0.5$ cal g^{-1} $°C^{-1}$; $S_{water} = 1.0$ cal g^{-1} $°C^{-1}$ and latent heat of fusion of ice is $L_f = 80$ cal g^{-1}. Assume no heat loss to the surrounding and calculate the value of m in grams.

Archives

JEE Main

Single Correct Answer Type

1. This question contains Statement 1 and Statement 2. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement 1: The temperature dependence of resistance is usually given as $R = R_0(1 + \alpha\Delta t)$. The resistance of a wire changes from 100 Ω to 150 Ω when its temperature is increased from 27°C to 227°C. This implies that $\alpha = 2.5 \times 10^{-3}$/°C.

Statement 2: $R = R_t(1 + \alpha\Delta T)$ is valid only when the change in the temperature ΔT is small and $\Delta R = (R - R_0) << R_0$.

(1) Statement 1 is true but statement 2 is false.

(2) Statement 1 is true, statement 2 is true; statement 2 is the correct explanation for statement 1.

(3) Statement 1 is true, statement 2 is true; statement 2 is not the a correct explanation for statement 1.

(4) Statement 1 is false but statement 2 is true.

(AIEEE 2009)

2. 100 g of water is heated from 30°C to 50°C. Ignoring the slight expansion of the water, the change in its internal energy is (specific heat of water is 4184 J/Kg/K)

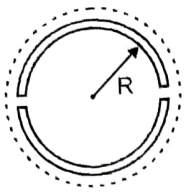
(1) 4.2 kJ (2) 8.4 kJ

(3) 84 kJ (4) 2.1 kJ

(AIEEE 2011)

3. A wooden wheel of radius R is made of two semicircular parts (see figure). The two parts are held together by a ring made of a metal strip of cross sectional area S and length L. L is slightly less than $2\pi R$. To fit the ring on the wheel, it is heated so that its temperature rises by ΔT and it just steps

over the wheel. As it cools down to surrounding temperature, it presses the semicircular parts together. If the coefficient of linear expansion of the metal is α, and its Young's modulus is Y, the force that one part of the wheel applies on the other part is



(1) $2\pi SY\alpha\Delta T$
(2) $SY\alpha\Delta T$
(3) $\pi SY\alpha\Delta T$
(4) $2SY\alpha\Delta T$ **(AIEEE 2012)**

4. The pressure that has to be applied to the ends of a steel wire of length 10 cm to keep its length constant when its temperature is raised by 100°C is:
(For steel Young's modulus is 2×10^{11} N m^{-2} and coefficient of thermal expansion is 1.1×10^{-5} K^{-1})
(1) 2.2×10^{7} Pa (2) 2.2×10^{6} Pa
(3) 2.2×10^{8} Pa (4) 2.2×10^{9} Pa **(JEE Main 2014)**

5. A pendulum clock loses 12 s a day if the temperature is 40°C and gains 4 s a day if the temperature is 20°C. The temperature at which the clock will show correct time, and the co-efficient of linear expansion (α) of the metal of the pendulum shaft are respectively:
(1) 25 °C ; $\alpha = 1.85 \times 10^{-5}$ /°C
(2) 60 °C ; $\alpha = 1.85 \times 10^{-4}$ /°C
(3) 30 °C ; $\alpha = 1.85 \times 10^{-3}$ /°C
(4) 55 °C ; $\alpha = 1.85 \times 10^{-2}$ /°C **(JEE Main 2016)**

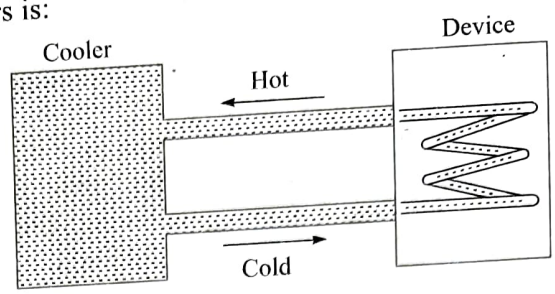
6. An external pressure P is applied on a cube at 0°C so that it is equally compressed from all sides. K is the bulk modulus of the material of the cube and α is its coefficient of linear expansion. Suppose we want to bring the cube to its original size by heating. The temperature should be raised by
(1) $\frac{3\alpha}{PK}$ (2) $3PK\alpha$
(c) $\frac{P}{3\alpha K}$ (4) $\frac{P}{\alpha K}$ **(JEE Main 2017)**

7. A copper ball of mass 100 g is at a temperature T. It is dropped in a copper calorimeter of mass 100 g, filled with 170 g of water at room temperature. Subsequently, the temperature of the system is found to be 75°C. T is given by (Given: room temperature = 30°C, specific heat of copper = 0.1 cal/g°C)
(1) 1250°C (2) 825°C
(3) 800°C (4) 885°C **(JEE Main 2017)**

JEE ADVANCED

Single Correct Answer Type

1. A water cooler of storage capacity 120 litres can cool water at constant rate of P watts. In a closed circulation system (as shown schematically in the figure), the water from the cooler is used to cool an external device that generates constantly 3 kW of heat (thermal load). The temperature of water fed into the device cannot exceed 30°C and the entire stored 120 litres of water is initially cooled to 10°C. The entire system is thermally insulated. The minimum value of P (in watts) for which the device can be operated for 3 hours is:

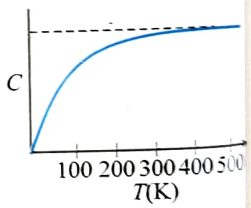


(Specific heat of water is 4.2 kJ kg^{-1} K^{-1} and the density of water is 1000 kg m^{-3})
(1) 1600 (2) 2067
(3) 2533 (4) 3933 **(JEE Advanced 2016)**

2. The ends Q and R of two thin wires, PQ and RS, are soldered (joined) together. Initially each of the wires has a length of 1m at 10°C. Now the end P is maintained at 10°C, while the end S is heated and maintained at 400°C. The system is thermally insulated from its surroundings. If the thermal conductivity of wire PQ is twice that of the wire RS and the coefficient of linear thermal expansion of PQ is 1.2×10^{-5} K^{-1}, the change in length of the wire PQ is.
(1) 0.78 mm (2) 0.90 mm
(3) 1.56 mm (4) 2.34 mm **(JEE Advanced 2016)**

Multiple Correct Answers Type

1. The figure shows the variation of specific heat capacity (C) of a solid as a function of temperature (T). The temperature is increased continuously from 0 to 500 K at a constant rate. Ignoring any volume change, the following statement(s) is (are) correct to a reasonable approximation.



(1) the rate at which heat is absorbed in the range 0–100 K varies linearly with temperature T.
(2) heat absorbed in increasing the temperature from 0–100 K is less than the heat required for increasing the temperature from 400 to 500 K.
(3) there is no change in the rate of heat absorption in range 400–500 K.
(4) the rate of heat absorption increases in the range 200–300 K. **(JEE Advanced 2013)**

Numerical Value Type

1. Steel wire of length 'L' at 40°C is suspended from the ceiling and then a mass 'm' is hung from its free end. The wire is cooled down from 40°C to 30°C to regain its original length 'L'. The coefficient of linear thermal expansion of the steel is 10^{-5}/°C. Young's modulus of steel is 10^{11} N/m^2 and radius of the wire is 1 mm. Assume that L >> diameter of the wire. Then the value of 'm' in kg is nearly. **(IIT-JEE 2011)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	2. (2)	3. (3)	4. (1)	5. (1)
6. (2)	7. (2)	8. (1)	9. (2)	10. (2)
11. (3)	12. (3)	13. (3)	14. (4)	15. (3)
16. (2)	17. (4)	18. (1)	19. (3)	20. (3)
21. (4)	22. (1)	23. (2)	24. (3)	25. (3)
26. (1)	27. (2)	28. (1)	29. (2)	30. (1)
31. (3)	32. (1)	33. (2)	34. (1)	35. (2)
36. (4)	37. (3)	38. (1)	39. (1)	40. (3)
41. (1)	42. (3)	43. (2)	44. (1)	45. (4)
46. (3)	47. (1)	48. (3)	49. (2)	50. (3)
51. (4)	52. (2)	53. (1)	54. (3)	55. (4)
56. (1)	57. (2)	58. (3)	59. (1)	60. (4)
61. (4)	62. (4)	63. (1)	64. (4)	65. (4)
66. (3)	67. (2)	68. (1)	69. (3)	70. (2)
71. (2)	72. (1)	73. (1)	74. (1)	

Multiple Correct Answers Type

1. (1),(3),(4)	2. (3),(4)	3. (2),(3)
4. (3),(4)	5. (3),(4)	6. (1),(3),(4)
7. (1),(4)	8. (2),(3)	9. (1),(2)
10. (2),(3)	11. (1),(4)	12. (2),(4)
13. (1),(2)	14. (1),(3)	15. (2),(3)
16. (1),(4)	17. (1),(3),(4)	

Linked Comprehension Type

1. (2)	2. (2)	3. (2)	4. (3)	5. (2)
6. (2)	7. (3)	8. (3)	9. (3)	10. (1)
11. (3)	12. (2)	13. (2)	14. (2)	15. (2)

Matrix Match Type

1. i. → c.; ii. → a.; iii. → d.; iv. → b.
2. i. → b.; ii. → a.; iii. → d.
3. i. → a.; ii. → c.; iii. → d.; iv. → b.
4. i. → e.; ii. → c.; iii. → b.; iv. → e.

Numerical Value Type

1. (8)	2. (6)	3. (3)	4. (8)	5. (2)
6. (1)	7. (4)	8. (0.076)	9. (500)	10. (1)
11. (6)	12. (50)	13. (120)	14. (160)	15. (60)
16. (2)	17. (12)			

ARCHIVES

JEE Main

Single Correct Answer Type

1. (4)	2. (2)	3. (4)	4. (3)	5. (1)
6. (3)	7. (4)			

JEE Advanced

Single Correct Answer Type

1. (2) 2. (1)

Multiple Correct Answers Type

1. (1),(2),(3),(4)

Numerical Value Type

1. (3)

2 Transmission of Heat

HEAT

Heat is the energy transferred from a body at higher temperature to a body at lower temperature. The transfer of heat from one body to another may take place by one of the following modes: **Conduction, Convection** and **Radiation.**

The process of transmission of heat energy in which heat is transferred from one particle of the medium to the other, but each particle of the medium stays at its own position is called **conduction**.

When heat is transferred from one point to the other through actual movement of heated particles, the process of heat transfer is called **convection**. In liquids and gases, some heat may be transported through conduction. But most of the transfer of heat in them occurs through the process of convection. Convection occurs through the aid of earth's gravity. Normally, the portion of fluid at greater temperature is less dense, while that at lower temperature is denser. Hence, hot fluid rises up while colder fluid sinks down, accounting for convection. In the absence of gravity convection would not be possible.

The process of the transfer of heat from one place to another place without heating the intervening medium is called **radiation**. The term radiation used here is another word for electromagnetic waves. These waves are formed due to the superposition of electric and magnetic fields perpendicular to each other and carry energy.

CONDUCTION

The process of heat energy in which the heat is transferred from one particle to other particle without dislocation of the particle from their equilibrium position is called **conduction** or **thermal conduction**. In this process, the transfer can be represented on an atomic scale as an exchange of kinetic energy between microscopic particles—molecules, atoms and free electrons—in which less-energetic particles gain energy in collisions with more-energetic particles.

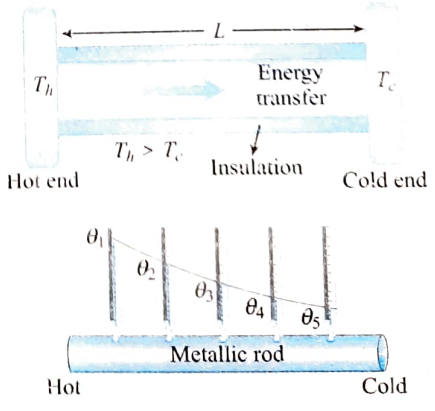
For example, if you hold one end of a long metal bar and insert the other end into a flame, you will find that the temperature of the metal in your hand soon increases. The energy reaches your hand by means of conduction. Initially, before the rod is inserted into the flame, the microscopic particles in the metal are vibrating about their equilibrium positions. As the flame raises the temperature of the rod, the particles near the flame begin to vibrate with greater amplitudes. These particles, in turn, collide with their neighbours and transfer some of their energy in the collisions. Slowly, the amplitudes of vibration of metal atoms and electrons farther from the flame increase until eventually those in the metal near your hand are affected. This increase in amplitude of vibration is detected by an increase in the temperature of the metal and of your potentially burned hand.

Important Points:

- Heat flows from hot end to cold end. Particles of the medium simply oscillate but do not leave their place.
- Medium is necessary for conduction. It is a slow process.
- The temperature of the medium increases through which heat flows.
- Conduction is a process which is possible in all states of matter.
- When liquid and gases are heated from the top, they conduct heat from top to bottom.
- In solids only conduction takes place.
- In non-metallic solids and fluids the conduction takes place only due to vibrations of molecules, therefore they are poor conductors.
- In metallic solids free electrons carry the heat energy, therefore they are good conductors of heat.

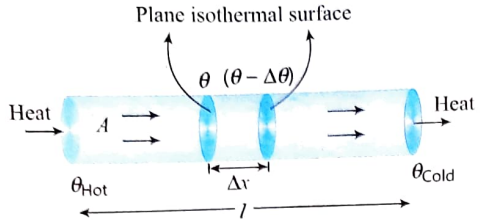
CONDUCTION IN METALLIC ROD

Consider that a metallic rod connects two bodies at temperature T_h and T_c ($T_h > T_c$). The sides of the rod are covered by some insulated material so that no heat flow takes place from the sides. When one end of a metallic rod is heated, heat flows by conduction from the hot end to the cold end. After sufficient long time, the temperature within the rod found to decrease from hot to cold face in uniform rate. At this time, the temperature of every cross-section of the rod becomes constant. In this state, no heat is absorbed by the rod. This state of the rod is called *steady state*.



TEMPERATURE GRADIENT

The surface (within a conductor) having its all points at the same temperature, is called **isothermal surface**. The direction of flow of heat through a conductor at any point is perpendicular to the isothermal surface passing through that point.



The rate of change of temperature with distance between two isothermal surfaces is called temperature gradient.

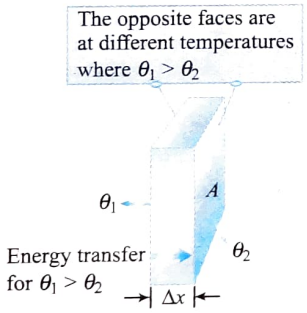
If the temperature of two isothermal surfaces be θ and $(\theta - \Delta\theta)$, and the perpendicular distance between them be Δx then

$$\text{Temperature gradient} = \frac{(\theta - \Delta\theta) - \theta}{\Delta x} = \frac{-\Delta\theta}{\Delta x}$$

The negative sign shows that temperature θ decreases as the distance x increases in the direction of heat flow.

LAW OF THERMAL CONDUCTIVITY

Consider a slab of material of thickness Δx and cross-sectional area A. One face of the slab is at a temperature θ_1 and the other face is at a temperature θ_2 $(\theta_1 > \theta_2)$. Experimentally, it is found that energy ΔQ transfers in a time interval Δt from the hotter face to the colder one.



The rate of heat flow or heat current $H\left(\frac{\Delta Q}{\Delta t}\right)$ at which this energy transfer occurs is found to be

(i) proportional to the cross-sectional area (A);

(ii) proportional to the temperature difference $\Delta\theta\,(\theta_1 - \theta_2)$ and

(iii) inversely proportional to the thickness Δx

Hence, $H = \frac{\Delta Q}{\Delta t} \propto \frac{A(\theta_1 - \theta_2)}{\Delta x}$

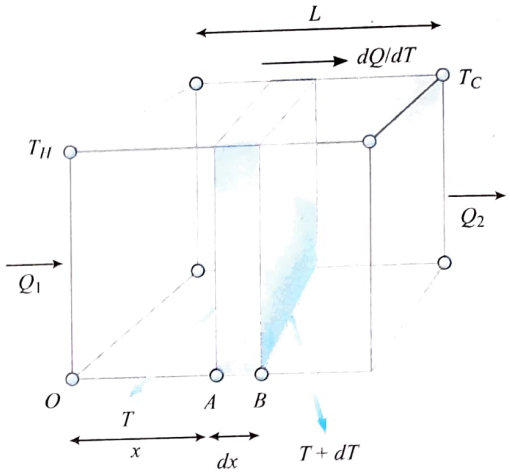
or $$H = \frac{\Delta Q}{\Delta t} = \frac{KA(\theta_1 - \theta_2)}{\Delta x} = KA\frac{\Delta\theta}{\Delta x}$$

Here, K is the coefficient of thermal conductivity of material of rod.

EXPRESSING RATE OF HEAT FLOW IN DIFFERENTIAL FORM

Consider a slab of face area A, lateral thickness L, whose faces have temperatures T_H and T_C $(T_H > T_C)$.

Now, consider two cross-sections in the slab at positions A and B separated by a lateral distance of dx. Let temperature of face A be T and that of face B be $T + dT$.



Then the rate at which heat crossing the area A of the slab at position x in time t is given by

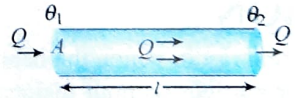
$$\frac{dQ}{dt} = -KA\frac{dT}{dx}$$

Here, K is a constant depending on the material of the slab and is named **thermal conductivity** of the material, and the quantity $\left(\frac{dT}{dx}\right)$ is called **temperature gradient**. The (–) sign in equation shows that heat flows from high to low temperature (dT is a –ve quantity)

Important Points:

- If a metallic rod of length l, cross-section area A and thermal conductivity K connects two bodies at temperature θ_1 and $\theta_2(\theta_1 > \theta_2)$, in steady state, then the amount of heat flowing from one face to the other face in time t is given by

$$Q = \frac{KA(\theta_1 - \theta_2)t}{l}$$



- In case of non-steady state or variable cross-section, a more general equation can be used to solve problems.
  Rate of heat flow or heat current,

$$\frac{dQ}{dt} = -KA\frac{d\theta}{dx}$$

ILLUSTRATION 2.1

A refrigerator door is 150 cm high, 80 cm wide, and 6 cm thick. If the coefficient of conductivity is 0.0005 cal/cm-s-°C, and the inner and outer surfaces are at 0°C and 30°C, respectively, what is the heat loss per minute through the door, in calories?

Sol. Apply the equation of thermal conductivity

$$Q = \frac{kA(t_h - t_c)(\text{time})}{d} = \frac{0.0005(150 \times 80)(30° - 0°)(60)}{6} = 1800 \text{ cal}$$

ILLUSTRATION 2.2

An ordinary refrigerator is thermally equivalent to a box of corkboard 90 mm thick and 5.6 m^2 in inner surface area. When the door is closed, the inside wall is kept, on the average, 22.2°C below the temperature of the outside wall. If the motor of the refrigerator runs 15% of the time while the door is closed, at what rate must heat be taken from the interior while the motor is running? The thermal conductivity of corkboard is $k = 0.05$ W/mK.

Sol. Consider a time interval Δt during which the door is closed. As approximation, take the heat conduction to be steady over Δt.

Then the rate of heat into the box is

$$\frac{\Delta Q}{\Delta t} = kA\left(\frac{\Delta T}{\Delta x}\right) = (0.05)(5.6)\left(\frac{22.2}{0.090}\right) = 69.1 \text{ W}$$

To remove this heat, the motor must, since it runs only for a time (0.15) Δt, cause heat to leave at the rate 69.1/0.15 = 460 W.

ILLUSTRATION 2.3

Water is being boiled in flat bottom kettle placed on a stove. The area of the bottom is 3000 cm^2 and the thickness is 2 mm. If the amount of steam produced is 1 g/min, calculate the difference of temperature between the inner and outer surface of the bottom. K for the material of kettle is 0.5 cal/°C/s/cm, and the latent heat of steam is 540 cal/g.

Sol. Mass of steam produced $= \frac{dm}{dt} = \frac{1}{60}$ g/s

Heat transferred per second $= \frac{dH}{dt} = L\frac{dm}{dt}$

$$\Rightarrow \quad \frac{dH}{dt} = 540 \times \frac{1}{60} \text{ cal/s} = 9 \text{ cal/s}$$

Area = 3000 cm^2; $K = 0.5$ cal/°C/s/cm

θ = temperature difference

d = thickness = 2 mm = 0.2 cm

$$\frac{dH}{dt} = \frac{KA\theta}{d} \quad \Rightarrow L\frac{dm}{dt} = \frac{KA\theta}{d}$$

$$\Rightarrow \quad 9 = \frac{0.5 \times 3000 \times \theta}{0.2} \quad \Rightarrow \theta = 1.2 \times 10^{-3}\text{°C}$$

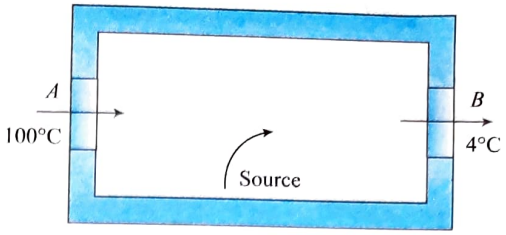
ILLUSTRATION 2.4

A closed cubical box made of perfectly insulating material has walls of thickness 8 cm and the only way for the heat to enter or leave the box is through the solid, cylindrical, metallic plugs each of cross-sectional area 12 cm^2 and length 8 cm fixed in the opposite walls of the box as shown in the given figure. The outer surface A is kept at 100°C while the outer surface B of other plug is kept at 4°C. K of the material of the plugs is 0.5 cal/s/°C/cm. A source of energy generating 36 cal/s is enclosed inside the box. Find the equilibrium temperature of the inner surface of the box assuming that it is same at all points on the inner surface.

Sol. Let θ be the temperature of inner surface of box.

Heat transfer per second through A + heat produced by source per second = Heat transfer per second through B.

$$\Rightarrow \quad \left(\frac{dH}{dt}\right)_A + 36 \text{ cal/s} = \left(\frac{dH}{dt}\right)_B$$



$$\Rightarrow \quad \frac{KA(100-\theta)}{d} + 36 = \frac{KA(\theta-4)}{d}$$

$$\Rightarrow \quad KA(\theta - 4 - 100 + \theta) = 36 \times d$$

Now, $d = 8$ cm, $A = 12$ cm^2, $K = 0.5$ cal/s/°C/cm.

$$\Rightarrow \quad 2\theta - 104 = \frac{36 \times 8}{12 \times 0.5} \quad \Rightarrow \theta = 76\text{°C}$$

MORE ABOUT COEFFICIENT OF THERMAL CONDUCTIVITY

It is the measure of the ability of a substance to conduct heat through it. Substances that are good thermal conductors have large thermal conductivity values, whereas good thermal insulators have low thermal conductivity values.

Notice that metals are generally better thermal conductors than non-metals. The magnitude of K depends only on nature of the material.

Substances in which heat flows quickly and easily are known as good conductors of heat. They possess large thermal conductivity due to large number of free electrons e.g., silver, brass etc. For perfect conductors, $K = \infty$.

Substances which do not permit easy flow of heat are called bad conductors. They possess low thermal conductivity due to very few free electrons e.g., glass, wood etc., and for perfect insulators, $K = 0$.

The thermal conductivity of pure metals decreases with rise in temperature but for alloys, thermal conductivity increases with increase of temperature.

Units: Cal/cm-sec-°C (in CGS), kcal/m-sec-K (in MKS) and W/m-K (in SI)

Dimensional formula: $[MLT^{-3}\theta^{-1}]$

THERMAL RESISTANCE TO CONDUCTION

If you are interested in insulating your house from cold weather or for that matter, keeping the meal hot in your tiffin-box, you are more interested in poor heat conductors, rather than good conductors. For this reason, the concept of thermal resistance R has been introduced. The thermal resistance of a body is a measure of its opposition to the flow of heat through it.

It is defined as the ratio of temperature difference to the heat current (= Rate of flow of heat). For a slab of cross-section A, lateral thickness l and thermal conductivity K,

$$R_T = \frac{\theta_1 - \theta_2}{H} = \frac{\theta_1 - \theta_2}{KA(\theta_1 - \theta_2)/l} = \frac{l}{KA}$$

Units: °C-sec/cal or K-sec/kcal

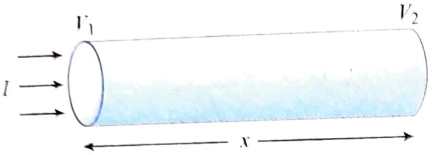
Dimensional formula: $[M^{-1}L^{-2}T^{3}\theta]$

In terms of R_T, we can express heat current or thermal current as:

$$H = \frac{Q}{t} = \frac{(\theta_H - \theta_L)}{R_T}$$

Electrical Analogy for Thermal Conduction

From Ohm's law, if current I flows through a resistance at potential difference $(V_1 - V_2)$, then $(V_1 - V_2) = IR$ or $I = \frac{(V_1 - V_2)}{R}$



Electric resistance, $R = \frac{\rho l}{A} = \frac{l}{\sigma A}$

Where R = resistance of conductor; σ = conductivity of the material; l = length of conductor.

$$I = \frac{\Delta Q}{\Delta t} = \frac{(V_1 - V_2)}{(l/\sigma A)}$$

$$\Rightarrow \quad I = \frac{\Delta Q}{\Delta t} = \frac{\sigma A(V_1 - V_2)}{l} \qquad \ldots(i)$$

If a metal rod of length l, cross-section area A and thermal conductivity K connects two bodies at temperature θ_1 and θ_2 $(\theta_1 > \theta_2)$. The rate of heat flow or heat current,

$$H = \frac{\Delta Q}{\Delta t} = \frac{(\theta_1 - \theta_2)}{R_T}$$

Thermal resistance of the rod, $R_T = \frac{l}{KA}$

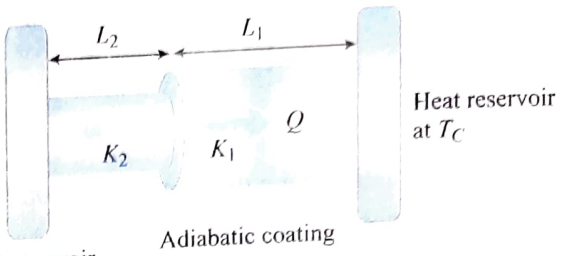
$$\therefore \quad H = \frac{\Delta Q}{\Delta t} = \frac{(\theta_1 - \theta_2)}{(l/KA)} = \frac{KA(\theta_1 - \theta_2)}{l} \qquad \ldots(ii)$$

Equation (ii) is mathematically equivalent to Eq. (i), with temperature donning the role of electric potential. Hence, results derived from Ohm's law are also valid for thermal conduction.

Electrical conduction	Thermal conduction
Electric charge flows from higher potential to lower potential.	Heat flows from higher temperature to lower temperature.
The rate of flow of charge is called the electric current, i.e., $I = \frac{dq}{dt}$	The rate of flow of heat may be called as heat current, i.e., $H = \frac{dQ}{dt}$
The relation between the electric current and the potential difference is given by Ohm's law, i.e., $I = \frac{V_1 - V_2}{R}$, where R is the electrical resistance of the conductor.	Similarly, the heat current may be related with the temperature difference as $H = \frac{\theta_1 - \theta_2}{R}$, where R is the thermal resistance of the conductor.
The electrical resistance is defined as $R = \frac{\rho l}{A} = \frac{l}{\sigma A}$, where ρ = Resistivity and σ = Electrical conductivity	The thermal resistance may be defined as $R = \frac{l}{KA}$, where K = Thermal conductivity of conductor

Slabs in Series Combination

Consider a composite slab consisting of two materials having different thickness L_1 and L_2, different cross-sectional areas A_1 and A_2 and different thermal conductivities K_1 and K_2, respectively. The temperatures at the outer surface of the slab are maintained at T_H and T_C, and all lateral surfaces are covered by an insulating coating.



Let temperature at the junction be T. Since steady state has been achieved, thermal current through each slab will be equal. Then thermal current through the first slab,

$$H = \frac{\Delta Q}{\Delta t} = \frac{T_H - T}{R_1}$$

or $\quad T_H - T = R_1 H$ ...(i)

And that through the second slab,

$$H = \frac{\Delta Q}{\Delta t} = \frac{T - T_C}{R_2}$$

or $\quad T - T_C = R_2 H$ ...(ii)

Adding Eq. (i) and Eq. (ii), $T_H - T_C = (R_1 + R_2)H$

or $\quad H = \frac{T_H - T_C}{R_1 + R_2}$ ...(iii)

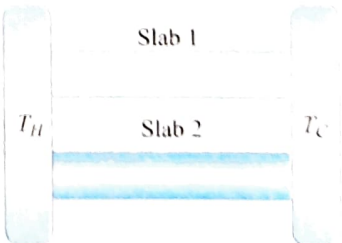
Thus these two slabs are equivalent to a single slab of thermal resistance $R_1 + R_2$.

If more than two slabs are joined in series and are allowed to attain steady state, then equivalent thermal resistance is given by

$$R = R_1 + R_2 + R_3 + \cdots$$

Slabs in Parallel Combination

Consider two slabs held between the same heat reservoirs, with their thermal conductivities K_1 and K_2 and cross-sectional areas A_1 and A_2.



Then $R_1 = \frac{L}{K_1 A_1}$ and $R_2 = \frac{L}{K_2 A_2}$

Thermal current through slab 1, $H_1 = \frac{T_H - T_C}{R_1}$

And that through slab 2, $H_2 = \frac{T_H - T_C}{R_2}$

Net heat current from the hot to cold reservoir,

$$H = H_1 + H_2 = (T_H - T_C)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

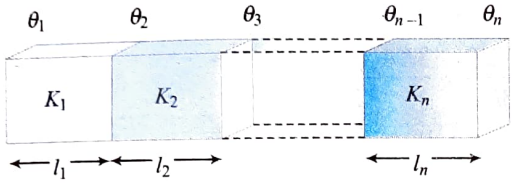
Comparing with $H = \frac{T_H - T_C}{R_{eq}}$, we get $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$

If more than two rods are joined in parallel, the equivalent thermal resistance is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

EQUIVALENT THERMAL CONDUCTIVITY

Series combination: Let n slabs each of cross-sectional area A, lengths l_1, l_2, l_3, ..., l_n and conductivities K_1, K_2, K_3, ..., K_n respectively, be connected in the series.



Equivalent thermal resistance, $R_{eq} = R_1 + R_2 + R_3 + \ldots$

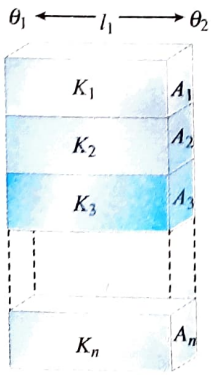
and $$\frac{l_1 + l_2 + \cdots + l_n}{K_{eq}A} = \frac{l_1}{K_1 A} + \frac{l_2}{K_2 A} + \cdots + \frac{l_n}{K_n A}$$

$$\Rightarrow \quad K_{eq} = \frac{l_1 + l_2 + \cdots + l_n}{\frac{l_1}{K_1} + \frac{l_2}{K_2} + \cdots + \frac{l_n}{K_n}}$$

Important Points:

- For n slabs of equal length, $K_{eq} = \dfrac{n}{\frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3} + \cdots + \frac{1}{K_n}}$
- For two slabs of equal length, $K_{eq} = \dfrac{2K_1K_2}{K_1 + K_2}$

Parallel combination: Let n slabs each of length l, areas A_1, A_2, A_3, ..., A_n and thermal conductivities K_1, K_2, K_3, ..., K_n, are connected in parallel.



Net heat current will be the sum of heat currents through individual slabs. i.e.,

$$H = H_1 + H_2 + H_3 + \cdots + H_n$$

$$\Rightarrow \quad \frac{K_{eq}(A_1 + A_2 + \cdots + A_n)(\theta_1 - \theta_2)}{l}$$

$$= \frac{K_1A_1(\theta_1 - \theta_2)}{l} + \frac{K_2A_2(\theta_1 - \theta_2)}{l} + \cdots + \frac{K_nA_n(\theta_1 - \theta_2)}{l}$$

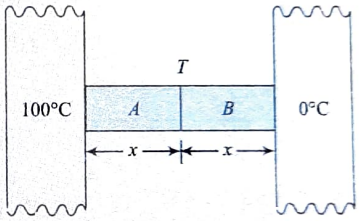
$$\Rightarrow \quad K_{eq} = \frac{K_1A_1 + K_2A_2 + K_3A_3 + \cdots + K_nA_n}{A_1 + A_2 + A_3 + \cdots + A_n}$$

Important Points:

- For n slabs of equal area, $K_{eq} = \dfrac{K_1 + K_2 + K_3 + \cdots + K_n}{n}$
- For two slabs of equal area, $K_{eq} = \dfrac{K_1 + K_2}{2}$

ILLUSTRATION 2.5

Two metal cubes A and B of same size are arranged as shown in the given figure. The extreme ends of the combination are maintained at the identical temperatures. The coefficients of thermal conductivity of A and B are 300 W/m°C and 200 W/m°C, respectively. After steady state is reached, what will be the temperature T of the interface?



Sol. In steady state, rate of flow of heat through A = Rate of flow of heat through B.

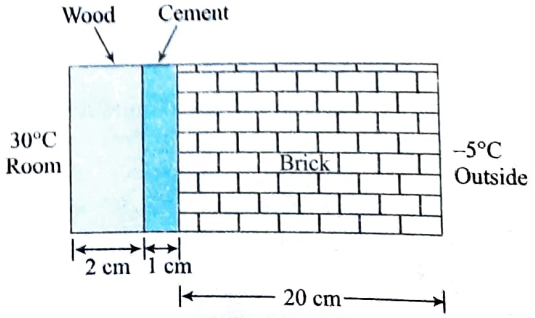
or $$K_1A\left(\frac{100 - T}{x}\right) = K_2A\frac{(T - 0)}{x}$$

or $\quad 300 - 3T = 2T$

$\therefore \quad T = 60°C$

ILLUSTRATION 2.6

An electric heater is used in a room of total wall area 150 m² to maintain a temperature of 30°C inside it, when the outside temperature is –5°C. The innermost layer is of wood of thickness 2.0 cm, the middle layer is of cement of thickness 1.0 cm and the outermost layer is of brick of thickness 20 cm. Find the power of the electric heater. Assume heat to flow only through the walls. The thermal conductivities of wood, cement and brick are 0.150, 1.75 and 1.0 W/m°C, respectively.



Sol. Evidently, the three layers (wood, cement and brick) are arranged in series.

So, the equivalent thermal resistance (R) of the combination will be $R = R_1 + R_2 + R_3$

$$= \frac{1}{150\ \text{m}^2}\left[\frac{2\times10^{-2}}{0.15}\frac{\text{m}^2\ \text{C}}{\text{W}} + \frac{1\times10^{-2}}{1.75}\frac{\text{m}^2\ \text{C}}{\text{W}} + \frac{20\times10^{-2}}{1.0}\frac{\text{m}^2\ \text{C}}{\text{W}}\right]$$

$$= 2.26 \times 10^{-3}\ ^\circ\text{C/W}$$

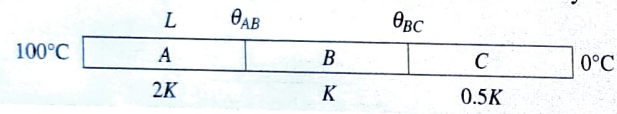
From, rate of heat loss $\frac{dQ}{dt} = \frac{\Delta T}{R}$, we have

$$\frac{dQ}{dt} = \frac{30-(-5)^\circ\text{C}}{2.26\times10^{-3}\,^\circ\text{C/W}} = 15.49\ \text{kW}$$

Since the heat lost from the inside of the room to the outside through the walls is compensated by the room heater, so the power of the heater is 15.49 kW.

ILLUSTRATION 2.7

Three cylindrical rods A, B and C of equal lengths and equal diameters are joined in series as shown in the given figure. Their thermal conductivities are $2K$, K and $0.5K$, respectively. In steady state, if the free ends of rods A and C are at 100°C and 0°C, respectively, calculate the temperature at the two junction points. Assume negligible loss by radiation through the curved surface. What will be the equivalent thermal conductivity?



Sol. As the rods are in series,

$$R_{eq} = R_A + R_B + R_C \text{ with } R = (L/KA)$$

i.e., $$R_{eq} = \frac{L}{2KA} + \frac{L}{KA} + \frac{L}{0.5KA} = \frac{7L}{2KA} \quad \ldots(i)$$

And hence, $$H = \frac{dQ}{dt} = \frac{\Delta\theta}{R} = \frac{(100-0)}{(7L/2KA)} = \frac{200KA}{7L}$$

Now in series, rate of flow of heat remains same,

i.e., $H = H_A = H_B = H_C$.

So for rod A, $\left[\frac{dQ}{dt}\right]_A = \left[\frac{dQ}{dt}\right]$

i.e., $$\frac{(100-\theta_{AB})2KA}{L} = \frac{200KA}{7L}$$

or $$\theta_{AB} = 100 - \left(\frac{100}{7}\right) = \left(\frac{600}{7}\right) = 87.7^\circ\text{C}$$

And for rod C, $\left[\frac{dQ}{dt}\right]_C = \left[\frac{dQ}{dt}\right]$

i.e., $$\frac{(\theta_{BC}-0)\times0.5KA}{L} = \frac{200KA}{7L} \quad \text{or } \theta_{BC} = \left(\frac{400}{7}\right) = 57.1^\circ\text{C}$$

Furthermore, if K_{eq} is equivalent thermal conductivity,

$$R_{eq} = \frac{L+L+L}{K_{eq}A} = \frac{7L}{2KA} \quad \text{[from Eq. (i)]}$$

i.e., $$K_{eq} = \left(\frac{6}{7}\right)K$$

ILLUSTRATION 2.8

Two walls of thickness in the ratio 1 : 3 and thermal conductivities in the ratio 3 : 2 form a composite wall of a building. If the free surfaces of the wall be at temperatures 30°C and 20°C, respectively, what is the temperature of the interface?

Sol. **1st method:** If the temperature difference across the first wall be ΔT°C, then that across the second wall will be $(10 - \Delta T)$°C (note).

In the steady state, the rate of heat flow across the two walls will be the same.

So, considering an area A normal to the flow of heat,

$$\frac{dQ}{dt} = \frac{K_1A(\Delta T)}{l_1} = \frac{K_2A(10-\Delta T)}{l_2}$$

where $K_1 = K$; $K_2 = 3K$ and $l_1 = 3l$; $l_2 = 2l$;

$$\frac{KA\Delta T}{3l} = \frac{3KA(10-\Delta T)}{2l}$$

$\Rightarrow$ $2\Delta T = 9(10 - \Delta T)$

$\Rightarrow$ $\Delta T = 8.18^\circ\text{C}$

So, the temperature of the interface will be $30 - \Delta T = 30 - 8.18 = 21.82^\circ\text{C}$

2nd method: Using the equation, temperature of the interface

$$T = \frac{(K_1T_1/l_1 + K_2T_2/l_2)}{(K_1/l_1 + K_2/l_2)}$$

We have $$T = \left(\frac{K(30)}{3l} + \frac{3K(20)}{2l}\right)\Big/\left(\frac{K}{3l} + \frac{3K}{2l}\right)$$

$$= \frac{(10+30)}{\left(\frac{1}{2}+\frac{3}{2}\right)} = 21.82^\circ\text{C}$$

3rd method: $R_1 = \frac{l_1}{K_1A} = \frac{3l}{KA}$ and $R_2 = \frac{l_2}{K_2A} = \frac{2l}{3KA}$

Effective thermal resistance $R = R_1 + R_2 = \frac{11}{3}\frac{l}{KA}$

Now, thermal current $i = \frac{\Delta T}{R} = \frac{30-20}{\frac{11l}{3KA}} = \frac{30KA}{11l}$

Also, thermal current through the first slab will be

$$i = \frac{(30-T)}{R_1} \text{ (where } T = \text{temperature of the interface)}$$

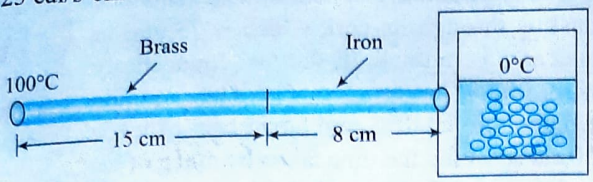
i.e., $$\frac{30KA}{11l} = \frac{(30-T)KA}{3l}$$

$\Rightarrow$ $90 = 330 - 11T$ $\Rightarrow$ $T = 21.82^\circ\text{C}$

ILLUSTRATION 2.9

One end of a uniform brass rod 15 cm long and 20 cm² cross-sectional area is kept at 100°C. The other end is at perfect contact with an iron rod of identical cross-section, but length 8 cm. The lateral surface of the composite rod is surrounded by a heat insulator and the free end of the iron rod is kept

in ice at 0°C. If 684 g of ice melts in 1 h, determine the thermal conductivity of iron. Thermal conductivity of brass = 0.25 cal/s-cm-°C and latent heat of fusion of ice = 80 cal/g.



Sol. Let the thermal conductivity of iron be K = cal/s-cm-°C. The thermal resistance of brass rod,

$$R_1 = \frac{l_1}{K_1 A} = \frac{15\,\text{cm}}{\frac{0.25\,\text{cal}}{\text{s-cm C}} \times 20\,\text{cm}^2} = 3\,\text{s-°C/cal}$$

And that of iron rod,

$$R_2 = \frac{l_2}{K_2 A} = \frac{8\,\text{cm}}{\frac{K\,\text{cal}}{\text{s-cm°C}} \times 20\,\text{cm}^2} = \frac{2}{5K}\,\text{s-°C/cal}$$

Since, the two rods are arranged in series, their effective thermal resistance is given by

$$R = R_1 + R_2 = \left(3 + \frac{2}{5K}\right)\text{s-°C/cal}$$

Now, rate of heat flow through the rods

$$\frac{\Delta Q}{\Delta t} = \frac{(684\,\text{g})}{3600\,\text{s}}\left(80\,\frac{\text{cal}}{\text{g}}\right) = 15.2\,\text{cal/s}$$

Since $\frac{\Delta Q}{\Delta t} = \frac{(T_1 - T_2)}{R}$

$$\therefore \quad R = \frac{(T_1 - T_2)}{(\Delta Q/\Delta t)}$$

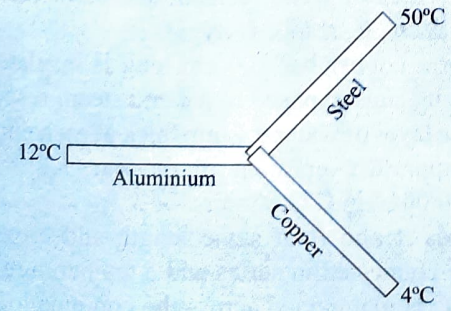
$$\Rightarrow \quad \left(3 + \frac{2}{5K}\right)\text{s°C/cal} = \frac{(100-0)°\text{C}}{15.2\,\text{cal/s}}$$

$$\Rightarrow \quad K = 111.8 \times 10^{-3}\ \text{cal/s-cm-°C}$$

ILLUSTRATION 2.10

Three identical rods of length 1 m each, having cross-sectional area of 1 cm² each and made of aluminium, copper and steel, respectively, are maintained at temperatures of 12°C, 4°C and 50°C, respectively, at their separate ends. Find the temperature of their common junction.

[K_{Cu} = 400 W/m-K, K_{Al} = 200 W/m-K, K_{steel} = 50 W/m-K]

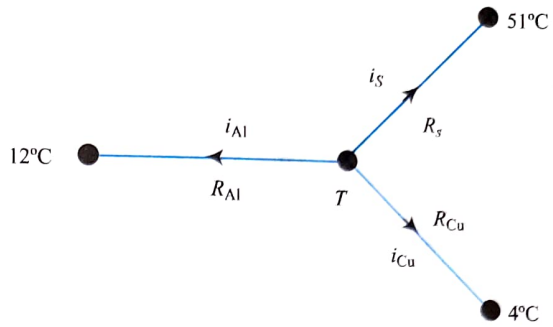


Sol. $R_{Al} = \frac{L}{KA} = \frac{1}{10^{-4} \times 200} = \frac{10^4}{200}$

Similarly $R_{steel} = \frac{10^4}{50}$ and $R_{copper} = \frac{10^4}{400}$

Let temperature of common junction be T. Then from Kirchhoff's current laws, $i_{Al} + i_{steel} + i_{Cu} = 0$

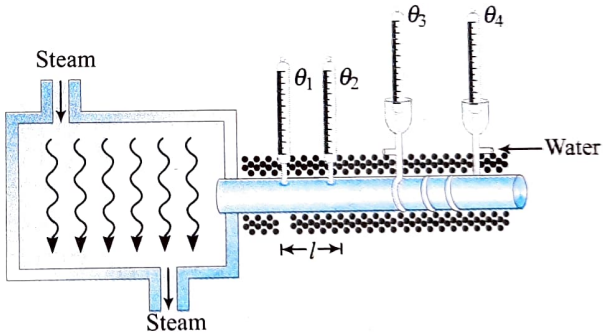
$$\Rightarrow \quad \frac{T-12}{R_{Al}} + \frac{T-50}{R_{steel}} + \frac{T-4}{R_{Cu}} = 0$$



$$\Rightarrow \quad (T-12)\,200 + (T-50)\,50 + (T-4)\,400 = 0$$
$$\Rightarrow \quad 4(T-12) + (T-50) + 8\,(T-4) = 0$$
$$\Rightarrow \quad 13T = 130$$
$$\Rightarrow \quad T = 10°\text{C}$$

SEARLE'S EXPERIMENT

It is a method of determining the measure of thermal conductivity of a material. A bar of material is being heated by steam on one side and the other side is cooled down by water while the length of the bar is thermally insulated.



In this experiment, a temperature difference $(\theta_1 - \theta_2)$ is maintained across a rod of length l and area of cross-section A. If the thermal conductivity of the material of the rod is K, then the amount of heat transmitted by the rod from the hot end to the cold end in time t is given by,

$$\Delta Q = \frac{KA(\theta_1 - \theta_2)t}{l} \qquad \ldots(\text{i})$$

The heat reaching the other end is utilized to raise the temperature of certain amount of water flowing through pipes circulating around the other end of the rod. If temperature of the water at the inlet is θ_3 and at the outlet is θ_4, then the amount of heat absorbed by water is given by,

$$\Delta Q = mc(\theta_4 - \theta_3) \qquad \ldots(\text{ii})$$

where m is the mass of the water which has absorbed this heat and temperature is raised and c is the specific heat of the water.

Equating Eqs. (i) and (ii), K can be determined, i.e.,

$$K = \frac{mc(\theta_4 - \theta_3)l}{A(\theta_1 - \theta_2)t}$$

Important Point:

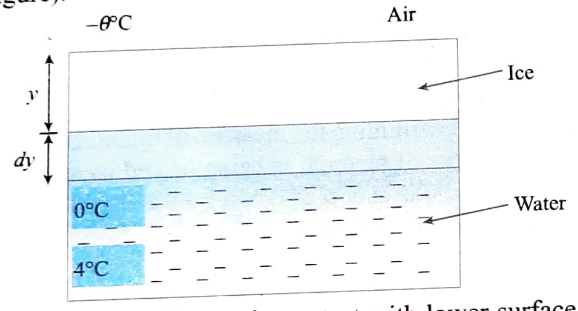
In some cases, we may have the situation where the amount of heat travelling to the other end may be required to do some other work, e.g., it may be required to melt the given amount of ice. In that case, Eq. (i) will have to be equated to mL.

i.e., $$mL = \frac{KA(\theta_1 - \theta_2)t}{l}$$

GROWTH OF ICE ON LAKE

When the temperature of the air is less than 0°C, the cold air near the surface of the pond takes heat (latent) from the water which freezes in the forms of layers.

Consequently, the thickness of the ice layer keeps increasing with time. Let y be the thickness of ice layer in the lake at any instant t and atmospheric temperature is $-\theta$°C. If the thickness is increased by dy in time dt, then the amount of heat flowing through the slab in time dt is given by $dQ_2 = mL = \rho(dy\,A)L$ (see figure).



The temperature of water in contact with lower surface of ice will be zero. If A is the area of lake, heat escaping through ice in time dt,

$$dQ_1 = \frac{KA[0-(-\theta)]dt}{y}$$

As $dQ_1 = dQ_2$, hence, rate of growth of ice will be $\frac{dy}{dt} = \frac{K\theta}{\rho L y}$

So, the time taken by ice to grow to a thickness y is

$$t = \frac{\rho L}{K\theta}\int_0^y y\,dy = \frac{\rho L}{2K\theta}y^2$$

If the thickness is increased from y_1 to y_2 then time taken,

$$t = \frac{\rho L}{K\theta}\int_{y_1}^{y_2} y dy = \frac{\rho L}{2K\theta}(y_2^2 - y_1^2)$$

Important Point:

Ice is a poor conductor of heat, therefore the rate of increase of thickness of ice on ponds decreases with time.

It follows from the above equation that time taken to double and triple the thickness, will be in the ratio of

$$t_1 : t_2 : t_3 :: 1^2 : 2^2 : 3^2$$

i.e., $t_1 : t_2 : t_3 :: 1 : 4 : 9$

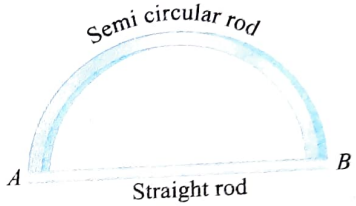
The time intervals to change the thickness from 0 to y, from y to $2y$ and so on will be in the ratio

$$\Delta t_1 : \Delta t_2 : \Delta t_3 :: (1^2 - 0^2) : (2^2 - 1^2) : (3^2 : 2^2)$$

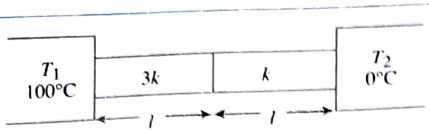
$$\Rightarrow \Delta t_1 : \Delta t_2 : \Delta t_3 :: 1 : 3 : 5$$

CONCEPT APPLICATION EXERCISE 2.1

1. The only possibility of heat flow in a thermos flask is through its cork which is 75 cm² in area and 5 cm thick. Its thermal conductivity is 0.0075 cal/cm-sec-°C. The outside temperature is 40°C and latent heat of ice is 80 cal-g^{-1}. Find the time taken by 500 g of ice at 0°C in the flask to melt into water at 0°C.
2. Two rods (one semi-circular and other straight) of same material and of same cross-sectional area are joined as shown in the figure. The points A and B are maintained at different temperatures. Find the ratio of the heat transferred through a cross-section of a semi-circular rod to the heat transferred through a cross-section of the straight rod in a given time.

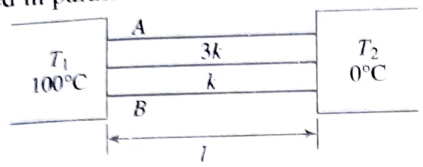


3. The opposite faces of a cubical block of iron of cross-section 4.0 cm² are kept in contact with steam and melting ice. Calculate the quantity of ice melted in 10 minutes (k for iron = 0.2 C.G.S. units).
4. A thermocole cubical icebox of side 30 cm has a thickness of 5.0 cm. If 4.0 kg of ice is put in the box, estimate the amount of ice remaining after 6 h. The outside temperature is 45°C and coefficient of thermal conductivity of thermocole = 0.01 J/s m-°C. Given heat of fusion of water = 335 × 10³ J/kg.
5. A brass boiler has a base area of 0.15 m² and thickness 1.0 cm. It boils water at the rate of 6.0 kg/min, when placed on a gas. Estimate the temperature of the part of the flame in contact with the boiler. Thermal conductivity of brass = 109 J/s-°C and heat of vapourization of water = 2256 J/g.
6. An electric heater is used in a room of total wall area 137 m² to maintain a temperature of + 20°C inside it, when the outside temperature is −10°C. The walls have three different layers of materials. The innermost layer is of wood of thickness 2.5 cm, the middle layer is of cement of thickness 1.0 cm and the outermost layer is of brick of thickness 25.0 cm. Find the power of electric heater. Assume that there is no heat loss through the floor and the ceiling. The thermal conductivities of wood, cement and brick are 0.125, 1.5 and 1.0 W/m-°C, respectively.
7. A uniform copper bar 100 cm long is insulated on side, and has its ends exposed to ice and steam respectively. If there is a layer of water 0.1 mm thick at each end, calculate the temperature gradient in the bar. K_{Cu} = 1.04 and K_{water} = 0.0014 in C.G.S. units.
8. Two rods A and B of same length and cross-sectional area are connected in series and a temperature difference of 100°C is maintained across the combination as shown in the given figure. If the thermal conductivity of the rod A is 3 k and that of rod B is k, then



(i) determine the thermal resistance of each rod.
(ii) determine the heat current flowing through each rod.
(iii) Find the temperature of the junction.
(iv) plot the variation of temperature along the length of the rod.

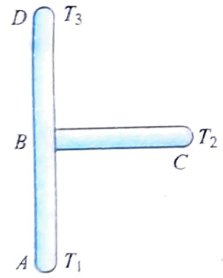
9. Two conductors A and B given in previous problem are connected in parallel as shown in the given figure.



(i) Determine the equivalent thermal resistance.
(ii) Determine the heat current in each rod.

10. One end of a brass rod of length 2.0 m and cross-section 1 cm^2 is kept in stream at 100°C and the other end in ice at 0°C. The lateral surface of the rod is covered by heat insulator. Determine the amount of ice melting per minute. Thermal conductivity of brass is 110 W/m-K and specific latent heat of fusion of ice is 80 cal/g.

11. Three rods AB, BC and BD having thermal conductivities in the ratio 1 : 2 : 3 and lengths in the ratio 2 : 1 : 1 are joined as shown in the given figure. The ends A, C and D are at temperatures T_1, T_2 and T_3, respectively. Find the temperature of the junction B. Assume steady state.



12. A cylinder is made up of two coaxial layers, one of radius R and the other of radius $2R$. The inner and outer portions are respectively made up of substances of thermal conductivities K and $2K$. Determine the effective thermal conductivity between the flat ends of the cylinder.

13. A cylinder of radius R and length l is made up of a substance whose thermal conductivity K varies with the distance x from the axis as $K = K_1x + K_2$. Determine the effective thermal conductivity between the flat faces of the cylinder.

ANSWERS

1. 2.47 hr **2.** $\frac{2}{\pi}$ **3.** 300 g **4.** 3.687 kg

5. 238 °C **6.** 9000 W **7.** 0.87 °C cm^{-1}

8. (i) $R_A = \frac{l}{3KA}, R_B = \frac{l}{KA}$ (ii) $73\left(\frac{kA}{l}\right)$ (iii) 75°C

9. (i) $\frac{1}{4kA}$ (ii) $400\left(\frac{kA}{l}\right)$ **10.** 0.098 g

11. $\frac{1}{11}(T_1+4T_2+6T_3)$ **12.** $\frac{7K}{4}$ **13.** $\frac{1}{3}(2K_1R+3K_2)$

RADIATION

When you stand in front of a big fire, you are warmed by absorbing thermal radiation from the fire; that is, your thermal energy increases as the fire's thermal energy decreases. No medium is required for heat transfer via radiation—the radiation can travel through vacuum from, say, the sun to you. The process of the transfer of heat from one place to another place without heating the intervening medium is called radiation.

It is electromagnetic energy transfer in the form of electromagnetic wave through any medium. It is possible even in vacuum e.g., the heat from the sun reaches the earth through radiation.

The wavelength of thermal radiations ranges from 7.8×10^{-7} m to 4×10^{-4} m. They belong to infra-red region of the electromagnetic spectrum. That is why thermal radiations are also called **infra-red** radiations. Every body whose temperature is above zero Kelvin emits thermal radiation.

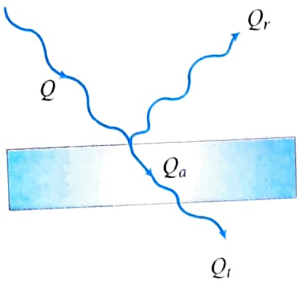
SOME IMPORTANT PROPERTIES OF RADIATION

- All objects emit radiations simply because their temperature is above absolute zero, and all objects absorb some of radiations.
- Maxwell on the basis of his electromagnetic theory proved that all radiations are electromagnetic waves and their sources are vibrations of charged particles in atoms and molecules.
- The intensity of radiation is inversely proportional to the square of distance of point of observation from the source (i.e., $I \propto 1/d^2$).
- Just as light waves, radiations follow the laws of reflection, refraction, interference, diffraction and polarization.
- When these radiations fall on a surface, they exert pressure on that surface which is known as radiation pressure.
- While travelling, these radiations travel just like photons of other electromagnetic waves. They manifest themselves as heat only when they are absorbed by a substance.
- Spectrum of these radiations cannot be obtained with the help of glass prism because it absorbs heat radiations. It is obtained by quartz or rock salt prism because these materials do not have free electrons and their interatomic vibrational frequency is greater than the radiation frequency, hence they do not absorb heat radiations.
- The wavelength corresponding to maximum emission of radiations shifts from the longer wavelength to the shorter wavelength as the temperature increases. Due to this, the colour of a body appears to be changing. Radiations from a body at NTP has predominantly infrared waves.
- When a body is heated, all radiations having wavelengths from zero to infinity are emitted.
- Radiations of longer wavelengths are predominant at lower temperature.
- The wavelength corresponding to maximum emission of radiations shifts from longer wavelength to shorter wavelength as the temperature increases. Due to this the colour of a body appears to be changing.

INTERACTION OF RADIATION WITH MATTER

When thermal radiations fall on a body, then a part of it gets absorbed, a part of it gets reflected and the remaining part gets

transmitted. Let thermal radiation (Q) incident on a body. If Q_a part of it gets absorbed, Q_r part gets reflected and Q_t part gets transmitted,



Then $Q = Q_a + Q_r + Q_t$

or $1 = \frac{Q_a}{Q} + \frac{Q_r}{Q} + \frac{Q_t}{Q}$

Let $a = \frac{Q_a}{Q}$ = Absorptance or absorbing power;

$r = \frac{Q_r}{Q}$ = Reflectance or reflecting power; and

$t = \frac{Q_t}{Q}$ = Transmittance or transmitting power

then $\frac{Q_a}{Q} + \frac{Q_r}{Q} + \frac{Q_t}{Q} = a + r + t = 1$

r, a and t all are the pure ratios so they have no unit and dimension.

Different Cases for a, r and t

- If $a = t = 0$ and $r = 1 \rightarrow$ body is perfect reflector
- If $r = t = 0$ and $a = 1 \rightarrow$ body is perfectly black body
- If, $a = r = 0$ and $t = 1 \rightarrow$ body is perfect transmitter
- If $t = 0 \Rightarrow r + a = 1$ or $a = 1 - r$, i.e., good reflectors are bad absorbers

Emissive Power, Absorptive Power and Emissivity

If temperature of a body is more than its surrounding then body emits thermal radiation.

Monochromatic emittance or spectral emissive power (E_λ): For a given surface, it is defined as the radiant energy emitted per second per unit area of the surface within a unit wavelength around λ.

$$\text{Spectral emissive power } (E_\lambda) = \frac{\text{Energy}}{\text{Area} \times \text{time} \times \text{wavelength}}$$

Unit: $\frac{\text{Joule}}{\text{m}^2 \times \text{sec} \times \text{Å}}$ and **Dimension:** $[ML^{-1}T^{-3}]$

Total emittance or total emissive power (E): The emissive power or coefficient of emission of a surface is defined as the radiant energy emitted per unit time per unit area of it.

Thus, if ΔQ be the amount of radiant energy emitted by a surface of area A in a time interval Δt, then its emissive power E will be given by

$$E = \frac{\Delta Q}{A \Delta t}$$

Experiments reveal that the emissive power of a surface depends upon the wavelength of radiation emitted. Thus, it would be wiser to speak in terms of the emissive power of a surface corresponding to a particular wavelength, known as the *spectral emissive power*.

Thus, the spectral emissive power of a surface is defined as the rate at which radiant energy is emitted per unit area of it, per unit wavelength range. If $E_\lambda d\lambda$ be the rate of heat emission of a surface, per unit area of it, then E_λ is the spectral emissive power corresponding to the wavelength λ.

Hence, we can define, $E = \int_0^\infty E_\lambda d\lambda$

Unit: $\frac{\text{Joule}}{\text{m}^2 \times \text{sec}}$ or $\frac{\text{Watt}}{\text{m}^2}$ and **Dimension:** $[MT^{-3}]$

Monochromatic absorptance or spectral absorptive power (a_λ): Experiments reveal that the absorptive power of a surface is wavelength dependent. Hence, we go for *spectral absorptive power*.

It is defined as the ratio of the amount of the energy absorbed in a certain time to the total heat energy incident upon it in the same time, both in the unit wavelength interval. It is dimensionless and unit less quantity. It is represented by a_λ.

Total absorptance or total absorptive power (a): The absorptive power a or the coefficient of absorption I of a surface is defined as the ratio of the amount of radiation absorbed by it in a given time interval to the total amount of radiation incident upon it in the same time interval. Notice that, the absorptive power of a surface is same as its absorptivity. Since absorptive power is the ratio of heats, it has no unit. Further, $0 \le a \le 1$.

Thus, if $a_\lambda d\lambda$, be the fraction of heat absorbed in the wavelength range $d\lambda$, then we define,

$$a = \int_0^\infty a_\lambda d\lambda$$

Emissivity (e): The ratio of emissive power of a surface to that of a perfectly black body under identical conditions is said to be the emissivity (e) of the surface. Since the emissive power of a surface is less than or equal to that of a perfectly black body, so $0 \le e \le 1$.

(i) For perfectly black body $e = 1$

(ii) For highly polished body $e = 0$

(iii) But for practical bodies emissivity (e) lies between zero and one ($0 < e < 1$).

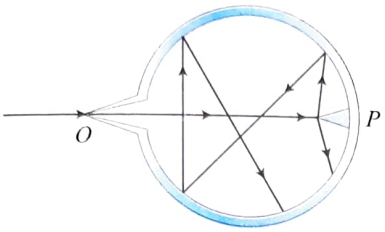
Perfectly Black Body

A perfectly black body is that which absorbs completely all the radiations incident on it. The colour of an opaque body is the colour (wavelength) of radiation reflected by it. As a black body reflects no wavelength so, it appears black, whatever be the colour of radiations incident on it.

As a perfectly black body neither reflects nor transmits any radiation, therefore the absorptance of a perfectly black body is unity, i.e., $t = 0$ and $r = 0 \Rightarrow a = 1$.

When perfectly black body is heated to a suitable high temperature, it emits radiations of all possible wavelengths. For example, temperature of the sun is very high (6000 K approx.) it emits all possible radiations so it is a black body.

Ferry's black body: A perfectly black body cannot be realised in practice. The simplest and most commonly used black body was designed by Ferry. It is a double walled evacuated spherical cavity whose inner wall is blackened. The space between the wall is evacuated to prevent the loss of heat by conduction and radiation. There is a fine hole in it. All the radiations incident upon this hole are absorbed by this black body. If this black body is heated to high temperature then it emits radiations of all wavelengths. It is the hole which is to be regarded as a black body and not the total enclosure



A perfectly black body cannot be realised in practice but materials like platinum black or lamp black come close to being ideal black bodies. Such materials absorb 85% to 96% of the incident radiations.

ILLUSTRATION 2.11

An electric heater of surface area 200 cm^2 emits radiant energy of 60 kJ at time interval of 1 min. Determine its emissive power. If its emissivity be 0.45, what would be the radiant energy emitted by a black body in one hour, identical to the electrical heater in all respects?

Sol. Given $\Delta Q = 60$ kJ; $A = 200 \times 10^{-4}$ m^2 and $\Delta t = 60$ s

Using, emissive power $E = \left(\frac{\Delta Q}{\Delta t}\right)\frac{1}{A}$, we have

$$E = \frac{60\,\text{kJ}}{200\times10^{-4}\,\text{m}^2\times60\,\text{s}} = 50\ \text{kW/m}^2$$

$\therefore$ Emissivity (of a surface)

$$e = \frac{\text{emissive power (of that surface)}}{\text{emissive power (of a black body)}}$$

$\therefore$ Emissive power of a black body

$$= \frac{50}{0.45}\text{kW/m}^2 = 111.11\ \text{kW/m}^2$$

$\therefore$ Heat radiated $\Delta Q = EA\Delta t$

$$= \left(111.11\frac{\text{kW}}{\text{m}^2}\right)(200\times10^{-4}\,\text{m}^2)(60\times60\,\text{s}) = 8000\,\text{kJ}$$

PREVOST'S THEORY OF HEAT EXCHANGES

Until the end of eighteenth century, the ideas regarding the radiant energy were confused. At that time, it was thought that a hot body gives out 'hot radiations' (or calorific rays) and the cold body radiates 'cold radiations' (or frigorific rays). It was the Swiss physicist, Pierre Prevost, who showed that this was not so, by putting forward a theory, known after him, as Prevost's theory of heat exchanges. The salient features of this theory are:

(i) All bodies at temperatures above 0 K emit thermal radiation irrespective of their surroundings.

(ii) The amount of radiation emitted increases with temperature, there is a continuous exchange of heat between a body and its surroundings.

(iv) The rise or fall in the temperature of the body is only due to this exchange.

(v) The exchange of heat between the body and the surroundings continues till a dynamic thermal equilibrium is established between them and their temperatures become equal.

Thus, according to this theory when we stand in front of fire, we feel hot due to the reason that we receive more radiation from the fire than we give to it. Reverse is the case when we stand in front of ice. Here, we lose more heat radiation than we gain from ice. Thus, we feel cold in front of ice.

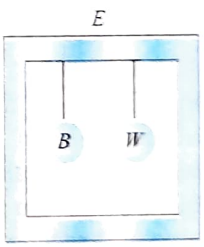
This theory is in full agreement with kinetic theory of matter according to which the molecules in matter are in a continuous state of agitation and as such emit thermal energy. This process of emission of thermal energy stops only at absolute zero when the molecular motion altogether stops.

Important Points:

- If $Q_{\text{emission}} > Q_{\text{absorption}} \rightarrow$ temperature of body decreases and consequently the body appears colder.
- If $Q_{\text{emission}} < Q_{\text{absorption}} \rightarrow$ temperature of body increases and it appears hotter.
- If $Q_{\text{emission}} = Q_{\text{absorption}} \rightarrow$ temperature of body remains constant (thermal equilibrium)

KIRCHHOFF'S LAW

One of the most important conclusions of the Prevost's theory of heat exchanges is the Kirchhoff's law. Let us try to arrive this law first qualitatively and then quantitatively. Let us suspend two balls, one black (B) and the other white (W) inside an enclosure E, with the help of insulating threads as shown in the figure. The chamber is maintained at a constant temperature T. The walls of the chamber are continuously emitting and absorbing radiation and the enclosure is full of radiation. Due to the exchange of radiation with the walls of the chamber (or with themselves), the balls will also attain the temperature of the enclosure, i.e., T. Their temperatures will remain constant as long as the temperature of the enclosure does not change.



We know that the black ball is a good absorber and as such it absorbs more radiation than the white ball. How are then their temperatures equal? It is possible only if the black ball also radiates more heat than the white ball. Thus, a good absorber (black ball) is a good emitter and vice-versa. It was Kirchhoff, who derived the exact quantities form of this statement which after him is known as Kirchhoff's law.

According to this law, the ratio of emissive power to absorptive power is same for all surfaces at the same temperature and is equal to the emissive power of a perfectly black body at that temperature. Hence,

$$\frac{e_1}{a_1} = \frac{e_2}{a_2} = \ldots = \left(\frac{E}{A}\right)_{\text{Perfectly black body}}$$

But for perfectly black body, $A = 1$, i.e., $\frac{e}{a} = E$

If emissive and absorptive powers are considered for a particular wavelength λ,

$$\left(\frac{e_\lambda}{a_\lambda}\right) = (E_\lambda)_{\text{black}}$$

Now since $(E_\lambda)_{\text{black}}$ is constant at a given temperature, according to this law, if a surface is a good absorber of a particular wavelength, then it is also a good emitter of that wavelength.

This in turn implies that a good absorber is a good emitter (or radiator).

STEFAN'S LAW

Based on several experimental observations on the rate of emission of radiant heat by surfaces of black bodies, Stefan gave this law which states that 'the emissive power (the rate of emission of radiant energy per unit area) of a perfectly black body varies directly as the fourth power of its absolute temperature.' This is known as Stefan's law.

Thus, if E be the emissive power of a black body maintained at a temperature (absolute) T then from Stefan's law,

$$E \propto T_0^4 \text{ or, } E = \sigma T_0^4 \quad \ldots(i)$$

where σ is a constant called Stefan's constant having dimension $[MT^{-3}\theta^{-4}]$ and value 5.67×10^{-8} W/m^{2}K^4.

Important Points:

Total radiant energy and power radiated by the ordinary body

- **Radiant energy:** Let A be the surface area of a body of emissivity e, at an absolute temperature T and Q be the total energy radiated by the ordinary body. Then

$$E = \frac{Q}{A \times t} = e\sigma T^4 \Rightarrow Q = Ae\sigma T^4 t$$

- **Radiant power (P):** It is defined as energy radiated per unit area, i.e., $P = \frac{Q}{t} = Ae\sigma T^4$.

The theoretical proof of Stefan's law was given by Boltzmann, by applying the usual laws of thermodynamics to a black body treating it as an isolated system. He also gave a general law suiting the emissive power for all types of bodies. If a perfectly black body be in thermal equilibrium with its surrounding at an absolute temperature T_0 then from Prevost's theory of exchanges, the rate of heat emission from the body per unit area will be same as the rate of heat absorption due to the surrounding.

i.e., Rate of heat emission per unit area = Rate of heat absorption per unit area = σT_0^4

Now, if the temperature of the body be raised to an absolute temperature T then, the rate of heat emission per unit area of the body will be σT^4.

Therefore, the net rate of heat emission per unit area of the body will be

$$E = \sigma(T^4 - T_0^4) \quad \ldots(ii)$$

In general, for an ordinary body (other than black body) of emissivity (or relative emittance) e, the above equation can be expressed as

$$E = e\sigma(T^4 - T_0^4) \quad \ldots(iii)$$

where $0 < e < 1$ and $e = 1$ for a perfectly black body.

The above equation is the mathematical expression of Stefan-Boltzmann law.

RATE OF LOSS OF HEAT (OR INITIAL RATE OF LOSS OF HEAT)

If an ordinary body of the surface area A, of emissivity e, and at an absolute temperature T, is kept in a surrounding of temperature T_0 ($T_0 < T$), then the heat lost by radiation is given by

$$\Delta Q = Q_{\text{emission}} - Q_{\text{absorption}} = Ae\sigma(T^4 - T_0^4) \times \Delta t$$

$$\text{Rate of loss of heat } (R_H) = \frac{dQ}{dt} = Ae\sigma(T^4 - T_0^4)$$

Important Points:

- If two bodies are made of same material, have same surface area and are at the same initial temperature

$$\text{then } \frac{dQ}{dt} \propto A \Rightarrow \frac{\left(\frac{dQ}{dt}\right)_1}{\left(\frac{dQ}{dt}\right)_2} = \frac{A_1}{A_2}$$

- Net power radiated by black body, $P_{\text{net}} = \sigma A(T^4 - T_0^4)$

For small temperature difference, $\Delta T = T - T_0$

$$P_{\text{net}} = \sigma A\left[(T_0 + \Delta T)^4 - T_0^4\right]$$

$$= \sigma A T_0^4\left[\left(1 + \frac{\Delta T}{T_0}\right)^4 - 1\right] = \sigma A T_0^4\left[\frac{4\Delta T}{T_0}\right]$$

$$P_{\text{net}} = 4\sigma A T_0^3(T - T_0) = 4\sigma A T_0^3 \Delta T = 4A\sigma T_0^4 \frac{\Delta T}{T_0}$$

INITIAL RATE OF FALL IN TEMPERATURE (RATE OF COOLING)

If a body of the surface area A, of emissivity e, at an absolute temperature T, is kept in a surrounding of temperature T_0, then the net rate of heat emission from the entire surface will be

$$\frac{dQ}{dt} = Ae\sigma(T^4 - T_0^4) \quad \ldots(i)$$

If 'c' be the heat capacity of the body, then the rate of heat loss will be

$$\frac{dQ}{dt} = c\left(-\frac{dT}{dt}\right) \quad \ldots(ii)$$

So, from Eqs. (i) and (ii), we have

$$-c\frac{dT}{dt} = Ae\sigma(T^4 - T_0^4)$$

$$-\frac{dT}{dt} = \frac{Ae\sigma}{c}(T^4 - T_0^4) \quad \ldots(iii)$$

If m, ρ, V and s be the mass, density, volume and specific heat capacity (of the substance) of the body, respectively, then $c = ms = \rho Vs$, so Eq. (iii) can be expressed also as

$$-\frac{dT}{dt} = \frac{Ae\sigma}{ms}(T^4 - T_0^4) = \frac{Ae\sigma}{\rho Vs}(T^4 - T_0^4) \quad \ldots(iv)$$

Equations (iii) and (iv) express, the rate of cooling (i.e., the rate of temperature drop or fall) of the body.

Important Point:

For two bodies of the same material under identical environments, the ratio of their rates of cooling is

$$\frac{(R_c)_1}{(R_c)_2} = \frac{A_1}{A_2}\cdot\frac{V_2}{V_1}$$

DEPENDENCE OF RATE OF COOLING

When a body cools by radiation, the rate of cooling depends on

(i) nature of radiating surface, i.e., greater the emissivity, faster will be the cooling.

(ii) area of radiating surface, i.e., greater the area of radiating surface, faster will be the cooling.

(iii) mass of radiating body, i.e., greater the mass of radiating body slower will be the cooling.

(iv) specific heat of radiating body, i.e., greater the specific heat of radiating body slower will be cooling.

(v) temperature of radiating body, i.e., greater the temperature of body faster will be cooling.

(vi) temperature of surrounding, i.e., greater the temperature of surrounding slower will be cooling.

ILLUSTRATION 2.12

A spherical body with radius 12 cm radiates 450 W power at 500 K. If the radius were halved and the temperature doubled, what would be the power radiated?

Sol. By Stefan's law, Power radiated $E = \varepsilon\sigma AT^4 = \varepsilon\sigma(4\pi r^2)\,T^4$

When radius is halved and temperature is doubled,

Power radiated $E' = e\sigma\left[4\pi\left(\frac{r}{2}\right)^2\right](2T)^4 = 4E = 4\times 450 = 1800\,\text{W}$

ILLUSTRATION 2.13

The operating temperature of an incandescent bulb (with a tungsten filament) of power 60 W is 3000 K. If the surface area of the filament be 25 mm^2, find its emissivity e.

Sol. Given $T = 3000$ K; Power $= \frac{dQ}{dt} = 60$ W

Area $A = 25\times10^{-6}\,\text{m}^2$, $e = ?$

Total radiation lost per second $= \frac{dQ}{dt} = Ae\sigma T^4$,

$$60\,\text{W} = (25\times10^{-6}\,\text{m}^2)e\left(5.67\times\frac{10^{-8}}{\text{m}^2-\text{K}^4}\right)(3000)^4\,\text{K}^4$$

$60 = 114.82\,e$

$e = 0.52$

ILLUSTRATION 2.14

A thin brass rectangular sheet of sides 15.0 and 12.0 cm is heated in a furnace to 600°C and taken out. How much electric power is needed to maintain the sheet at this temperature, given that its emissivity is 0.250? Neglect heat loss due to convection (Stefan–Boltzmann constant, $\sigma = 5.67\times10^{-8}$ W/m^2-K^4).

Sol. Area of the both sides of the plate

$A = 2\times(15.0)\times(12.0)\times10^{-4}\,\text{m}^2 = 3.60\times10^{-2}\,\text{m}^2$

The energy radiated by the plate $= \varepsilon\sigma AT^4$

$= 0.250\times5.67\times10^{-8}\times3.60\times10^{-2}\times(600+273)^4$

$= 5.10\times10^{-12}\times873^4 = 296.4$ W

ILLUSTRATION 2.15

The emissivity of tungsten is approximately 0.35. A tungsten sphere 1 cm in radius is suspended within a large evacuated enclosure whose walls are at 300 K. What power input is required to maintain the sphere at a temperature of 3000 K if heat conduction along the supports is neglected? $\sigma = 5.67\times10^{-8}$ SI units.

Sol. Net heat lost by sphere per second $H_{net} = e\sigma A(T^4 - T_0^4)$

where T = temperature of sphere = 3000 K

T_0 = temperature of surrounding = 300 K

$\Rightarrow \quad A = 4\pi r^2 = 4\pi(0.01)^2$

To maintain constant temperature, power input required = net heat loss per second from the surface

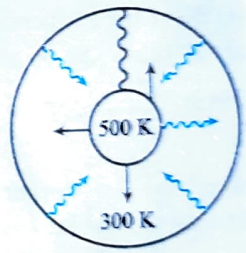
$$P_{input} = e\sigma A(T^4 - T_0^4)$$

$$P_{input} = 0.35\times5.67\times10^{-8}\times4\pi(0.01)^2\times(3000^4 - 300^4)$$

$$= 2019.8\,\text{W}$$

ILLUSTRATION 2.16

A copper sphere is suspended in an evacuated chamber maintained at 300 K. The sphere is maintained at a constant temperature of 500 K by heating it electrically. A total of 210 W of electric power is needed to do it. When the surface of the copper sphere is completely blackened, 700 W is needed to maintain the same temperature of the sphere. Calculate the emissivity of copper.



Sol. Let ε be the emissivity of the sphere. Then by Stefan's law

$$E = \varepsilon\sigma A(T^4 - T_0^4)$$

or $\quad 210 = \varepsilon\sigma A(500^4 - 300^4)$ ...(i)

When sphere is blackened, it behaves like a perfectly black body, so we have

$$700 = \sigma A\,(500^4 - 300^4)$$ ...(ii)

Dividing Eq. (i) by Eq. (ii), we get $\varepsilon = 0.3$.

NEWTON'S LAW OF COOLING

It is a common experience that when hot bodies say, soup, tea, milk, etc., are kept exposed to the surrounding, they lose heat and finally attain the temperature of the surrounding. The rate at which the temperature falls is given by Newton's law of cooling, which states that, the rate of cooling of a hot body kept in a cold surrounding varies directly as the temperature difference between the hot body and the cold surrounding, provided it is not too large.

Thus, if T be the instantaneous temperature of the hot body, then the rate of cooling $= -dT/dt$

If T_0 be the temperature of the surrounding (which remains fixed), then the instantaneous temperature difference between the body and the surrounding will be $(T - T_0)$.

Assuming $T - T_0$ to be small, from Newton's law of cooling,

Rate of cooling $\propto$ temperature difference between the body and the surrounding,

Mathematically, $\frac{-dT}{dt} \propto (T - T_0)$

Hence we can write rate of cooling,

$$R = \frac{dT}{dt} = -k(T - T_0) \quad \ldots(i)$$

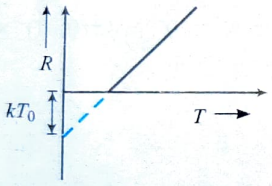
where k is a positive constant, which depends upon the heat capacity of the body and nature and extent of its exposed area.

Important Point:

If we plot curve between the rate of cooling (R) and body temperature (θ) the graph will be straight line.

As $R = k(T - T_0) = kT - kT_0$

This is a straight line with positive slope and intercept with R-axis at $-kT_0$.



If, the initial temperature of the body be T_1. Then temperature T at any time t can be obtained by integrating the expression.

Since, $\frac{dT}{dt} = -k(T - T_0)$

so, $\frac{dT}{(T - T_0)} = -kdt$

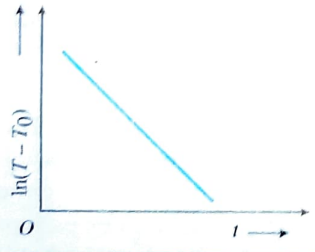
$\therefore \quad \int_{T_1}^{T} \frac{dT}{(T - T_0)} = \int_0^t -kdt \quad \Rightarrow \quad [\ln (T - T_0)]_{T_1}^{T} = -k[t]_0^t$

From, $\ln(T - T_0) = -kt + \ln(T_1 - T_0)$ $\quad \ldots(ii)$

Important Point:

If $\ln (T - T_0)$ be plotted against t, then the equation assumes the form $y = mx + c$; where $m = -k$ and $C = \ln(T_1 - T_0)$.

This is a straight line with negative slope.

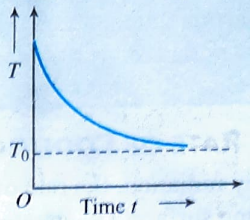


We have $\ln\left(\frac{T - T_0}{T_1 - T_0}\right) = -kt$ or $\left(\frac{T - T_0}{T_1 - T_0}\right) = e^{-kt}$

or $\quad T = T_0 + (T_1 - T_0)e^{-kt} \quad \ldots(iii)$

Important Point:

If we plot a curve between temperature of body and time, the graph will be exponential decreasing curve.



Alternatively, without using calculus, Newton's law of cooling can yet be stated in another way. This approach can be employed to solve numerical problems.

From, Newton's law of cooling, (Average rate of cooling $= -k$ (Average temperature difference between the body and the surrounding)

Let T_0 be the temperature of the surrounding. If the initial temperature of the body be T_1 and T_2 be the temperature after time t, then,

Average rate of cooling $= \frac{T_2 - T_1}{t}$

Average temperature difference between the body and the surrounding $= \frac{T_1 + T_2}{2} - T_0$

$$\Rightarrow \quad \frac{T_2 - T_1}{t} = -k\left[\left(\frac{T_1 + T_2}{2}\right) - T_0\right]$$

$$\Rightarrow \quad \frac{T_1 - T_2}{t} = k\left[\frac{(T_1 + T_2)}{2} - T_0\right] \quad \ldots(iv)$$

Also, we can verify Newton's law of cooling using Stefan's law. When the temperature difference between the body and its surrounding is not very large, i.e., $T - T_0 = \Delta T$ then $T^4 - T_0^4$ may be approximated as $4T_0^3 \Delta T$.

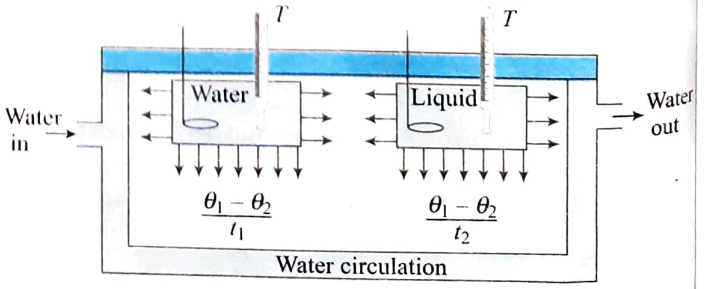
By Stefan's law, $\frac{dT}{dt} = \frac{A\varepsilon\sigma}{mc}[T^4 - T_0^4]$

Hence, $\frac{dT}{dt} = \frac{A\varepsilon\sigma}{mc} 4T_0^3 \Delta T \quad \Rightarrow \quad \frac{dT}{dt} \propto \Delta T$ or $\frac{dT}{dt} \propto T - T_0$

i.e., if the temperature of body is not very different from surrounding, rate of cooling is proportional to temperature difference between the body and its surrounding. This is called Newton's law of cooling.

DETERMINATION OF SPECIFIC HEAT OF LIQUID

If volume, radiating surface area, nature of surface, initial temperature and surrounding of water and given liquid are equal and they are allowed to cool down (by radiation) then rate of loss of heat and fall in temperature of both will be same.



i.e., $\left(\frac{dQ}{dt}\right)_{water} = \left(\frac{dQ}{dt}\right)_{liquid}$

or $(m_W c_W + W)\frac{(\theta_1 - \theta_2)}{t_1} = (m_l c_l + W)\frac{(\theta_1 - \theta_2)}{t_2}$

or $\left[\frac{m_W c_W + W}{t_1}\right] = \left[\frac{m_l c_l + W}{t_2}\right]$

$W = m_c c_c$ = Water equivalent of calorimeter,

where m_c and c_c are mass and specific heat of calorimeter.

If densities of water and liquid are ρ, and θ', respectively, then $m_W = V\rho_W$ and $m_l = V\rho_l$.

Specific heat of liquid, $c_l = \frac{1}{m_l}\left[\frac{t_l}{t_W}(m_W c_W + W) - W\right]$

ILLUSTRATION 2.17

Two solid copper spheres of radii $r_1 = 15$ cm and $r_2 = 20$ cm are both at a temperature of 60°C. If the temperature of surrounding is 50°C, then find

(a) the ratio of the heat loss per second from their surfaces initially.

(b) the ratio of rates of cooling initially.

Sol.

(a) Ratio of heat loss $= \frac{H_1}{H_2} = \frac{(dQ/dt)_1}{(dQ/dt)_2}$

$\Rightarrow \quad \frac{H_1}{H_2} = \frac{KA_1(60-50)}{KA_2(60-50)}$

by using Newton's law of cooling

$$\frac{H_1}{H_2} = \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} = \left(\frac{15}{20}\right)^2 = \frac{9}{16}$$

(b) The ratio of initial rates of cooling $= \frac{(d\theta/dt)_1}{(d\theta/dt)_2}$

we have $\frac{H_1}{H_2} = \left(\frac{r_1}{r_2}\right)^2 \Rightarrow \frac{M_1 s(d\theta/dt)_1}{M_2 s(d\theta/dt)_2} = \left(\frac{r_1}{r_2}\right)^2$

As the spheres have the same densities, the ratio of their masses is equal to the ratio of their volumes.

$$\Rightarrow \quad \frac{(d\theta/dt)_1}{(d\theta/dt)_2} = \left(\frac{r_1}{r_2}\right)^2 \frac{M_2}{M_1} = \left(\frac{r_1}{r_2}\right)^2 \left(\frac{r_2}{r_1}\right)^3 = \frac{r_2}{r_1} = \frac{20}{15} = \frac{4}{3}$$

ILLUSTRATION 2.18

A body cools down from 60°C to 55°C in 30 s. Using Newton's law of cooling, calculate the time taken by same body to cool down from 55°C to 50°C. Assume that the temperature of surrounding is 45°C.

Sol. Assume that a body cools down from temperature θ_1 to θ_2 in t seconds, and θ_c is the temperature of surroundings. Applying Newton's law of cooling, According to Newton's law of cooling

$$\frac{d\theta}{dt} = -K(\theta - \theta_0); \quad \theta_0 = 45°C; \quad \int \frac{d\theta}{\theta - 45} = -k\int dt$$

(K is constant)

from $t = 0$ s to $t = 30$ s, θ changes from 60°C to 55°C.

$$\int_{55}^{60} \frac{d\theta}{\theta - 45} = -K\int_{30}^{0} dt$$

$$\Rightarrow \quad \ln\left(\frac{50-45}{60-45}\right) = -K(30-0) \qquad ...(i)$$

$$\int_{55}^{50} \frac{d\theta}{\theta - 45} = -K\int_{30}^{t} dt$$

$$\Rightarrow \quad \ln\left(\frac{50-45}{55-45}\right) = -K(t-30) \qquad ...(ii)$$

Divide Eq. (ii) by Eq. (i) to get

$$\frac{t-30}{30} = \frac{\ln\frac{50-45}{55-45}}{\ln\frac{55-45}{60-45}} \Rightarrow \frac{t-30}{30} = \frac{\ln 2}{\ln 3/2}$$

$$\Rightarrow \quad t = 30\left(1 + \frac{\ln 2}{\ln 3/2}\right) = 81.28 \text{ s}$$

Time taken to cool down from $\theta = 55°C$ to $\theta = 50°C$ is $(t - 30)$ = (81.28–30) = 51.28 s.

ILLUSTRATION 2.19

Two identical spheres A and B are suspended in an air chamber which is maintained at a temperature of 50°C. Find the ratio of the heat lost per second from the surface of the spheres if

(a) A and B are at temperatures 60°C and 55°C, respectively.

(b) A and B are at temperatures 250°C and 200°C, respectively.

Sol. From Stefan's Law

Net heat lost per second per unit area = $e\sigma(T^4 - T_0^4)$

If $T - T_0$ is small as compared to the temperature of surroundings, we have from Newton's law of cooling, Net heat lost per second per unit area = (constant) × $(T - T_0)$

(a) Here the temperature difference is small and hence we can use Newton's law of cooling.

$$E_A \propto (60 - 50) \quad \text{and} \quad E_B \propto (55 - 50)$$

Hence, $\frac{E_A}{E_B} = \frac{10}{5} = 2$

(b) As the temperature difference is not negligible as compared to the temperature of surrounding, we use Stefan's law for accurate answer.

$$E_A \propto (250+273)^4 - (50+273)^4$$

$$\Rightarrow \quad E_B \propto (200+273)^4 - (50+273)^4$$

Hence, $\frac{E_A}{E_B} = \frac{523^4 - 323^4}{423^4 - 323^4} = 1.632$

ILLUSTRATION 2.20

A body cools from 60°C to 50°C in 10 min. Find its temperature at the end of next 10 min if the room temperature is 25°C. Assume Newton's law of cooling holds.

Sol. According to Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

or $\log(\theta - \theta_0) = -kt + c$

Given at $t = 0$, $\theta = 60°C$

$\therefore \quad \log(60 - 25) = c$

$\therefore \quad \log(\theta - 25) = -kt + \log 35$

or $kt = \log 35 - \log(\theta - 25)$

At $t = 10$ min $\theta = 50°C$

$\therefore \quad k10 = \log 35 - \log 25 \Rightarrow k = \frac{1}{10}\log\frac{7}{5}$

$\therefore \quad \left(\frac{1}{10}\log\frac{7}{5}\right)t = \log 35 - \log(\theta - 25)$

When $t = 20$ min $\theta = ?$

$\therefore \quad \left(\frac{1}{10}\log\frac{7}{5}\right)20 = \log\frac{35}{\theta - 25}$

$\Rightarrow \quad \log\left(\frac{7}{5}\right)^2 = \log\frac{35}{\theta - 25} \Rightarrow \left(\frac{7}{5}\right)^2 = \frac{35}{\theta - 25}$

$\Rightarrow \quad \theta = 42.8°C$

ILLUSTRATION 2.21

A hot body placed in the air is cooled down according to Newton's law of cooling, the rate of decrease of temperature being k times the temperature difference from the surrounding. Starting from $t = 0$, find the time in which the body will lose half the maximum heat it can lose.

Sol. We have, $-\frac{dT}{dt} = k(T - T_0)$

where T_0 is the temperature of the surrounding. If T_1 is the initial temperature and T is the temperature at any time t, then

$$\int_{T_1}^{T} \frac{dT}{(T - T_0)} = -k\int_0^t dt$$

or $\left|\ln(T - T_0)\right|_{T_1}^{T} = -kt$ or $\ln\left[\frac{T - T_0}{T_1 - T_0}\right] = -kt$

or $T = T_0 + (T_1 - T_0)e^{-kt}$ ...(i)

The body continues to lose heat till its temperature becomes equal to that of the surrounding. The loss of heat

$Q = mc(T_1 - T_0)$

If the body lose half of the maximum loss that it can, then decrease in temperature

$$\frac{Q}{2} = mc\left(\frac{T_1 - T_0}{2}\right)$$

If body loses this heat in time t, then its temperature at time t' will be

$$T_1 - \left(\frac{T_1 + T_0}{2}\right) = \frac{T_1 + T_0}{2}$$

Putting these values in Eq. (i), we have $\frac{T_1 + T_0}{2} = T_0 + (T_1 - T_0)e^{-kt'}$

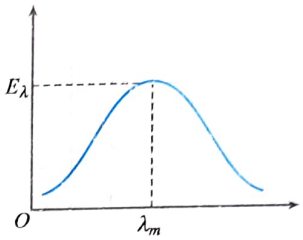
or $\frac{T_1 - T_0}{2} = (T_1 - T_0)e^{-kt'}$

or $e^{-kt'} = \frac{1}{2}$ or $t' = \frac{\ln 2}{k}$

DISTRIBUTION OF ENERGY IN THE SPECTRUM OF BLACK BODY

Langley and later on Lummer and Pringsheim investigated the distribution of energy amongst the different wavelengths in the thermal spectrum of a black body radiation. A perfectly black body emits radiation of all possible wavelength.

The results obtained are shown in the following figure. From these curves, it is clear that at a given temperature energy is not uniformly distributed among different wavelengths.



At a given temperature, intensity of heat radiation increases with wavelength, reaches a maximum at a particular wavelength and with further increase in wavelength it decreases. For all wavelengths, an increase in temperature causes an increase in intensity.

The area under the curve will represent the total intensity of radiation at a particular temperature,

i.e., Area $= E = \int E_\lambda d\lambda$

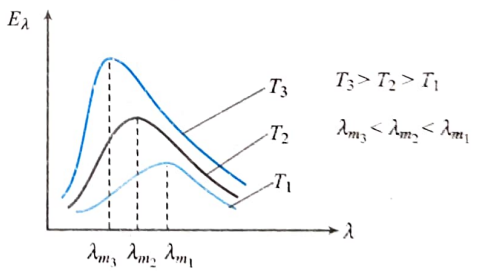
From Stefan's law, $E = \sigma T^4$

$\Rightarrow$ Area under E_λ–λ curve, $E \propto T^4$

WIEN'S DISPLACEMENT LAW

According to Wien's law, the product of wavelength corresponding to maximum intensity of radiation and temperature of body (in Kelvin) is constant, i.e., $\lambda_m T = b$ = constant where b is Wien's constant and has value 2.89×10^{-3} m-K.

As the temperature of the body increases, the wavelength at which the spectral intensity (E_λ) is maximum shifts towards left. It is also called Wien's displacement law.



This law is of great importance in astrophysics as through the analysis of radiations coming from a distant star by finding λ_m the temperature of the star T $(= b/\lambda_m)$ is determined.

ILLUSTRATION 2.22

The spectral energy distribution of the sun has a maximum at 4753 Å. If the temperature of the sun is 6050 K, what is the temperature of a star for which this maximum is at 9506 Å?

Sol. Given $\lambda_m = 4753$ Å, $T = 6050$ K

$\lambda'_m = 9506$ Å

If T' is the temperature of star, then

$$\lambda_m T = \lambda'_m T' \quad \text{or} \quad T' = \frac{\lambda_m T}{\lambda'_m} = \frac{4753 \times 6050}{9506} = 3025 \text{ K}$$

ILLUSTRATION 2.23

The solar radiation spectrum reveals that the intensity corresponding to a wavelength of 4750 Å is maximum. Estimate the surface temperature of the sun. (Given Wien's constant = 2.89×10^{-3} m-K)

Sol. From Wien's displacement law, the temperature T of a body corresponding to maximum intensity wavelength λ_m is given by

$$T = \frac{b}{\lambda_m}$$

$$T = \frac{2.89 \times 10^{-3} \text{ m} - \text{K}}{4750 \times 10^{-10} \text{ m}} = 6084 \text{ K}$$

This temperature corresponds to the chromosphere (surface) of the sun.

ILLUSTRATION 2.24

A hot black body emits the energy at the rate of 16 J m^{-2} s^{-1} and its most intense radiation corresponds to 20,000 Å. When the temperature of this body is further increased and its most intense radiation corresponds to 10,000 Å, then find the value of energy radiated in Jm^{-2} s^{-1}.

Sol. Wien's displacement law is $\lambda_m T = b$ i.e. $T \propto \frac{1}{\lambda_m}$

Here, λ_m becomes half, so temperature doubles.

Also $e = \sigma T^4$

$$\Rightarrow \quad \frac{e_1}{e_2} = \left(\frac{T_1}{T_2}\right)^4$$

$$\Rightarrow \quad e_2 = \left(\frac{T_1}{T_2}\right)^4 \cdot e_1 = (2)^4 \times 16 = 16 \times 16 = 256 \text{ J m}^{-2} \text{ s}^{-1}$$

ILLUSTRATION 2.25

If the filament of a 100 W bulb has an area 0.25 cm^2 and behaves as a perfect black body. Find the wavelength corresponding to the maximum in its energy distribution. Given that Stefan's constant is $\rho = 5.67 \times 10^{-8}$ J/m^2 s K^4.

Sol. In the bulb filament given, the energy radiated per second per m^2 of its surface area is given as

$$E = \frac{P}{A} = \frac{100}{0.25 \times 10^{-4}} = 4 \times 10^6 \text{ J/s-m}^2$$

If T is the temperature of the filament then according to Stefan's law, we have

$$E = \sigma T^4$$

or $\quad 4 \times 10^6 = 5.67 \times 10^{-8} \times T^4$

or $\quad T^4 = \dfrac{4 \times 10^6}{5.67 \times 10^{-8}} = 7.055 \times 10^{13}$

or $\quad T = [7.055 \times 10^{13}]^{1/4} = 2898.14$ K

If the filament radiates the maximum energy at a wavelength λ_m, from Wien's displacement law, we have

$$\lambda_m T = b$$

or $\quad \lambda_m = \dfrac{b}{T} = \dfrac{2.89 \times 10^{-3}}{2898.14} = 9971.9$ Å

ILLUSTRATION 2.26

Two bodies A and B have thermal emissivity of 0.01 and 0.81, respectively. The outer surface areas of the two bodies are the same. The two bodies emit total radiant power at the same rate. The wavelength of B corresponding to maximum spectral radiancy in the radiation from B is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from A by 1.00 μm. The temperature of A is 5802 K. Find temperature of B and wavelength corresponding to maximum spectral radiancy.

Sol. If T_A and T_B are the temperature of the bodies A and B, respectively, then $\varepsilon_A \sigma T_A^4 = \varepsilon_B \sigma T_B^4$

$$\therefore \quad T_B = \left(\frac{e_A}{e_B}\right) T_A = \left(\frac{0.01}{0.81}\right)^{1/4} \times 5802 = 1934 \text{ K}$$

By Wien's displacement law, we have $\lambda_A T_A = \lambda_B T_B$

or $\quad \lambda_A \times 5802 = \lambda_B \times 1934$

or $\quad \lambda_A = \dfrac{\lambda_B}{3}$ ...(i)

It is also given $\lambda_B - \lambda_A = 10^{-6}$ ...(ii)

Solving Eqs. (i) and (ii), we get $\lambda_B = 1.5 \times 10^{-6}$ m = 1.5 μm

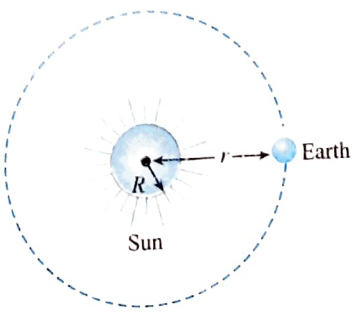
TEMPERATURE OF THE SUN AND SOLAR CONSTANT

If R is the radius of the sun and T is its temperature, then the energy emitted by the sun per second through radiation in accordance with Stefan's law will be given by

$$P = A\sigma T^4 = 4\pi R^2 \sigma T^4$$

By the time it reaches the earth, this energy will spread over a sphere of radius r (= average distance between sun and earth); so the intensity of solar radiation at the surface of earth (called solar constant S) will be given by

$$S = \frac{P}{4\pi r^2} = \frac{4\pi R^2 \sigma T^4}{4\pi r^2}$$



i.e., $$T=\left[\left(\frac{r}{R}\right)^2\frac{S}{\sigma}\right]^{1/4}$$

$$=\left[\left(\frac{1.5\times10^8}{7\times10^5}\right)^2\times\frac{1.4\times10^3}{5.67\times10^{-8}}\right]^{1/4}\simeq 5800\ \text{K}$$

As $r=1.5\times10^8$ km, $R=7\times10^5$ km,

$$S=2\frac{\text{cal}}{\text{cm}^2\text{-min}}=1.4\frac{\text{kW}}{\text{m}^2}\ \text{and}\ \sigma=5.67\times10^{-8}\frac{\text{W}}{\text{m}^2\text{-K}^4}$$

This result is in good agreement with the experimental value of temperature of sun, i.e., 6000 K.

ILLUSTRATION 2.27

The intensity of solar radiation just outside the earth's atmosphere is measured to be 1.4 kW/m^2. If the radius of the sun 7×10^8 m, while the earth–sun distance is 150×10^6 km, then find

(a) the intensity of solar radiation at the surface of the sun,

(b) the temperature at the surface of the sun assuming it to be a black body,

(c) the most probable wavelength in solar radiation.

Sol. Assuming the sun to be a 'black body' at a temperature T_0, we can write, W = intensity of solar radiation on the sun's surface = σT_0^4, where σ is the Stefan–Boltzmann constant.

(a) The radiation emitted from the solar surface per unit time is spread over the surface of a sphere having a radius equal to earth–sun distance where it is received on the earth (just outside the atmosphere)

$$W\times4\pi R_S^2=I_0\times4\pi D_{SE}^2$$

where D_{SE} is the distance between the sun and the earth, and I_0 is the intensity outside the earth's atmosphere.

$$I_0=W\times\left(\frac{R_S}{D_{SE}}\right)^2$$

Now, $R_S=7\times10^8$ m, $D_{SE}=150\times10^9$ m

and $I_0=1.4\times10^3$ W/m^2

$$\therefore\quad 1.4\times10^3=W\times\left(\frac{7\times10^8}{150\times10^6}\right)^2=W\times\frac{49}{225}\times10^{-4}$$

or $$W=6.4\times10^7\ \text{W/m}^2$$

(b) Assuming the sun to be a black body,

$$6.4\times10^7=\sigma T_0^4=(5.67\times10^{-8})T_0^4$$

$$\therefore\quad T_0^4=\frac{6.4}{5.67}\times10^{15}$$

or $$T_0\approx0.58\times10^4\ \text{K}=5800\ \text{K}$$

(c) Using Wien's displacement law,

$$\lambda_{mp}T_0=0.29\ \text{cm-K}=2.9\times10^{-3}\ \text{m-K}$$

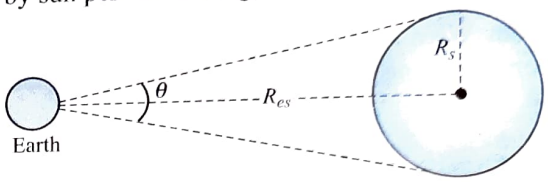
or $$\lambda_{mp}=\frac{2.0\times10^{-3}}{5800}=5\times10^{-7}\ \text{m}=5000\ \text{Å}$$

Note: λ_{mp} is also referred to as λ_{max}.

ILLUSTRATION 2.28

The earth receives solar energy at the rate of 2 cal cm^2 per min. Assuming the radiation to be black body in character, estimate the surface temperature of the sun. Given that $s=5.67\times10^{-8}$ W/m^2/K^4 and angular diameter of the sun = 32 min of arc.

Sol. Let surface temperature of sun be T_s. Then total energy radiated by sun per second is given as $E=\sigma T_s^4\cdot(4\pi R_s^2)$



Energy received by the earth per second per square metre is given as

$$\frac{dQ}{dt}=\frac{E}{4\pi R_{es}^2}=\sigma T_s^4\left(\frac{R_s}{R_{es}}\right)^2 \quad ...(i)$$

It is given that angular diameter of sun as observed from earth is 32 min of arc. Thus, we have

$$\frac{R_s}{R_{es}}=\frac{\theta}{2}=\frac{1}{2}\times\frac{32}{60}=4.655\times10^{-3}$$

and it is given that $$\frac{dQ}{dt}=\frac{2\times4.2\times10^4}{60}\ \text{J/s-m}^2 \quad ...(ii)$$

Now from Eqs. (i) and (ii), we have

$$5.67\times10^{-8}\times T_s^4\times[4.655\times10^{-3}]^2=\frac{2\times4.2\times10^4}{60}$$

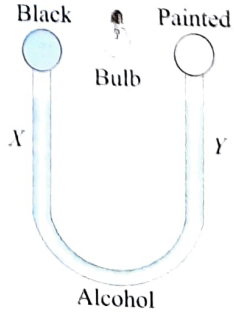
or $$T_s^4=\frac{2\times4.2\times10^4}{60\times5.67\times10^{-8}\times(4.655\times10^{-3})^2}$$

or $$T_s^4=1.14\times10^{15}$$

or $$T_s=5810.67\ \text{K}$$

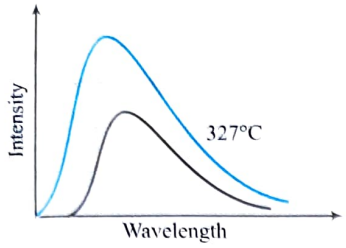
CONCEPT APPLICATION EXERCISE 2.2

1. The following figure shows two air-filled bulbs connected by a U-tube partly filled with alcohol. What happens to the levels of alcohol in the limbs X and Y when an electric bulb placed midway between the bulbs is lighted?



2. A calorimeter of mass 0.2 kg and specific heat 900 J/kg-K contains 0.5 kg of a liquid of specific heat 2400J /kg-K. Its temperature falls from 60°C to 55°C in one minute. Find the rate of cooling.

3. The initial temperature of a body is 80°C. If its temperature falls to 64°C in 5 minutes and to 52°C in 10 minutes. Find the temperature of surrounding.
4. The spectrum of a black body at two temperatures 27°C and 327°C is shown in the figure. Let A_1 and A_2 be the areas under the two curves, respectively. Find the value of A_2/A_1.



5. In Newton's law of cooling in the form $\frac{d\Delta\theta}{dt} = -k\Delta\theta$ what factors does the constant k depend upon? What are the dimensions of k?
6. An indirectly heated filament is radiating maximum energy of wavelength 2.16×10^{-5} cm. Find the net amount of heat energy lost per second per unit area. The temperature of the surrounding air is 13°C. Given $b = 0.288$ cm-K. $\sigma = 5.77 \times 10^{-5}$ erg/s-cm^2-K^4.
7. A sphere, a cube and a thin circular pate are heated to the same temperature. If they are made of same material and have equal masses, determine which of these three objects cools the fastest and which one cools the slowest?
8. A cube and a sphere of equal edge and radius, made of the same substance are allowed to cool under identical conditions. Determine which of the two will cool at a faster rate.
9. A spherical ball of radius 1 cm coated with a metal having emissivity 0.3 is maintained at 1000 K temperature and suspended in a vacuum chamber whose walls are maintained at 300 K temperature. Find rate at which electrical energy is to be supplied to the ball to keep its temperature constant.
10. A body emits maximum energy at 4253 Å and the same body at some other temperature emits maximum energy at 2342 Å. Find the ratio of the maximum energy radiated by the body in a short wavelength range.
11. A black body at 1500 K emits maximum energy of wavelength 20000 Å. If sun emits maximum energy of wavelength 5500 Å, what would be the temperature of sun?

ANSWERS

1. The level of limb X falls while the level of limb Y will rises
2. 115 J sec^{-1} 3. 94°C 4. $\frac{16}{1}$ 5. $[T^{-1}]$
6. 1.824×10^{12} erg s^{-1} cm^{-2}
7. The circular plate cools fastest and sphere cools slowest.
8. Cube 9. 21.1 W 10. 0.0506 11. 5454.54 K

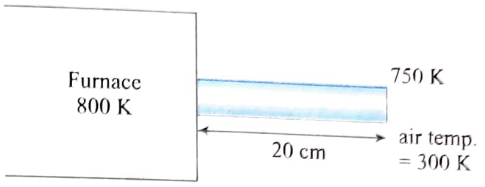
Solved Examples

EXAMPLE 2.1

One end of a rod of length 20 cm is inserted in a furnace at 800 K. The sides of the rod are covered with an insulating material and the other end emits radiation like a blackbody. The temperature of this end is 750 K in the steady state. The temperature of the surrounding air is 300 K. Assuming radiation to be the only important mode of energy transfer between the surrounding and the open end of the rod, find the thermal conductivity of the rod. Stefan's constants, $\sigma = 6.0 \times 10^{-8}$ W/m^2-K^4.

Sol. Quantity of heat flowing through the rod in steady state,

$$\frac{dQ}{dt} = \frac{K \cdot A \cdot d\theta}{x} \quad ...(i)$$



Quantity of heat radiated from the end of the rod in steady state,

$$\frac{dQ}{dt} = A\sigma(T^4 - T_o^4) \quad ...(ii)$$

From Eqs. (i) and (ii), $\frac{K \cdot d\theta}{x} = \sigma(T^4 - T_o^4)$

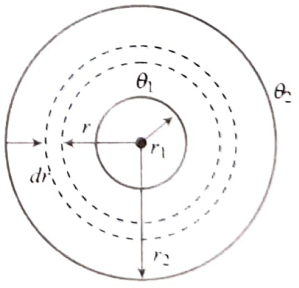
or $\frac{K \times 50}{0.2} = 6.0 \times 10^{-8}[(7.5)^4 - (3)^4] \times 10^8$

or $K = 74$ W/m-K.

EXAMPLE 2.2

Find temperature as a function of radius r in case of spherical shell. Inner and outer surfaces' temperatures are fixed at θ_1 and θ_2, respectively.

Sol. Consider a spherical shell of inner radius r_1 and outer radius r_2, maintained at temperatures θ_1 and θ_2, respectively $(\theta_2 < \theta_1)$.



Consider an elementary spherical shell of thickness dr at a temperature difference $d\theta$.

Rate of radial flow of heat in steady state, $H = \frac{dQ}{dt} = -K(4\pi r^2)\frac{d\theta}{dr}$

or $\int_{r_1}^{r_2} \frac{dr}{r^2} = -\frac{K4\pi}{H}\int_{\theta_1}^{\theta_2} d\theta$

or $\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{K.4\pi}{H}[\theta_1 - \theta_2]$

Rate of flow, $H = \frac{dQ}{dt} = \frac{K4\pi(\theta_1 - \theta_2)}{\left[\frac{1}{r_1} - \frac{1}{r_2}\right]} = \frac{4\pi K r_1 r_2(\theta_1 - \theta_2)}{(r_2 - r_1)}$

Consider the temperature of the layer as θ at a distance r from the centre.

$$\int_{r_1}^{r}\frac{dr}{r^2}=-\frac{K4\pi}{H}\int_{\theta_1}^{\theta}d\theta$$

or $$\left[\frac{1}{r_1}-\frac{1}{r}\right]=\frac{K.4\pi}{H}[\theta_1-\theta] \quad ...(i)$$

and $$\int_{r_1}^{r_2}\frac{dr}{r^2}=-\frac{K.4\pi}{H}\int_{\theta_1}^{\theta_2}d\theta$$

or $$\left[\frac{1}{r_1}-\frac{1}{r_2}\right]=\frac{K.4\pi}{H}[\theta_1-\theta_2] \quad ...(ii)$$

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{\frac{1}{r_1}-\frac{1}{r}}{\frac{1}{r_1}-\frac{1}{r_2}}=\frac{\theta_1-\theta}{\theta_1-\theta_2}$$

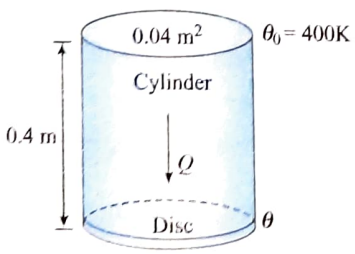
$$\Rightarrow \quad \theta=\theta_1-\frac{\frac{1}{r_1}-\frac{1}{r}}{\frac{1}{r_1}-\frac{1}{r_2}}(\theta_1-\theta_2)$$

EXAMPLE 2.3

A cylindrical block of length 0.4 m and area of cross-section 0.04 m^2 is placed coaxially on a thin metallic disc of mass 0.4 kg and same cross-section. The upper face of the cylinder is maintained at a constant temperature of 400 K and the initial temperature of the disc is 300 K. If the thermal conductivity of the material of the cylinder is 10 w/mK and the specific heat of the material of the disc is 600 J/kg K, how long will it take for the temperature of the disc to reach to 350 K? Assume for purpose of calculation, the thermal conductivity of the disc to be very high and the system to be thermally insulated except for the upper and lower faces of the cylinder.

Sol. If the temperature of disc at time t is θ, the heat conducted by the cylinder to the disc in time dt,

$$dQ=KA\frac{(\theta_0-\theta)}{L}dt \quad ...(i)$$



This heat increases the temperature of disc of specific heat c and mass m from θ to $\theta+d\theta$

So, $dQ=mc\,d\theta$ ...(ii)

So, from Eqs. (i) and (ii),

$$KA\frac{(\theta_0-\theta)}{L}dt=mc\,d\theta \quad \text{or} \quad dt=\frac{mcL}{KA}\frac{d\theta}{(\theta_0-\theta)}$$

Integrating the above equation between limit, $\theta=\theta_1$ to $\theta=\theta_2$,

$$t=\frac{mcL}{KA}\left[-\ln(\theta_0-\theta)\right]_{\theta_1}^{\theta_2}=\frac{mcL}{KA}\ln\left[\frac{\theta_0-\theta_1}{\theta_0-\theta_2}\right]$$

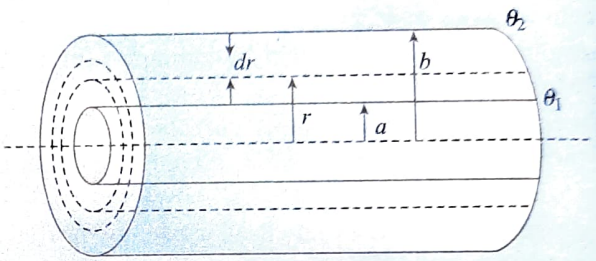
Substituting the given data, we get

$$t=\frac{0.4\times600\times0.4}{10\times0.04}\ln\left[\frac{400-300}{400-350}\right]=240\ln2$$

$t=240\times0.7=168$ sec $=2$ min 48 sec [as $\ln2\approx0.7$]

EXAMPLE 2.4

A 2 m long wire of resistance 4 ohm and diameter 0.64 mm is coated with plastic insulation of thickness 0.06 mm. When a current of 5 ampere flows through the wire, find the temperature difference across the insulation in steady state if [$K=0.16\times10^{-2}$cal/cm-°C sec].



Sol. Consider a concentric cylindrical shell of radius r and thickness dr as shown in the figure. The radial rate of flow of heat through this shell in steady state will be

$$H=\frac{dQ}{dt}=-KA\frac{d\theta}{dr}$$

Negative sign is used as with increase in r, θ decreases.
Now as for cylindrical shell, $A=2\pi rL$.

Then $$H=-2\pi rLK\frac{d\theta}{dr}$$

or $$\int_a^b\frac{dr}{r}=-\frac{-2\pi LK}{H}\int_{\theta_1}^{\theta_2}d\theta$$

Which on integration and simplification gives

$$H=\frac{dQ}{dt}=-\frac{-2\pi LK(\theta_1-\theta_2)}{\ln(b/a)} \quad ...(i)$$

Hence, $$H=\frac{I^2R}{4.2}=\frac{(5)^2\times4}{4.2}=24\frac{\text{cal}}{\text{sec}}$$

$L=2\,\text{m}=200\,\text{cm}$

$r_1=(0.64/2)$ mm $=0.032$ cm

and $r_2=r_1+d=0.032+0.006=0.038$ cm

So, $$(\theta_1-\theta_2)=\frac{24\times\ln(38/32)}{2\times3.14\times200\times0.16\times10^{-2}}$$

$$=\frac{24\times2.3026\left[\log_{10}38-\log_{10}32\right]}{3.14\times0.64}$$

$$=\frac{55\times[1.57-1.50]}{2}=2°\text{C}$$

EXAMPLE 2.5

Two bodies A and B have thermal emissivities of 0.01 and 0.81, respectively. The outer surface areas of the two bodies are same. The two bodies emit total radiant power at the same rate. The wavelength λ_B corresponding to maximum spectral radiancy in the radiation from B is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from A by 1.00 μm. If the temperature of A is 5802 K, calculate **(a)** temperature of B and **(b)** wavelength λ_B.

Sol.

(a) According to Stefan's law, the power radiated by a body is given by $-P = e\sigma AT^4$.

According to the given problem, $P_A = P_B$ with $A_A = A_B$.

So, $e_A T_A^4 = e_B T_B^4$

or $0.01 \times (5802)^4 = 0.8(T_B)^4$

or $T_B = (1/3)(5802) = 1934$ K

(b) According to Wien's displacement law, $\lambda_A T_A = \lambda_B T_B$

or $\lambda_B = (5802/1934)\lambda_A$

or $\lambda_B = 3\lambda_A$ and also, $\lambda_B - \lambda_A = 1$ μm (given)

or $\lambda_B - (1/3)\lambda_B = 1$ μm

or $\lambda_B = 1.5$ μm

EXAMPLE 2.6

According to Stefan's law of radiation, a black body radiates energy σT^4 from its unit surface area every second where T is the surface temperature of the black body and $\sigma = 5.67 \times 10^{-8}$ W/m^{2}K^4 is known as Stefan's constant.

A nuclear weapon may be thought of as a ball of radius 0.5 m. When detonated, it reaches temperature of 10^6 K and can be treated as a black body.

(a) Estimate the power it radiates.

(b) If surrounding has water at 30°C, how much water can 10% of the energy produced evaporate in 1 s?

[$S_W = 4186.0$ J/kg K and $L_V = 22.6 \times 10^5$ J/kg]

(c) If all this energy U is in the form of radiation, corresponding momentum is $p = U/c$. How much momentum per unit time does it impart on unit area at a distance of 1 km?

Sol. Given, $\sigma = 5.67 \times 10^{-8}$ W/m^{2}kg

Radius $= R = 0.5$ m, $T = 10^6$ K

(a) Power radiated by Stefan's law,

$$P = \sigma AT^4 = \sigma(4\pi R^2)T^4$$
$$= (5.67 \times 10^{-8}) \times 4 \times (3.14) \times (0.5)^2 \times (10^6)^4$$
$$= 1.78 \times 10^{17} \text{ J/s} = 1.8 \times 10^{17} \text{ J/s}$$

(b) Energy available per second,

$$U = 1.8 \times 10^{17} \text{ J/s} = 1.8 \times 10^{16} \text{ J/s}$$

Actual energy required to evaporate water

$$= 10\% \text{ of } 1.8 \times 10^{17} \text{ J/s} = 18 \times 10^{16} \text{ J/s}$$

Energy used per second to raise the temperature of m kg of water from 30°C to 100°C and then into vapour at 100°C

$$= ms_w \Delta\theta + mL_V$$
$$= m \times 4186 \times (100 - 30) + m \times 22.6 \times 10^5$$
$$= 2.93 \times 10^5 m + 22.6 \times 10^5 m = 25.53 \times 10^5 \text{ m J/s}$$

As per question, $25.53 \times 10^5 \text{ m} = 1.8 \times 10^{16}$

or $$m = \frac{1.8 \times 10^{16}}{25.33 \times 10^5} = 7.0 \times 10^9 \text{ kg}$$

(c) Momentum per unit time,

$$p = \frac{U}{c} = \frac{1.8 \times 10^{17}}{3 \times 10^8} = 6 \times 10^8 \text{ kg-m/s}^2$$

$$\begin{bmatrix} p = \text{momentum} \\ U = \text{energy} \\ c = \text{velocity of light} \end{bmatrix}$$

Momentum per unit time per unit area,

$$= \frac{p}{4\pi R^2} = \frac{6 \times 10^8}{4 \times 3.14 \times (10^3)^2}$$
$$= 47.7 \text{ N/m}^2 \qquad [4\pi R^2 = \text{surface area}]$$

EXAMPLE 2.7

A body cools in a cold surrounding obeying Newton's law of cooling. It is found that from $t = 0$ to $t = t_1$, it loses half the total heat. In what time, from $t = t_1$ will it loose one fourth the total heat?

Sol. Let the initial temperature of the body be T_1 and that of the surrounding be T_0. If 'c' be the heat capacity of the body, then, total heat lost (after a long time) by the body $= c(T_1 - T_0)$. Now, from Newton's law of cooling,

$$\frac{dQ}{dt} = -k(T - T_0)$$

or $$\frac{cdT}{dt} = -k(T - T_0) \quad \text{or} \quad \frac{cdT}{T - T_0} = -kdt$$

or $$\int_{T_1}^{T} \frac{cdT}{T - T_0} = \int_0^t -kdt$$

or $$c \ln\left[\frac{T - T_0}{T_1 - T_0}\right] = -kt \qquad \ldots(i)$$

Since heat lost by the body $= c(T_1 - T)$, It is given that at $t = t_1$

$$c(T_1 - T) = \frac{c}{2}(T_1 - T_0) \quad \text{or} \quad T = \frac{T_1 + T_0}{2}$$

Now, Eq. (i) becomes $$c \ln\left[\frac{\frac{T_1 + T_0}{2} - T_0}{T_1 - T_0}\right] = -kt_1$$

or $$kt_1 = c \ln(2) \qquad \ldots(ii)$$

Now, for loss of one fourth the total heat lost by the body the temperature (T) will be given by

$$c\left[\frac{T_1+T_0}{2}-T\right]=\frac{c}{4}(T_1-T_0) \quad \text{or} \quad T=\frac{T_1+3T_0}{4}$$

Now, Eq. (i) becomes $c \ln\left[\frac{\frac{T_1+3T_0}{4}-T_0}{T_1-T_0}\right]=-kt$

or $kt = c\ln(4)$ ...(iii)

Dividing Eq. (iii) by (ii), $\frac{t}{t_1}=\frac{\ln 4}{\ln 2}$ or $(t-t_1)=t_1$

$\therefore$ Required time $=(t-t_1)=t_1$

EXAMPLE 2.8

A cylindrical rod of heat capacity 120 J/K in a room temperature 27°C is heated internally by heater of power 250 W. The steady state temperature attained by the rod is 37°C. Find the following:

(a) The initial rate of increase in temperature

(b) The steady state rate of emission of radiant heat. If the heater is switched off, find

(c) The initial rate of decrease in temperature

(d) The rate of decrease in temperature of the cylinder when its temperature falls to 31°C and

(e) The maximum amount of heat lost by the cylinder.

Sol. Initially, when the rod is in thermal equilibrium with the room at 27°C, from Prevost's theory of exchanges, the rate of total radiant heat emitted by the rod to the surrounding is same as the rate of total radiant energy absorbed by it from the surrounding. So, as soon as the heater is switched on, the entire heat provided by it is completely absorbed by the rod,

$$c\frac{dT}{dt}=P=\text{Power of heater}$$

$$120\ \frac{\text{J}}{\text{K}}\left(\frac{dT}{dt}\right)=250\ \frac{\text{J}}{\text{s}}$$

$$\frac{dT}{dt}=2.08\ \text{K/s}$$

(a) Thus, the initial rate of increase in temperature of the rod is 2.08 K/s.

(b) In the steady state, the absorption of the radiant energy by the rod ceases, which evidently implies that the heat energy liberated by the heater per second is emitted by the rod per second.

So, the steady state rate of emission of radiant heat from the rod = power of heater = 250 W.

(c) Immediately after the heater is switched off, the rod continues emitting heat energy at the same rate, as during the steady state, by virtue of its temperature. However, the heater stops supplying the heat to the rod completely. As a result, the temperature of the rod starts falling.

From $-c\frac{dT}{dt}=\frac{dQ}{dt}$, we have $\frac{-dT}{dt}=\frac{250\ \text{W}}{120\ \text{J/K}}=2.08\ \text{K/s}$

(d) Since temperature difference between the rod and the room is not too large during the process, so Newton's law of cooling becomes valid.

$$\frac{-dT}{dt}=k(T-T_0)$$

Initially, $\left(\frac{-dT}{dt}\right)_i=k(37-27)^\circ\text{C}=10k\ ^\circ\text{C}$ ...(i)

Finally, at $T = 31^\circ$C

$$\left(\frac{-dT}{dt}\right)_f=k(31-27)^\circ\text{C}=4k\ ^\circ\text{C}$$ ...(ii)

From Eqs. (i) and (ii), we get

$$\left(\frac{dT}{dt}\right)_f=\frac{4}{10}(2.08\ \text{K/s}) \quad \left[\because \left(\frac{-dT}{dt}\right)_i=2.08\ \text{K/s}\right]$$

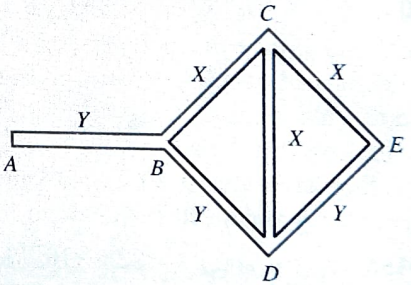
$$=0.83\ \text{K/s}$$

(e) Since a body cannot be cooled below the temperature of the surroundings, so maximum fall in temperature of the rod can be 10°C. So, the maximum heat that the rod can lose after the heater is switched off, by the time its temperature falls to that of the room, will be $c(\Delta T)_{\max}$

$$=120\frac{\text{J}}{\text{K}}(10\ \text{K})=1200\ \text{J}$$

EXAMPLE 2.9

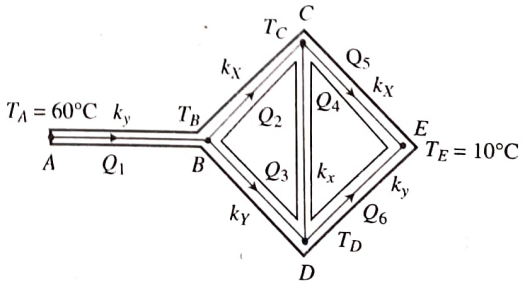
Three rods of material X and three rods of material Y are connected as shown in the given figure. All the rods are of identical lengths and cross-sectional areas. If the rod end A is maintained at 60°C and the junction E at 10°C, calculate the temperature of junctions B, C and D. the thermal conductivity of X is 9.2×10^{-2} kcal/m/s/°C and that of Y is 4.6×10^{-2} kcal/m/s/°C.



Sol. Let k_x and k_y be the thermal conductivities of X and Y respectively, and let T_B, T_C and T_D be the temperatures of junctions B, C and D, respectively. Given $T_A = 60^\circ$C and $T_E = 10^\circ$C.

In order to solve this problem, we will apply the principle that, in the steady state, the rate at which heat enters a junction is equal to the rate at which heat leaves that junction. In the figure shown below, the direction of flow of heat is given by arrows.

For junction B, we have



$$\frac{k_y A(T_A - T_B)}{d} = \frac{k_x A(T_B - T_C)}{d} + \frac{k_y A(T_B - T_D)}{d}$$

or $k_y(T_A - T_B) = k_x(T_B - T_C) + k_y(T_B - T_D)$

or $(T_A - T_B) = \frac{k_x}{k_y}(T_B - T_C) + (T_B - T_D)$

Given $\frac{k_x}{k_y} = \frac{9.2\times10^{-2}}{4.6\times10^{-2}} = 2$ and $T_A = 60°C$. Therefore

$(60 - T_B) = 2(T_B - T_C) + (T_B - T_D)$

or $4T_B - 2T_C - T_D = 60$ ...(i)

For junction C, we have

$$\frac{k_x A(T_B - T_C)}{d} = \frac{k_x A(T_C - T_D)}{d} + \frac{K_x A(T_C - T_E)}{d}$$

Solving we get $-T_B + 3T_C - T_D = 10$ [At $T_E = 10°C$] ...(ii)

For junction D, we have

$$\frac{k_y A(T_B - T_D)}{d} + \frac{k_x A(T_C - T_D)}{d} = \frac{k_y A(T_D - T_E)}{d}$$

or $(T_B - T_D) + \frac{k_x}{k_y} = 2$ and $T_E = 10°C$.

In this equation, we have $(T_B - T_D) + 2(T_C - T_D) = (T_D - 10)$

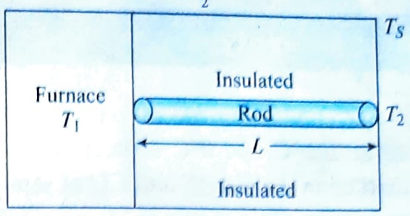
$T_B + 2T_C - 4T_D = -10$ ...(iii)

Solving Eqs. (i), (ii) and (iii), we get

$T_B = 30°C$ and $T_C = T_D = 20°C$

EXAMPLE 2.10

One end of a rod of length L and cross-sectional area A is kept in a furnace at temperature T_1. The other end of the rod is kept at a temperature T_2. The thermal conductivity of the material of the rod is K and emissivity of the rod is e. It is given that $T_2 = T_S + \Delta T$, where $\Delta T << T_S$, T_S is the temperature of the surroundings. If $\Delta T \propto (T_1 - T_2)$, find the proportional constant. Consider that heat is lost only by radiation at the end where the temperature of the rod is T_2.



Sol. Rate of heat conduction through rod = rate of the heat lost from right end of the rod.

$$\frac{KA(T_1 - T_2)}{L} = eA\sigma(T_2^4 - T_S^4) \quad ...(i)$$

Given that $T_2 = T_S + \Delta T$

$$T_2^4 = (T_S + \Delta T)^4 = T_S^4\left(1 + \frac{\Delta T}{T_S}\right)^4$$

Using binomial expansion, we have

$$T_2^4 = T_S^4\left(1 + 4\frac{\Delta T}{T_S}\right) \quad (\text{as } \Delta T << T_S)$$

$$T_2^4 - T_S^4 = 4(\Delta T)(T_S^3)$$

Substituting in Eq. (i), we have $\frac{K(T_1 - T_S - \Delta T)}{L} = 4e\sigma T_S^3 \Delta T$

$$\Rightarrow \quad \frac{K(T_1 - T_S)}{L} = \left(4e\sigma T_S^3 + \frac{K}{L}\right)\Delta T$$

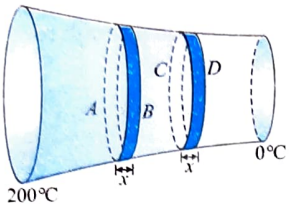
$$\Rightarrow \quad \Delta T = \frac{K(T_1 - T_S)}{(4e\sigma L T_S^3 + K)}$$

Comparing with the given relation, proportionality constant $= \frac{K}{4e\sigma L T_S^3 + K}$

Exercises

Single Correct Answer Type

1. Two ends of a conducting rod of varying cross-section are maintained at 200°C and 0°C respectively. There are two sections marked in the rod AB and CD of same thickness. In steady state:



(1) Temperature difference across AB and CD is equal

(2) Temperature difference across AB is greater than that of across CD

(3) Temperature difference across AB is less than that of across CD

(4) Temperature difference may be equal or different depending on the thermal conductivity of the rod

2. If the temperature of the sun becomes twice its present temperature, then

(1) radiated energy would be predominantly in infrared

(2) radiated energy would be predominantly in ultraviolet

(3) radiated energy would be predominantly in X-ray region

(4) radiated energy would become twice the present radiated energy

3. Two identical objects A and B are at temperatures T_A and T_B, respectively. Both objects are placed in a room with perfectly absorbing walls maintained at temperatures T ($T_A > T > T_B$). The objects A and B attain temperature T eventually. Which one of the following is correct statement?

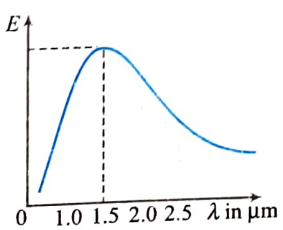
(1) A only emits radiations while B only absorbs them until both attain temperature T.

(2) A loses more radiations than it absorbs while B absorbs more radiations that it emits until temperature T is attained.

(3) Both A and B only absorb radiations until they attain temperature T.

(4) Both A and B only emit radiations until they attain temperature T.

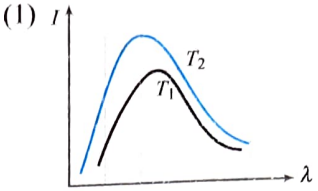
4. In the figure, the distribution of energy density of the radiation emitted by a black body at a given temperature is shown. The possible temperature of the black body is

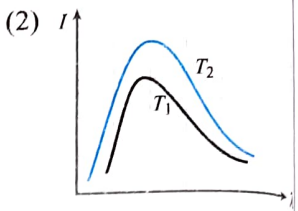


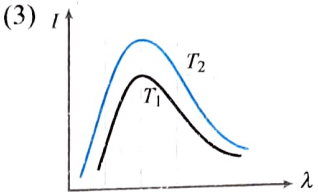
(1) 1500 K (2) 2000 K

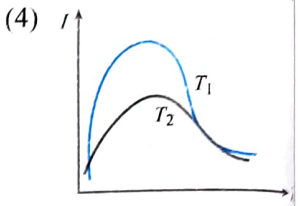
(3) 2500 K (4) 3000 K

5. Shown below are the black body radiation curves temperatures T_1 and T_2 ($T_2 > T_1$). Which of the followi plots is correct?

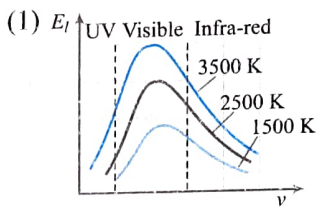
(1) 

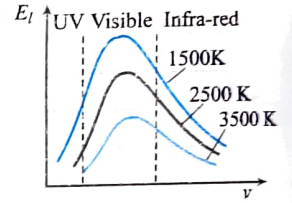
(2) 

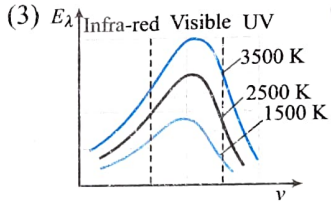
(3) 

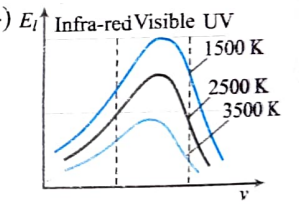
(4) 

6. Which of the following graphs shows the correct variati in intensity of heat radiations by black body and frequenc at a fixed temperature?

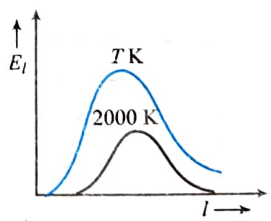
(1) 

(2) 

(3) 

(4) 

7. The adjoining diagram shows the spectral energy densi distribution E_λ of a black body at two different temperature If the areas under the curves are in the ratio 16 : 1, the valu of temperature T is



(1) 32,000 K (2) 16,000 K

(3) 8,000 K (4) 4,000 K

8. The rates of cooling of two different liquids put in exact similar calorimeters and kept in identical surroundings a the same if

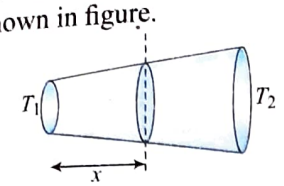
(1) the masses of the liquids are equal

(2) equal masses of the liquids at the same temperature a taken

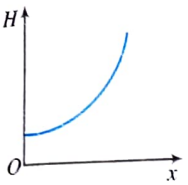
(3) different volumes of the liquids at the same temperatu are taken

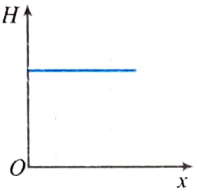
(4) equal volumes of the liquids at the same temperature a taken

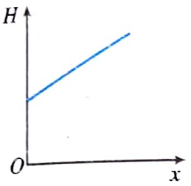
9. Radius of a conductor increases uniformly from left end to right end as shown in figure.

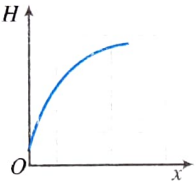


Material of the conductor is isotropic and its curved surface is thermally isolated from surrounding. Its ends are maintained at temperatures T_1 and T_2 ($T_1 > T_2$). If, in steady state, heat flow rate is equal to H, then which of the following graphs is correct?

(1) 

(2) 

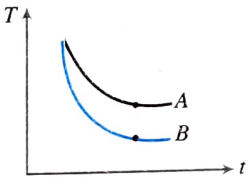
(3) 

(4) 

10. A hot metallic sphere of radius r radiates heat. It's rate of cooling is

(1) independent of r
(2) proportional to r
(3) proportional to r^2
(4) proportional to $1/r$

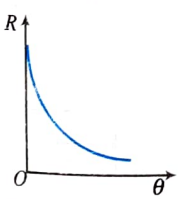
11. Water and turpentine oil (of specific heat less than that of water) are both heated to same temperature. Equal amounts of these placed in identical calorimeters are then left in air.

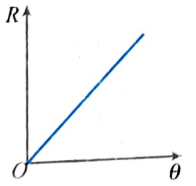


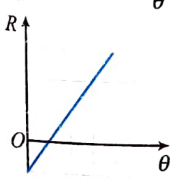
Then

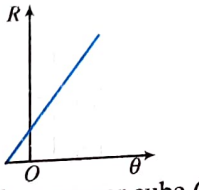
(1) Their cooling curves will be identical.
(2) A and B will represent cooling curves of water and oil, respectively.
(3) B and A will represent cooling curves of water and oil, respectively.
(4) None of these

12. For a small temperature difference between the body and the surroundings, the relation between the rate of heat loss R and the temperature of the body is depicted by

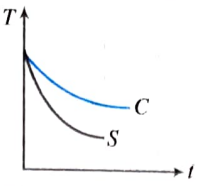
(1) 

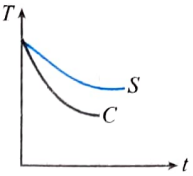
(2) 

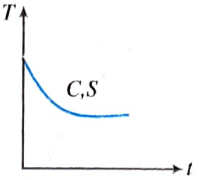
(3) 

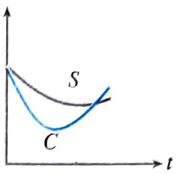
(4) 

13. A hollow copper sphere S and a hollow copper cube C, both of negligibly thick walls of same area, are filled with water at 90°C and allowed to cool in the same environment. The graph that correctly represents their cooling is

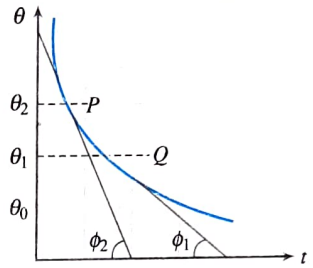
(1) 

(2) 

(3) 

(4) 

14. A body cools in a surrounding which is at a constant temperature of θ_0. Assume that it obeys Newton's law of cooling. Its temperature θ is plotted against time t. Tangents are drawn to the curve at the points $P(\theta = \theta_2)$ and $Q(\theta = \theta_1)$. These tangents meet the time axis at angles of ϕ_2 and ϕ_1, as shown

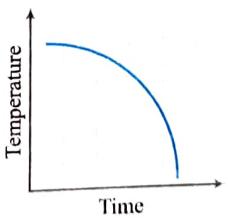


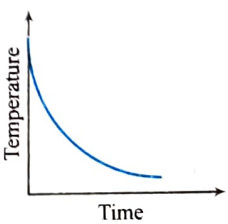
(1) $\dfrac{\tan\phi_2}{\tan\phi_1} = \dfrac{\theta_1 - \theta_0}{\theta_2 - \theta_0}$
(2) $\dfrac{\tan\phi_2}{\tan\phi_1} = \dfrac{\theta_2 - \theta_0}{\theta_1 - \theta_0}$
(3) $\dfrac{\tan\phi_1}{\tan\phi_2} = \dfrac{\theta_1}{\theta_2}$
(4) $\dfrac{\tan\phi_1}{\tan\phi_2} = \dfrac{\theta_1}{\theta_1}$

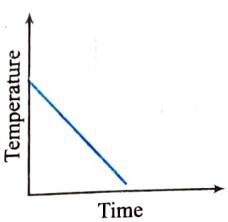
15. A solid sphere and a hollow sphere of the same material and size are heated to the same temperature and allowed to cool in the same surroundings. If the temperature difference between each sphere and its surroundings is T, then

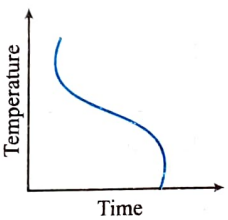
(1) the hollow sphere will cool at a faster rate for all values of T
(2) the solid sphere will cool at a faster rate for all values of T
(3) both spheres will cool at the same rate for all values of T
(4) both spheres will cool at the same rate only for small values of T

16. A block of metal is heated to a temperature much higher than the room temperature and allowed to cool in a room free from air currents. Which of the following curves correctly represents the rate of cooling?

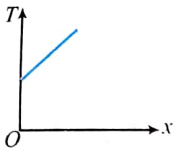
(1) 

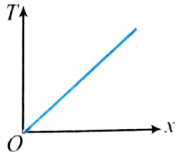
(2) 

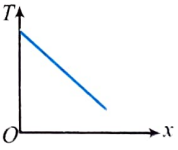
(3) 

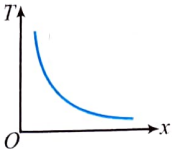
(4) 

17. Heat is flowing through a conductor of length l from $x = 0$ to $x = l$. If its thermal resistance per unit length is uniform, which of the following graphs is correct?

(1) 

(2) 

(3) 

(4) 

18. Two rectangular blocks A and B of different metals have same length and same area of cross-section. They are kept in such a way that their cross-sections touch each other. The temperature at one end of A is 100°C and that of B at the other end is 0°C. If the ratio of their thermal conductivities is 1 : 3, then under steady state, the temperature of the junction in contact will be

(1) 25°C (2) 50°C

(3) 75°C (4) 100°C

19. Two rods of same length and material transfer a given amount of heat in 12 seconds, when they are joined end to end. But when they are joined along their lengths, then they will transfer same heat in same conditions in

(1) 24 s (2) 3 s

(3) 1.5 s (4) 48 s

20. The coefficients of thermal conductivity of copper, mercury and glass are respectively K_c, K_m and K_g such that $K_c > K_m > K_g$. If the same quantity of heat is to flow per second per unit area of each and corresponding temperature gradients are X_c, X_m and X_g, then

(1) $X_c = X_m = X_g$ (2) $X_c > X_m > X_g$

(3) $X_c < X_m < X_g$ (4) $X_m < X_c < X_g$

21. The ends of two rods of different materials with their thermal conductivities, radii of cross-sections and lengths all are in the ratio 1 : 2 are maintained at the same temperature difference. If the rate of flow of heat in the longer rod is 4 cal/sec, then that in the shorter rod in cal/sec will be

(1) 1 (2) 2

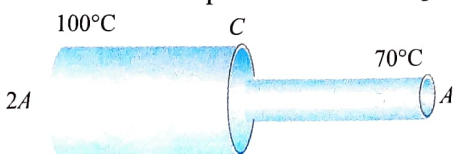
(3) 8 (4) 16

22. Two spheres of different materials one with double the radius and one-fourth wall thickness of the other, are filled with ice. If the time taken for melting complete ice in the larger sphere is 25 minutes and that for smaller one is 16 minutes, the ratio of thermal conductivity of the material of larger sphere to that of the smaller sphere is

(1) 4 : 5 (2) 5 : 4

(3) 25 : 1 (4) 1 : 25

23. A metallic rod of length 2 m has cross-sectional area $2A$ and A as shown in figure. The ends are maintained at temperatures 100°C and 70°C. The temperature at middle point C is



(1) 80°C (2) 85°C

(3) 90°C (4) 95°C

24. In a steady state of thermal conduction, temperatu of the ends A and B of a 20 cm long rod are 100°C a 0°C, respectively. What will be the temperature of t rod at a point at a distance of 6 cm from the end A of t rod?

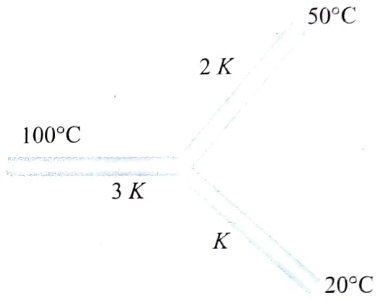
(1) –30°C

(2) 70°C

(3) 5°C

(4) None of the above

25. Three rods of the same dimension have thermal conductiviti $3K$, $2K$ and K. They are arranged as shown in figure giv below, with their ends at 100°C, 50°C and 20°C. Th temperature of their junction is



(1) 60°C (2) 70°C

(3) 50°C (4) 35°C

26. Four identical rods of same material are joined end to en to form a square. If the temperature difference between th ends of a diagonal is 100°C, then the temperature differenc between the ends of other diagonal will be

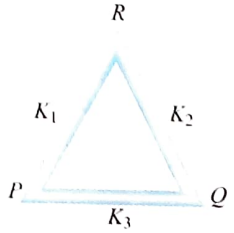
(1) 0°C

(2) $\frac{100}{l}$ °C; where l is the length of each rod

(3) $\frac{100}{2l}$ °C

(4) 100°C

27. Three rods of same dimensions are arranged as shown i figure. They have thermal conductivities K_1, K_2 and K Points P and Q are maintained at different temperatures fo the heat to flow at the same rate along PRQ and PQ.

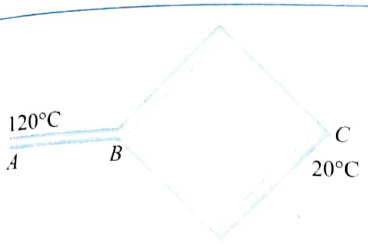


Which of the following options is correct?

(1) $K_3 = \frac{1}{2}(K_1 + K_2)$ (2) $K_3 = K_1 + K_2$

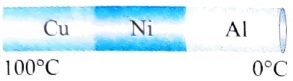
(3) $K_3 = \frac{K_1 K_2}{K_1 + K_2}$ (4) $K_3 = 2(K_1 + K_2)$

28. Five identical rods are joined as shown in figure. Point and C are maintained at temperature 120°C and 20°C respectively. The temperature of junction B will be



(1) 100°C (2) 80°C
(3) 70°C (4) 0°C

29. A composite metallic bar of uniform cross-section is made up of length 25 cm of copper, 10 cm of nickel and 15 cm of aluminium, each part being in perfect thermal contact with the adjoining part. The copper end of the composite rod is maintained at 100°C and the aluminium end at 0°C. The whole rod is covered with belt so that there is no heat loss at the sides. If $K_{Cu} = 2\,K_{Al}$ and $K_{Al} = 3\,K_{Ni}$, then what will be the temperatures of Cu–Ni and Ni–Al junctions, respectively?



(1) 23.33°C and 78/8°C (2) 83.33° and 20°C
(3) 50°C and 30°C (4) 30°C and 50°C

30. A room is maintained at 20°C by a heater of resistance 20 ohm connected to 200 volt mains. The temperature is uniform throughout the room and heat is transmitted through a glass window of area 1 m^2 and thickness 0.2 cm. What will be the temperature outside? Given that thermal conductivity K for glass is 0.2 cal/m/°C/sec and $J = 42$ J/cal
(1) 15.24°C (2) 15.00°C
(3) 24.15°C (4) None of these

31. A solid cube and a solid sphere of the same material have equal surface area. Both are at the same temperature 120°C. Then
(1) both the cube and the sphere cool down at the same rate
(2) the cube cools down faster than the sphere
(3) the sphere cools down faster than the cube
(4) whichever is having more mass will cool down faster

32. A black metal foil is warmed by radiation from a small sphere at temperature T and at a distance d. It is found that the power received by the foil is P. If both the temperature and the distance are doubled, the power received by the foil will be
(1) 16 P (2) 4 P
(3) 2 P (4) P

33. There is a formation of x cm thick layer of snow on water, when the temperature of air is $-\theta$°C (less than freezing point). If the thickness of layer increases from x to y in the time t, then the value of t is given by
(1) $\frac{(x+y)(x-y)\rho L}{2k\theta}$ (2) $\frac{(x-y)\rho L}{2k\theta}$
(3) $\frac{(x+y)(x-y)\rho L}{k\theta}$ (4) $\frac{(x-y)\rho Lk}{2\theta}$

34. The total energy radiated from a black body source is collected for one minute and is used to heat a quantity of water. The temperature of water is found to increase form 20°C to 20.5°C. If the absolute temperature of the black body is doubled and the experiment is repeated with the same quantity of water at 20°C, the temperature of water will be
(1) 21°C (2) 22°C
(3) 24°C (4) 28°C

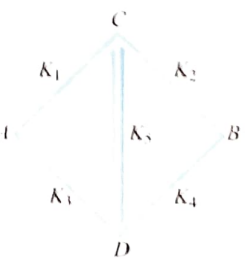
35. A solid copper cube of edge 1 cm is suspended in an evacuated enclosure. Its temperature is found to fall from 100°C to 99°C in 100 s. Another solid copper cube of edge 2 cm, with similar surface nature, is suspended in a similar manner. The time required for this cube to cool from 100°C to 99°C will be, approximately,
(1) 25 s (2) 50 s
(3) 200 s (4) 400 s

36. A cylindrical rod with one end in a steam chamber and the other end in ice results in melting of 0.1 g of ice per second. If the rod is replaced by another with half the length and double the radius of the first and if the thermal conductivity of material of second rod is 1/4th that of the first, the rate at which ice melts in gm/sec will be
(1) 3.2 (2) 1.6
(3) 0.2 (4) 0.1

37. One end of a copper rod of length 1.0 m and cross-sectional area 10^{-3} m^2 is immersed in boiling water and the other end in ice. If the coefficient of thermal conductivity of copper is 92 cal/m-s-°C and the latent heat of ice is 8×10^4 cal/kg, then the amount of ice which will melt in one minute is
(1) 9.2×10^{-3} kg (2) 8×10^{-3} kg
(3) 6.9×10^{-3} kg (4) 5.4×10^{-3} kg

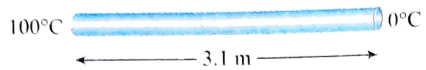
38. An ice box used for keeping eatables cold has a total wall area of 1 m^2 and a wall thickness of 5.0 cm. The thermal conductivity of the ice box is K = 0.01 Joule/metre-°C. It is filled with ice at 0°C along with eatables on a day when the temperature is 30°C. The latent heat of fusion of ice is 334×10^3 Joules/kg. The amount of ice melted in one day is
(1 day = 86,400 s)
(1) 776 g (2) 7760 g
(3) 11520 g (4) 1552 g

39. Five rods of same dimensions are arranged as shown in the figure. They have thermal conductivities K_1, K_2, K_3, K_4 and K_5. When points A and B are maintained at different temperatures, no heat flows through the central rod if



(1) $K_1 = K_4$ and $K_2 = K_3$ (2) $K_1 K_4 = K_2 K_3$
(3) $K_1 K_2 = K_3 K_4$ (4) $\frac{K_1}{K_4} = \frac{K_2}{K_3}$

40. One end of a copper rod of uniform cross-section and of length 3.1 m is kept in contact with ice and the other end with water at 100°C. At what point along its length should a temperature of 200°C be maintained so that in steady state, the mass of ice melting be equal that of the steam produced in the same interval of time? Assume that the whole system is insulated from the surroundings. Latent heat of fusion of ice and vaporisation of water are 80 cal/g and 540 cal/g, respectively



(1) 40 cm from 100°C end
(2) 40 cm from 0°C end
(3) 125 cm from 100°C end
(4) 125 cm from 0°C end

41. A sphere and a cube of same material and same volume are heated upto same temperature and allowed to cool in the same surroundings. The ratio of the amounts of radiations emitted will be

(1) 1 : 1
(2) $\frac{4\pi}{3} : 1$
(3) $\left(\frac{\pi}{6}\right)^{1/3} : 1$
(4) $\frac{1}{2}\left(\frac{4\pi}{3}\right)^{2/3} : 1$

42. If between wavelength λ and $\lambda + d_\lambda$, e_λ, and a_λ are the emissive and absorptive powers of a body and E_λ is the emissive power of a perfectly black body, then according to Kirchhoff's law, which is true?

(1) $e_\lambda = a_\lambda = E_\lambda$
(2) $e_\lambda E_\lambda = a_\lambda$
(3) $e_\lambda = a_\lambda E_\lambda$
(4) $e_\lambda a_\lambda E_\lambda$ = constant

43. The energy spectrum of a black body exhibits maximum energy around a wavelength λ_o. The temperature of the black body is now changed such that the energy is maximum around a wavelength $\frac{3\lambda_o}{4}$. The power radiated by the black body will now increase by a factor of

(1) 256/81
(2) 64/27
(3) 16/9
(4) 4/3

44. Hot water cools from 60°C to 50°C in the first 10 minutes and to 42°C in the next 10 minutes. The temperature of the surrounding is

(1) 5°C
(2) 10°C
(3) 15°C
(4) 20°C

45. A bucket full of hot water cools from 75°C to 70°C in time T_1, from 70°C to 65°C in time T_2 and from 65°C to 60°C in time T_3, then

(1) $T_1 = T_2 = T_3$
(2) $T_1 > T_2 > T_3$
(3) $T_1 < T_2 < T_3$
(4) $T_1 > T_2 < T_3$

46. In Newton's experiment of cooling, the water equivalent of two similar calorimeters is 10 g each. They are filled with 350 g of water and 300 g of a liquid (equal volumes) separately. The time taken by water and liquid to cool from 70°C to 60°C is 3 min and 95 sec, respectively. The specific heat of the liquid will be

(1) 0.3 Cal/g × °C
(2) 0.5 Cal/g × °C
(3) 0.6 Cal/g × °C
(4) 0.8 Cal/g × °C

47. Liquid is filled in a vessel which is kept in a room w temperature 20°C. When the temperature of the liquid 80°C, then it loses heat at the rate of 60 cal/sec. What be the rate of loss of heat when the temperature of the liq is 40°C?

(1) 180 cal/sec
(2) 40 cal/sec
(3) 30 cal/sec
(4) 20 cal/sec

48. A liquid cools down from 70°C to 60°C in 5 minutes. T time taken to cool it from 60°C to 50°C will be

(1) 5 minutes
(2) Lesser than 5 minutes
(3) Greater than 5 minutes
(4) Lesser or greater than 5 minutes depending upon density of the liquid

49. If a metallic sphere gets cooled from 62°C to 50°C in minutes and in the next 10 minutes gets cooled to 42°C, th the temperature of the surroundings is

(1) 30°C
(2) 36°C
(3) 26°C
(4) 20°C

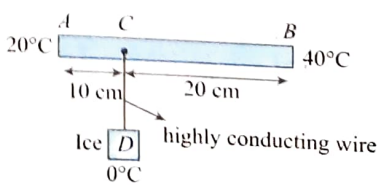
50. There are two spherical balls A and B of the same materi with same surface, but the diameter of A is half that of B. A and B are heated to the same temperature and then allowe to cool, then

(1) rate of cooling is same in both
(2) rate of cooling of A is four times that of B
(3) rate of cooling of A is twice that of B
(4) rate of cooling of A is $\frac{1}{4}$ times that of B

51. There are two thin spheres A and B of the same materi and same thickness. They behave like black bodies. Radiu of A is double that of B and both have same temperatur T. When A and B are kept in a room of temperatur T_0 ($< T$), the ratio of their rates of cooling is (assum negligible heat exchange between A and B)

(1) 2:1
(2) 1:1
(3) 4:1
(4) 8:1

52. As shown in figure, AB is a rod of length 30 cm an area of cross-section 1.0 cm^2 and thermal conductivity 336 units. The ends A and B are maintained at temperatures 20° and 40°C, respectively. A point C of this rod is connecte to a box D, containing ice at 0°C through a highl conducting wire of negligible heat capacity. The rate which ice melts in the box is (assume latent heat of fusio for ice $L_f = 80$ cal/g)



(1) 84 mg/s
(2) 84 g/s
(3) 20 mg/s
(4) 40 mg/s

3. A rod of length l and cross-sectional area A has a variable conductivity given by $K = \alpha T$, where α is a positive constant and T is temperature in kelvin. Two ends of the rod are maintained at temperatures T_1 and T_2 ($T_1 > T_2$). Heat current flowing through the rod will be

(1) $\frac{A\alpha(T_1^2 - T_2^2)}{l}$ (2) $\frac{A\alpha(T_1^2 + T_2^2)}{l}$

(3) $\frac{A\alpha(T_1^2 + T_2^2)}{3l}$ (4) $\frac{A\alpha(T_1^2 - T_2^2)}{2l}$

4. A refrigerator is thermally equivalent to a box of cork board 90 mm thick and 6 m^2 in inner surface area, the thermal conductivity of cork being 0.05 W/mK. The motor of the refrigerator runs 15% of the time while the door is closed. The inside wall of the door, when it is closed, is kept, on an average, 22°C below the temperature of the outside wall. The rate at which heat is taken from the interior wall while the motor is running is

(1) 400 W (2) 500 W

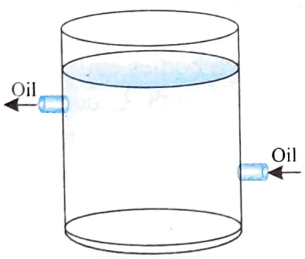
(3) 300 W (4) 250 W

5. Power radiated by a black body is P_0 and the wavelength corresponding to maximum energy is around λ_0. On changing the temperature of the black body, it was observed that the power radiated becomes $\frac{256}{81}P_0$. The shift in wavelength corresponding to the maximum energy will be

(1) $+\frac{\lambda_0}{4}$ (2) $+\frac{\lambda_0}{2}$

(3) $-\frac{\lambda_0}{4}$ (4) $-\frac{\lambda_0}{2}$

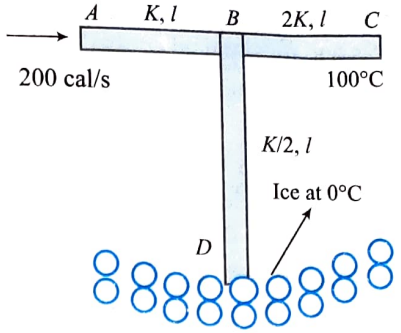
56. The top of insulated cylindrical container is covered by a disc having emissivity 0.6 and thickness 1 cm. The temperature is maintained by circulating oil as shown in figure. If temperature of upper surface of disc is 127°C and temperature of surrounding is 27°C, then the radiation lost to the surroundings will be (Take $\sigma = \frac{17}{3} \times 10^{-8}$ W/m$^2K^4$)



(1) 595 J/m^2 × sec (2) 595 cal/m^2 × sec

(3) 991.0 J/m^2 × sec (4) 440 J/m^2 × sec

57. Three rods AB, BC and BD of same length l and cross-sectional area A are arranged as shown. The end D is immersed in ice whose mass is 440 g and is at 0°C. The end C is maintained at 100°C. Heat is supplied at constant rate of 200 cal/s. Thermal conductivities of AB, BC and BD are K, $2K$ and $K/2$, respectively. Time after which whole ice will melt is (K = 100 cal/m-s-°C, A = 10 cm^2, l = 1 m)



(1) 400 s (2) 600 s

(3) 700 s (4) 800 s

58. A system receives heat continuously at the rate of 10 W per sq. unit area. The temperature of the system becomes constant at 70°C when the temperature of the surroundings is 30°C. After the heater is switched off, the system cools from 50°C to 49.9°C in 1 min. The heat capacity of the system is

(1) 1000 J/°C (2) 1500 J/°C

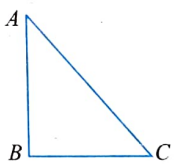
(3) 3000 J/°C (4) None of these

59. The earth receives on its surface radiation from the sun at the rate of 1400 W/m^2. The distance of the centre of the sun from the surface of the earth is 1.5×10^{11} m and the radius of the sun is 7.0×10^8 m. Treating sun as a black body, it follows from the above data that its surface temperature is

(1) 5801 K (2) 10^6 K

(3) 50.1 K (4) 5801°C

60. Three rods of identical cross-sectional area and made from the same metal form the sides of an isosceles triangle ABC right angled at B. The points A and B are maintained at temperatures T and $\sqrt{2}T$, respectively, in the steady state. Assuming that only heat conduction takes place, temperature of point C is



(1) $\frac{3T}{\sqrt{2}+1}$ (2) $\frac{T}{\sqrt{2}+1}$

(3) $\frac{T}{3(\sqrt{2}-1)}$ (4) $\frac{T}{\sqrt{2}-1}$

61. Two hollow spheres of different materials, one with double the radius and one-fourth wall thickness of the other, are filled with ice. If the times taken for complete melting of ice in the larger to the smaller one are in the ratio of 25 : 16, then their corresponding thermal conductivities are in the ratio

(1) 4:5 (2) 5:4

(3) 8:25 (4) 25:8

62. A room at 20°C is heated by a heater of resistance 20 ohm connected to 200 V mains. The temperature is uniform throughout the room and the heat is transmitted through a glass window of area 1 m^2 and thickness 0.2 cm. Calculate

the temperature outside. Thermal conductivity of glass is 0.2 cal/m C° s and mechanical equivalent of heat is 4.2 J/cal.

(1) 13.69°C (2) 15.24°C
(3) 17.85°C (4) 19.96°C

63. When the temperature of a black body increases, it is observed that the wavelength corresponding to maximum energy changes from 0.26 μm to 0.13 μm. The ratio of the emissive powers of the body at the respective temperatures is

(1) $\frac{16}{1}$ (2) $\frac{4}{1}$
(3) $\frac{1}{4}$ (4) $\frac{1}{16}$

64. The radiation emitted by a star A is 10000 times that of the sun. If the surface temperatures of the sun and star A are 6000 K and 2000 K, respectively, the ratio of the radii of the star A and the sun is

(1) 300:1 (2) 600:1
(3) 900:1 (4) 1200:1

65. A planet radiates heat at a rate proportional to the fourth power of its surface temperature T. If such a steady temperature of the planet is due to an exactly equal amount of heat received from the sun then which of the following statements is true?

(1) The planet's surface temperature varies inversely as the distance of the sun.
(2) The planet's surface temperature varies directly as the square of its distance from the sun.
(3) The planet's surface temperature varies inversely as the square root of its distance from the sun.
(4) The planet's surface temperature is proportional to the fourth power of distance from the sun.

66. A black body is at a temperature of 2880 K. The energy of radiation emitted by this object with wavelength between 499 nm and 500 nm is U_1, between 999 nm and 1000 nm is U_2 and between 1499 nm and 1500 nm is U_3. Wien's constant $b = 2.88 \times 10^6$ nm-K, Then

(1) $U_1 = 0$ (2) $U_2 = 0$
(3) $U_1 = U_2$ (4) $U_2 > U_1$

67. A closed cubical box is made of a perfectly insulating material walls of thickness 8 cm and the only way for heat to enter or leave the box is through two solid metallic cylindrical plugs, each of cross-sectional area 12 cm^2 and length 8 cm, fixed in the opposite walls of the box. The outer surface A on one plug is maintained at 100°C, while the outer surface of B of the plug is maintained at 4°C. The thermal conductivity of the material of the plug is 0.5 cal/°C/cm. A source of energy generating 36 cal/s is enclosed inside the box. Assuming the temperature to be the same at all points on the inner surface, the equilibrium temperature of the inner surface of the box is

(1) 62°C (2) 46°C
(3) 76°C (4) 52°C

68. Two models of a windowpane are made. In one model, two identical glass panes of thickness 3 mm are separated with an air gap of 3 mm. This composite system is fixed in the window of a room. The other model consists of a sing glass pane of thickness 3 mm, the temperature differen being the same as for the first model. The ratio of the he flow for the double pane to that for the single pane is

$\left(\text{Given } \frac{K_{glass}}{K_{air}} = 48\right)$

(1) 1/20 (2) 1/70
(3) 1/100 (4) 1/50

Multiple Correct Answers Type

1. During heat exchange, temperature of a solid mass does no change. In this process, heat

(1) is not being supplied to the mass
(2) is not being taken out from the mass
(3) may have been supplied to the mass
(4) may have been taken out from the mass

2. A polished metallic piece and a black painted wooden piec are kept in open in bright sun for a long time. Which of th following statements is/are corect?

(1) The wooden piece will absorb less heat than the metalli piece.
(2) The wooden piece will have a lower temperature than the metallic piece.
(3) If touched, the metallic piece will be felt hotter than the wooden piece.
(4) When the two pieces are removed from the open to a cold room, the wooden piece will lose heat at a faster rate than the metallic piece.

3. Choose the correct statements from the following:

(1) Good reflectors are good emitters of thermal radiation
(2) Burns caused by water at 100°C are more severe than those caused by steam at 100°C
(3) If the earth did not have atmosphere, it would become intolerably cold
(4) It is impossible to construct a heat engine of 100% efficiency

4. Two bodies A and B have thermal emissivities of 0.01 and 0.81, respectively. The outer surface areas of the two bodies are the same. The two bodies emit total radiant power at the same rate. The wavelength λ_B corresponding to maximum spectral radiancy in the radiation from B is shifted from the wavelength corresponding to maximum spectral radiancy in the radiation from A, by 1.00 μm. If the temperature of A is 5820 K, then

(1) the temperature of B is 1934 K
(2) $\lambda_B = 1.5$ μm
(3) the temperature of B is 11604 K
(4) the temperature of B is 2901 K

5. A hollow and a solid sphere of same material and identical outer surface are heated to the same temperature.

(1) In the beginning both will emit equal amount of radiation per unit time
(2) In the beginning both will absorb equal amount of radiation per unit time

(3) Both spheres will have same rate of fall of temperature (dT/dt)

(4) Both spheres will have equal temperatures at any moment.

6. A heated body emits radiation which has maximum intensity at frequency ν_m. If the temperature of the body is doubled then

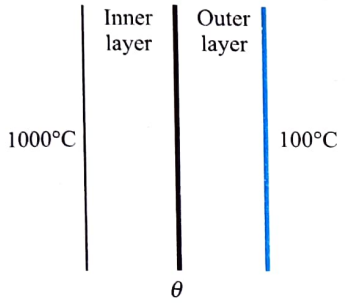
(1) the maximum intensity radiation will be at frequency $2\nu_m$

(2) the maximum intensity radiation will be at frequency $(1/2)\,\nu_m$

(3) the total emitted energy will increase by a factor of 16

(4) the total emitted energy will increase by a factor of 2

7. The temperature drops through a two-layer furnace wall by 900°C. Each layer is of equal area of cross-section. Which of the following actions will result in lowering the temperature θ of the interface?



(1) By increasing the thermal conductivity of outer layer.

(2) By increasing the thermal conductivity of inner layer.

(3) By increasing thickness of outer layer.

(4) By increasing thickness of inner layer.

8. Choose the correct statements from the following:

(1) A temperature change which increases the length of a steel rod by 0.1% will increase its volume by nearly 0.3%

(2) The specific heat of a solid is different when the solid is heated at (i) constant pressure and (ii) the constant volume

(3) The thermal conductivity of air being less than that for wool, we prefer wool to air for thermal insulation

(4) When the distance between two fixed points is measured with a steel tape, the observed reading will be less on a hot day than on a cold day

9. Two identical objects A and B are at temperatures T_A and T_B, respectively. Both objects are placed in a room with perfectly absorbing walls maintained at a temperature T $(T_A > T > T_B)$. The objects A and B attain the temperature T eventually. Select the correct statements from the following:

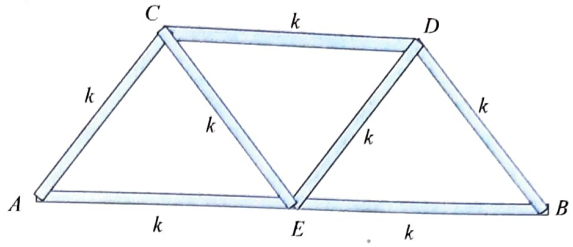
(1) A only emits radiation, while B only absorbs it until both attain the temperature T

(2) A loses more heat by radiation than it absorbs, while B absorbs more radiation than it emits, until they attain the temperature T

(3) Both A and B only absorb radiation, but do not emit it, until they attain the temperature T

(4) Each object continuous to emit and absorb radiation even after attaining the temperature T

10. Seven identical rods of material of thermal conductivity k are connected as shown in the given figure. All the rods are of identical length l and cross-sectional area A. If the one end B is kept at 100°C and the other end is kept at 0°C, what would be the temperatures of the junctions C, D and E (θ_C, θ_D and θ_E) in the steady state?



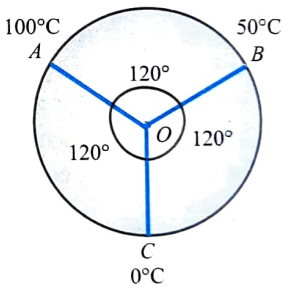
(1) $\theta_C > \theta_E > \theta_D$

(2) $\theta_E = 50°C$ and $\theta_D = 37.5\ °C$

(3) $\theta_E = 50°C$, $\theta_C = 62.5°C$ and $\theta_D = 37.5°C$

(4) $\theta_E = 50°C$, $\theta_C = 60°C$ and $\theta_D = 40°C$

11. A circular ring (centre O) of radius a, and of uniform cross-section is made up of three different metallic rods AB, BC and CA (joined together at the points A, B and C in pairs) of thermal conductivities α_1, α_2 and α_3, respectively (see diagram). The junctions A, B and C are maintained at the temperatures 100°C, 50°C and 0°C, respectively. All the rods are of equal lengths and cross-sections. Under steady state conditions, assume that no heat is lost from the sides of the rods. Let Q_1, Q_2 and Q_3 be the rates of transmission of heat along the three rods AB, BC and CA. Then



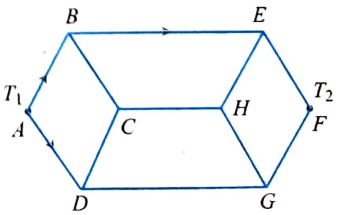
(1) $Q_1 = Q_2 = Q_3$ and all are transmitted in the clockwise sense

(2) Q_1 and Q_2 flow in clockwise sense and Q_3 in the anticlockwise sense

(3) $Q_1 : Q_2 : Q_3 :: \alpha_1 : \alpha_2 : 2\alpha_3$

(4) $\dfrac{Q_1}{\alpha_1} + \dfrac{Q_2}{\alpha_2} = \dfrac{Q_3}{\alpha_3}$

12. Eleven identical rods are arranged as shown in figure. Each rod has length l, cross-sectional area A and thermal conductivity of material L. Ends A and F are maintained at temperatures T_1 and T_2 $(< T_1)$, respectively. If lateral surface of each rod is thermally insulated, the rate of heat transfer $\left(\dfrac{dQ}{dt}\right)$ in each rod is



(1) $\left(\dfrac{dQ}{dt}\right)_{AB} = \left(\dfrac{dQ}{dt}\right)_{CD}$

(2) $\left(\dfrac{dQ}{dt}\right)_{BE} = \dfrac{2}{7}\dfrac{(T_1 - T_2)\,KA}{l}$

(3) $\left(\frac{dQ}{dt}\right)_{CH} \neq \left(\frac{dQ}{dt}\right)_{DG}$

(4) $\left(\frac{dQ}{dt}\right)_{BC} = \left(\frac{dQ}{dt}\right)_{DC}$

13. Two spheres A and B have same radius but the heat capacity of A is greater than that of B. The surfaces of both are painted black. They are heated to the same temperature and allowed to cool. Then:

(1) A cools faster than B

(2) Both A and B cool at the same rate

(3) At any temperature the ratio of their rates of cooling is a constant

(4) B cools faster than A

14. The two ends of a uniform rod of thermal conductivity k are maintained at different but constant temperatures. The temperature gradient at any point on the rod is $\frac{d\theta}{dl}$ (equal to the difference in temperature per unit length). The heat flow per unit time per unit cross-section of the rod is I then which of the following statements is/are correct:

(1) $\frac{d\theta}{dl}$ is the same for all points on the rod

(2) I will decrease as we move from higher to lower temperature

(3) $I \propto K \cdot \frac{d\theta}{dl}$

(4) All the above options are incorrect

15. Two ends of area A of a uniform rod of thermal conductivity K are maintained at different but constant temperatures. At any point on the rod, the temperature gradient is $\frac{dT}{dl}$. If I be the thermal current in the rod, then:

(1) $I \propto A$ (2) $I \propto \frac{dT}{d\ell}$

(3) $I \propto A^0$ (4) $I \propto \frac{I}{\left(\frac{dT}{d\ell}\right)}$

16. A metal cylinder of mass 0.5 kg is heated electrically by a 12 W heater in a room at 15°C. The cylinder temperature rises uniformly to 20°C in 5 min and finally becomes constant at 45°C. Assuming that the rate of heat loss is proportional to the excess temperature over the surroundings:

(1) The rate of loss of heat of the cylinder to surrounding at 20°C is 2 W

(2) The rate of loss of heat of the cylinder to surrounding at 45°C is l2 W

(3) The rate of loss of heat of the cylinder to surrounding at 20°C is 5 W

(4) The rate of loss of heat of the cylinder to surrounding at 45°C is 30 W

17. Two spherical black bodies A and B, having radii r_A and $r_B = 2r_A$ emit radiation with peak intensities at wavelength 400 nm and 800 nm respectively. If their temperature are T_A and T_B respectively in Kelvin scale, their emissive powe are E_A and E_B then:

(1) $\frac{T_A}{T_B} = 2$ (2) $\frac{T_A}{T_B} = 4$

(3) $\frac{E_A}{E_B} = 8$ (4) $\frac{E_A}{E_B} = 4$

Linked Comprehension Type

For Problems 1–3

Assume that the thermal conductivity of copper is twice that aluminium and four times that of brass. Three metal rods mad of copper, aluminium and brass are each 15 cm long and 2 c in diameter. These rods are placed end to end, with aluminiu between the other two. The free ends of the copper and bra rods are maintained at 100°C and 0°C, respectively. The syste is allowed to reach the steady state condition. Assume there is n loss of heat anywhere.

1. When steady state condition is reached everywhere, whic of the following statements is true?
   (1) No heat is transmitted across the copper–aluminium o aluminium–brass junctions.
   (2) More heat is transmitted across the copper–aluminiu junction than across the aluminium–brass junction.
   (3) More heat is transmitted across the aluminium–bras junction than the copper–aluminium junction.
   (4) Equal amount of heat is transmitted at the copper- aluminium and aluminium–brass junctions.
2. Under steady state condition, the equilibrium temperature o the copper–aluminium junction will be
   (1) 86°C (2) 18.8°C
   (3) 57°C (4) 73°C
3. Under steady state condition, the equilibrium temperature o the aluminium–brass junction will be
   (1) 57°C (2) 35°C
   (3) 18.8°C (4) 28.5°C

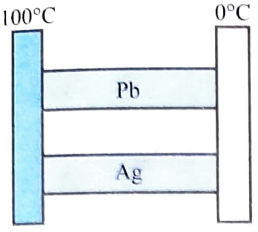
For Problems 4–6

A body of area 0.8×10^{-2} m^2 and mass 5×10^{-4} kg directly face the sun on a clear day. The body has an emissivity of 0.8 an specific heat of 0.8 cal/kg K. The surroundings are at 27°C. (sol constant = 1.4 kW/m^2).

4. The rate of rise of the body's temperature is nearly
   (1) 0.36°C/s (2) 3.6 K/s
   (3) 36°C/s (4) 72 K/s
5. The maximum attainable temperature of the body is
   (1) 396 K (2) 396°C
   (3) 85°C (4) 85 K
6. The temperature that the body would reach if it lost all i heat by radiation is
   (1) 396 K (2) 296°C
   (3) 85°C (4) 85 K

For Problems 7–9

Two insulated metal bars each of length 5 cm and rectangular cross-sections with sides 2 cm and 3 cm are wedged between two walls, one held at 100°C and the other at 0°C. The bars are made of lead and silver. $K_{Pb} = 350$ W/mK, $K_{Ag} = 425$ W/mK.



7. Thermal current through lead bar is
   (1) 210 W (2) 420 W
   (3) 510 W (4) 930 W
8. Total thermal current through the two-bar system is
   (1) 210 W (2) 420 W
   (3) 510 W (4) 930 W
9. Equivalent thermal resistance of the two bar system is
   (1) 0.1 K/W
   (2) 0.23 K/W
   (3) 0.19 K/W
   (4) 0.42 K/W

For Problems 10–12

A thin copper rod of uniform cross-section A square metres and of length L metres has a spherical metal sphere of radius r metre at its one end symmetrically attached to the copper rod. The thermal conductivity of copper is K and the emissivity of the spherical surface of the sphere is ε. The free end of the copper rod is maintained at the temperature T Kelvin by supplying thermal energy from a P watt source. Steady state conditions are allowed to be established while the rod is properly insulated against heat loss from its lateral surface. Surroundings are at 0°C. Stefan's constant, $a = 5.67 \times 10^{-8}$ W/m^2 K^4.

10. After the steady state conditions are reached, the temperature of the spherical end of the rod, T_S is
    (1) $T_S = T - \dfrac{PL}{KA}$ (2) $T_S = 0°C$
    (3) $T_S = \dfrac{PL}{KA}$ (4) $T_S = T - \dfrac{P(L+r)}{KA}$
11. The net power that will be radiated out, P_S, from the sphere after steady state conditions are reached is
    (1) $P_S = P$ (2) $P_S = \dfrac{PA}{4\pi r^2}$
    (3) $P_S = 0$ (4) $P_S = \sigma \varepsilon T_S^4$
12. If the metal sphere attached at the end of the copper rod is made of brass, whose thermal conductivity is $K_b < K$, then which of the following statements is true?
    (1) The temperature of the sphere will, under steady state conditions, continue to be T_B
    (2) The power that will be radiated out from the sphere will still be P_S
    (3) It will take smaller time for steady state conditions to be reached
    (4) The rate of thermal energy transmitted across the copper rod, under steady state, will be reduced

For Problems 13–15

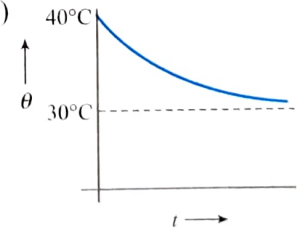
A wire of length 1 m and radius 10^{-3} m is carrying a heavy current and is assumed to radiate as a black body. At equilibrium, its temperature is 900 K while that of surrounding is 300 K. The resistivity of the material of the wire at 300 K is $\pi^2 \times 10^{-8}$ ohm-m and its temperature coefficient of resistance is 7.8×10^{-3}/°C (Stefan's constant $\sigma = 5.68 \times 10^{-8}$ W/m^{2}K^4).

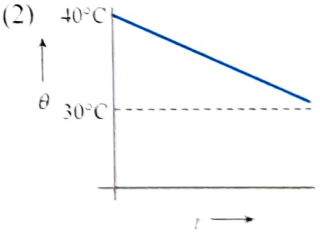
13. The resistivity of wire at 900 K is nearly
    (1) 2.4×10^{7} ohm-m
    (2) 2.4×10^{-7} ohm-m
    (3) 1.2×10^{-7} ohm-m
    (4) 1.2×10^{7} ohm-m
14. Heat radiated per second by the wire is nearly
    (1) 23 W
    (2) 230 W
    (3) 2300 W
    (4) 23000 W
15. The current in the wire is nearly
    (1) 0.555 A (2) 5.5 A
    (3) 55 A (4) 550 A

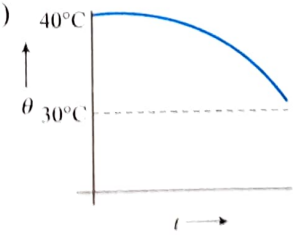
For Problems 16–18

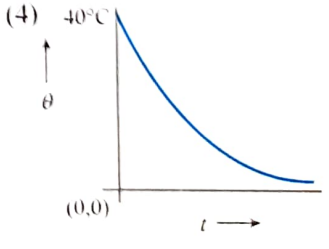
A body cools in a surrounding of constant temperature 30°C. Its heat capacity is 2 J/°C. Initial temperature of the body is 40°C. Assume Newton's law of cooling is valid. The body cools to 38°C in 10 min.

16. In further 10 min it will cool from 38°C to _____:
    (1) 36°C (2) 36.4°C
    (3) 37°C (4) 37.5°C
17. The temperature of the body in °C denoted by θ. The variation of θ versus time t is best denoted as

(1)


(2)


(3)


(4)


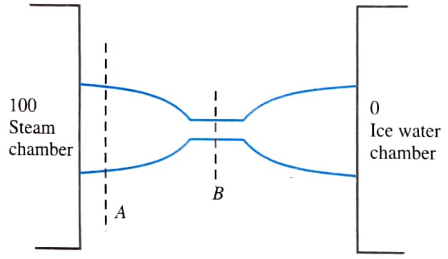
18. When the body temperature has reached 38°C, it is heated again so that it reaches 40°C in 10 min. The heat required from a heater by the body is
    (1) 3.6 J (2) 0.364 J
    (3) 8 J (4) 4 J

Matrix Match Type

1. The surface of a household radiator has an emissivity of 0.55 and an area of 1.5 m^2. Its equilibrium temperature is 50°C and the surroundings are at 22°C ($\sigma = 5.67 \times 10^{-8}$ W/m^2K^4).

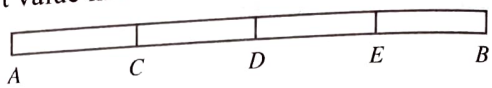
Column I	Column II
i. Rate at which radiation is emitted by the radiator is	a. 155
ii. Rate at which radiation is absorbed by the radiator is	b. 509
iii. Net value of radiation from the radiator is	c. 354

2. A copper rod (initially at room temperature 20°C) of non-uniform cross-section is placed between a steam chamber at 100°C and ice water chamber at 0°C. A and B are cross-sections as shown in the following figure. Then match the statements in Column I with results in Column II comparing only between cross-sections A and B. (The mathematical expressions in Column I have their usual meanings in heat transfer.)



Column I	Column II
i. Initially, rate of heat flow $\left(\frac{dQ}{dt}\right)$ will be	a. maximum at section A
ii. At steady state, rate of heat flow $\left(\frac{dQ}{dt}\right)$ will be	b. maximum at section B
iii. At steady state temperature, gradient $\left\|\left(\frac{dT}{dx}\right)\right\|$ will be	c. minimum at section B
iv. At steady state, rate of change of temperature $\left(\frac{dT}{dt}\right)$ will be	d. same for all sections

3. A rod AB of uniform cross-section consists of four sections AC, CD, DE and EB of different metals with thermal conductivities K, $0.8K$, $1.2K$ and $1.50K$, respectively. Their lengths are respectively L, $1.2L$, $1.5L$ and $0.6L$. They are joined rigidly in succession at C, D and E to form the rod AB. The end A is maintained at 100°C and the end B is maintained at 0°C. The steady state temperatures of the joints C, D and E are respectively T_C, T_D and T_E. Column I lists the temperature differences $(T_A - T_C)$, $(T_C - T_D)$, $(T_D - T_E)$ and $(T_E - T_B)$ in the four sections and Column II their values jumbled up. Match each item in Column I with correct value in Column II.

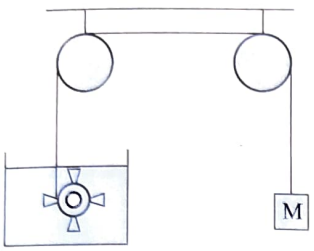


Column I	Column II
i. $(T_A - T_C)$	a. 9.6
ii. $(T_C - T_D)$	b. 30.1
iii. $(T_D - T_E)$	c. 24.1
iv. $(T_E - T_B)$	d. 36.2

Numerical Value Type

1. A spherical black body with a radius of 12 cm placed in space radiates 450 W power at 500 K. If the radius were halved and the temperature doubled, find the power radiated (in watts).
2. A block having some emissivity is maintained at 500 K temperature in a surrounding of 300 K temperature. It is observed that, to maintain the temperature of the block 210 W external power is required to be supplied to it. If instead of this block a black body of same geometry and size is used, 700 W external power is needed for the same. Find the emissivity of the material of the block.
3. A calorimeter of negligible heat capacity contains 100 g waters at 40°C. The water cools to 35°C in 5 minutes. If the water is now replaced by a liquid of same volume as that of water at same initial temperature, it cools to 35°C in 2 minutes. Given specific heats of water and that liquid are 4200 J/kg°C and 2100 J/kg°C respectively. Find the density of the liquid in kg/m^3.
4. The maximum in the energy distribution spectrum of the sun is at wavelength 4753 Å and its temperature is 6050 K. What will be the temperature of the star(in K) whose energy distribution shows a maximum at wavelength 9506 Å?
5. When the centre of earth is at a distance of 1.6×10^{11} m from the centre of sun, the intensity of solar radiation reaching at the earth's surface is 2.4 kW/m^2. There is a spherical cloud of cosmic dust, containing iron particles. The melting point for iron particles in the cloud is 2000 K. Find the distance of iron particles from the centre of sun at which the iron particle starts melting (in $\times 10^9$ m). (Assume sun and cloud as a black body, $\sigma = 6.0 \times 10^{-8}$ $W/m^2 K^4$):
6. A uniform rod of thermal conductivity of 65 J/m-s-°C is surrounded by an insulator on its sides. One of its ends is put in a furnace, while the other end is kept exposed. If the temperature gradient of the rod is −75°C/m, find the emissive power of the exposed end.
7. A slab of stone of area 3600 cm^2 and thickness 10 cm is exposed on the lower surface to steam at 100°C. A block of ice at 0°C rests on the upper surface of the slab. If in 1 h 4.8 kg of ice is melted, calculate the thermal conductivity of the stone.
8. The given figure shows water in a container having 2.0 mm thick walls made of a material of thermal conductivity 0.50 W/m-°C. The container is kept in a melting ice bath at 0°C.

The total surface area in contact with water is 0.05 m². A wheel is clamped inside the water and is coupled to a block of mass M as shown in the figure. As the block goes down, the wheel rotates. It is found that after some time a steady state is reached in which the block goes down with a constant speed of 10 cm/s and the temperature of the water remains constant at 1.0°C. Find the mass M of the block. Assume that the heat flows out of the water only through the walls in contact. Take $g = 10$ m/s².



9. Let us assume that sun radiates like a black body with surface temperature at $T_0 = 6000$ K and earth absorbs radiations coming from sun only. If both earth and sun are considered perfect spheres with distance between centre of earth and centre of sun to be 200 times the radius of sun, find the temperature (in Kelvin) of surface of earth in steady state (assume radiation incident on earth to be almost parallel).

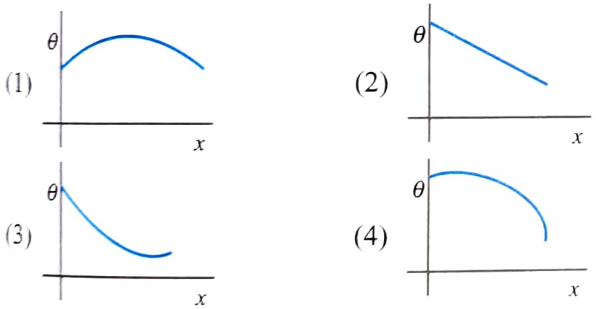
10. A cylindrical rod of length 50 cm and cross-sectional area 1 cm² is fitted between a large ice chamber at 0°C and an evacuated chamber maintained at 27°C as shown in figure. Only small portions of the rod are inside the chambers and the rest is thermally insulated from the surrounding. The cross-section going into the evacuated chamber is blackened so that it completely absorbs any radiation falling on it. The temperature of the blackened end is 17°C when steady state is reached. Stefan constant $\sigma = 6 \times 10^{-8}$ W/m²-K⁴. Find the thermal conductivity of the material of the rod. (in W/m °C)

Archives

JEE MAIN

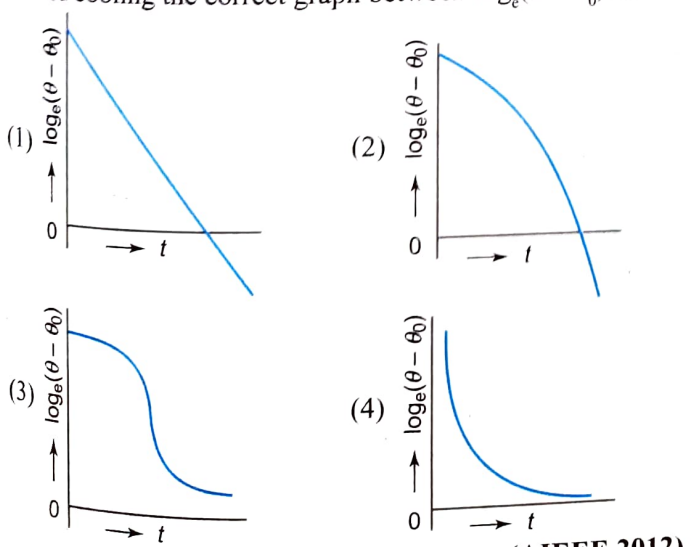
Single Correct Answer Type

1. A long metallic bar is carrying heat from one of its ends to the other end under steady state. The variation of temperature θ along the length x of the bar from its hot end is best described by which of the following figures.



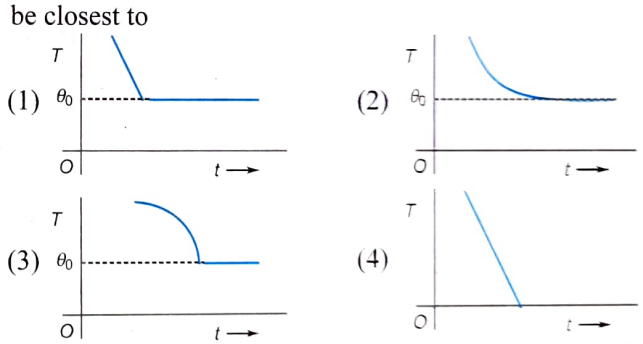
(AIEEE 2009)

2. A liquid in a beaker has temperature $\theta(t)$ at time t and θ_0 is temperature of surroundings, then according to Newton's law of cooling the correct graph between $\log_e(\theta - \theta_0)$ and t



(AIEEE 2012)

3. If a piece of metal is heated to temperature θ and then allowed to cool in a room which is at temperature θ_0, the graph between the temperature T of the metal and time t will be closest to



(JEE Main 2013)

4. Three rods of copper, brass and steel are welded together to form a Y-shaped structure. Area of cross-section of each rod = 4 cm². The end of copper rod is maintained at 100°C where as ends of brass and steel are kept at 0°C. Lengths of the copper, brass and steel rods are 46, 13 and 12 cm respectively. The rods are thermally insulated from surroundings except at ends. Thermal conductivities of copper, brass and steel are 0.92, 0.26 and 0.12 CGS units respectively. The rate of heat flow through copper rod is

(1) 4.8 cal/s (2) 6.0 cal/s

(3) 1.2 cal/s (4) 2.4 cal/s

(JEE Main 2014)

JEE ADVANCED

Single Correct Answer Type

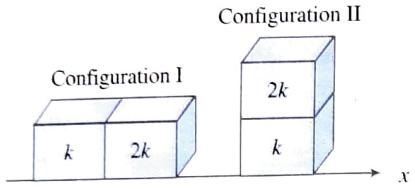
1. Three very large plates of same area are kept parallel and close to each other. They are considered as ideal black surfaces and have very high thermal conductivity. The first and third plates are maintained at temperatures $2T$ and $3T$, respectively. The temperature of the middle (i.e., second) plate under steady state condition is

(1) $\left(\frac{65}{2}\right)^{1/4} T$ (2) $\left(\frac{97}{4}\right)^{1/4} T$

(3) $\left(\frac{97}{2}\right)^{1/4} T$ (4) $(97)^{1/4} T$

(IIT-JEE 2012)

2. Two rectangular blocks, having identical dimensions, can be arranged either in configuration I or in configuration II as shown in the figure. One of the blocks has thermal conductivity k and the other $2k$. The temperature difference between the ends along the x-axis is the same in both the configurations. It takes 9 s to transport a certain amount of heat from the hot end to the cold end in the configuration I. The time to transport the same amount of heat in the configuration II is



(1) 2.0 s (2) 3.0 s

(3) 4.5 s (4) 6.0 s

(JEE Advanced 2013)

3. Parallel rays of light of intensity $I = 912\ \text{Wm}^{-2}$ are incident on a spherical black body kept in surroundings of temperature 300 K. Take Stefan-Boltzmann constant $\sigma = 5.7 \times 10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$ and assume that the energy exchange with the surroundings is only through radiation. The final steady state temperature of the black body is close to

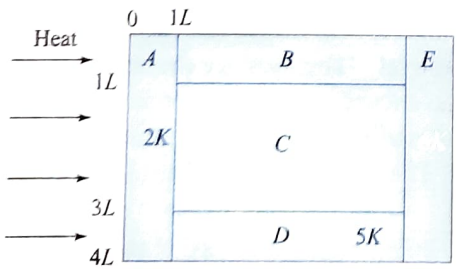
(1) 330 K (2) 660 K

(3) 990 K (4) 1550 K

(JEE Advanced 2014)

Multiple Correct Answers Type

1. A composite block is made of slabs A, B, C, D and E of different thermal conductivities (given in terms of a constant K) and sizes (given in terms of length, L) as shown in the figure. All slabs are of same width.



(1) Heat flow through slabs A and E is same

(2) Heat flow through slab E is maximum

(3) Temperature difference across slab E is smallest.

(4) Heat flow through C = heat flow through B + heat flow through D. **(IIT-JEE 2011)**

2. An incandescent bulb has a thin filament of tungsten that is heated to high temperature by passing an electric current. The hot filament emits black-body radiation. The filament is observed to break up at random locations after a sufficiently long time of operation due to non-uniform evaporation of tungsten from the filament. If the bulb is powered at constant voltage, which of the following statement(s) is(are) true?

(1) The temperature distribution over the filament is uniform

(2) The resistance over small sections of the filament decreases with time

(3) The filament emits more light at higher band of frequencies before it breaks up

(4) The filament consumes less electrical power towards the end of the life of the bulb **(JEE Advanced 2016)**

3. A human body has a surface area of approximately 1 m^2. The normal body temperature is 10 K above the surrounding room temperature T_0. Take the room temperature to be $T_0 = 300$ K. For $T_0 = 300$ K, the value of $\sigma T_0^4 = 460\ \text{Wm}^{-2}$ (where σ is the Stefan-Boltzmann constant). Which of the following options is/are correct?

(1) Reducing the exposed surface area of the body (e.g. by curling up) allows humans to maintain the same body temperature while reducing the energy lost by radiation

(2) If the body temperature rises significantly then the peak in the spectrum of electromagnetic radiation emitted by the body would shift to longer wavelengths

(3) The amount of energy radiated by the body in 1 second is close to 60 Joules

(4) If the surrounding temperature reduces by a small amount $\Delta T_0 \ll T_0$, then to maintain the same body temperature the same (living) human being needs to radiate $\Delta W = 4\sigma T_0^3 \Delta T_0$ more energy per unit time

(JEE Advanced 2017)

Numerical Value Type

1. A metal rod AB of length $10x$ has its one end A in ice at 0°C and the other end B in water at 100°C. If a point P on the rod is maintained at 400°C, then it is found that equal amounts of water and ice evaporate and melt per unit time. The latent heat of evaporation of water is 540 cal/g and latent heat of melting of ice is 80 cal/g. If the point P is at a distance of λx from the ice end A, find the value of λ. (Neglect any heat loss to the surrounding.) **(IIT-JEE 2009)**

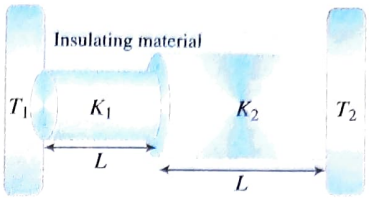
2. Two spherical bodies A (radius 6 cm) and B (radius 18 cm) are at temperature T_1 and T_2 respectively. The maximum intensity in the emission spectrum of A is at 500 nm and that of B is at 1500 nm. Considering them to be black bodies, what will be the ratio of the rate of total energy radiated by A to that of B? **(IIT-JEE 2010)**

3. Two spherical stars A and B emit blackbody radiation. The radius of A is 400 times that of B and A emits 10^4 times the power emitted from B. The ratio $\left(\frac{\lambda_A}{\lambda_B}\right)$ of their wavelengths λ_A and λ_B at which the peaks occur in their respective radiation curves is _____. **(JEE Advanced 2015)**

4. A metal is heated in a furnace where a sensor is kept above the metal surface to read the power radiated (P) by the metal. The sensor has a scale that displays $\log_2 (P/P_0)$, where P_0 is a constant. When the metal surface is at a temperature of 487°C, the sensor shows a value 1. Assume that the emissivity of the metallic surface remains constant. What is the value displayed by the sensor when the temperature of the metal surface is raised to 2767°C?

(JEE Advanced 2016)

5. Two conducting cylinders of equal length but different radii are connected in series between two heat baths kept at temperatures T_1 = 300 K and T_2 = 100 K, as shown in the figure. The radius of the bigger cylinder is twice that of the smaller one and the thermal conductivities of the materials of the smaller and the larger cylinders are K_1 and K_2 respectively. If the temperature at the junction of the two cylinders in the steady state is 200 K, then K_1/K_2 = _____.



(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	2. (2)	3. (2)	4. (2)	5. (1)
6. (3)	7. (4)	8. (4)	9. (2)	10. (4)
11. (2)	12. (3)	13. (3)	14. (2)	15. (1)
16. (2)	17. (3)	18. (1)	19. (2)	20. (3)
21. (1)	22. (4)	23. (3)	24. (2)	25. (2)
26. (1)	27. (3)	28. (3)	29. (2)	30. (1)
31. (2)	32. (2)	33. (1)	34. (4)	35. (3)
36. (3)	37. (3)	38. (4)	39. (2)	40. (1)
41. (3)	42. (3)	43. (1)	44. (2)	45. (3)
46. (3)	47. (4)	48. (3)	49. (3)	50. (3)
51. (2)	52. (4)	53. (4)	54. (2)	55. (3)
56. (1)	57. (4)	58. (3)	59. (1)	60. (1)
61. (3)	62. (2)	63. (4)	64. (3)	65. (3)
66. (4)	67. (3)	68. (4)		

Multiple Correct Answers Type

1. (3),(4)	2. (3),(4)	3. (3),(4)
4. (1),(2)	5. (1),(2)	6. (1),(3)
7. (1),(4)	8. (1),(2),(4)	9. (2),(4)
10. (1),(3)	11. (2),(3),(4)	12. (2),(3),(4)
13. (3),(4)	14. (1),(3)	15. (1),(2)
16. (1),(2)	17. (1),(4)	

Linked Comprehension Type

1. (4)	2. (1)	3. (1)	4. (2)	5. (1)
6. (3)	7. (2)	8. (4)	9. (1)	10. (4)

11. (1)	12. (2)	13. (2)	14. (2)	15. (3)
16. (2)	17. (1)	18. (3)		

Matrix Match Type

1. i. → b.; ii. → c.; iii. → a.
2. i. → a., c.; ii. → d.; iii. → b.; iv. → d.
3. i. → c.; ii. → d.; iii. → b.; iv. → a.

Numerical Value Type

1. (1800)	2. (0.3)	3. (800)	4. (3025)	5. (8.0)
6. (4875)	7. (1.24)	8. (12.5)	9. (300)	10. (1.8)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)	2. (1)	3. (2)	4. (1)

JEE Advanced

Single Correct Answer Type

1. (3)	2. (1)	3. (1)

Multiple Correct Answers Type

1. (1),(3),(4)	2. (3),(4)	3. (1),(3)

Numerical Value Type

1. (9)	2. (9)	3. (2)	4. (9)	5. (4)

3 Kinetic Theory of Gases

INTRODUCTION

A gas consists of atoms (either individually or bound together as molecules) that fill their container's volume and exert pressure on the container's walls. We can usually assign a temperature to such a contained gas. These three variables associated with a gas—volume, pressure and temperature—are all a consequence of the motion of the atoms. The volume is a result of the freedom that atoms have to spread throughout the container, the pressure is a result of the collisions of the atoms with the container's walls and the temperature has to do with the kinetic energy of the atoms. The focus of this chapter relates the motion of the atoms to the volume, pressure and temperature of the gas.

AVOGADRO'S NUMBER

The first step in our discussion of the kinetic theory of gases deals with measuring the amount of a gas present in a sample, for which we use Avogadro's number. When our thinking is slanted towards atoms and molecules, it makes sense to measure the sizes of our samples in moles. Mole is one of the seven SI base units. Avogadro suggested that all gases occupying the same volume under the same conditions of temperature and pressure contain the same number of atoms or molecules. One mole of a substance contains N_A elementary units (usually atoms or molecules), where N_A is found experimentally to be 6.02×10^{23} mol^{-1}. The number N_A is called Avogadro's number.

One molar mass M of any substance is the mass of one mole of the substance. A mole is related to the mass m of the individual molecules of the substance by $M = mN_A$.

ILLUSTRATION 3.1

Calculate the number of atoms in 39.4 g gold. Molar mass of gold is 197 g mole^{-1}.

Sol. We know that, Molar mass = Mass of Avogadro's number of atoms (Molecules) = 6.023×10^{23} atoms

Given, molar mass of gold = 197 g/mol

Now,

$\because$ 197 g of gold contains 6.023×10^{23} atoms

$\therefore$ 1 g of gold contains $\dfrac{6.023 \times 10^{23}}{197}$ atoms

$\therefore$ 39.4 g of gold contains $\dfrac{6.023 \times 10^{23} \times 39.4}{197}$ atoms

Hence, 39.4 g of gold contains 1.20×10^{23} atoms.

ASSUMPTION OF IDEAL GASES (OR KINETIC THEORY OF GASES)

Kinetic theory of gases relates the macroscopic properties of gases (such as pressure, temperature etc.) to the microscopic properties of the gas molecules (such as speed, momentum, kinetic energy of molecule etc.).

Actually, it attempts to develop a model of the molecular behaviour which should result in the observed behaviour of an ideal gas. This model treats an ideal gas as a collection of molecules with the following assumptions:

(i) Every gas consists of extremely small particles known as molecules. Their size is negligible in comparison to intermolecular distance (10^{-9} m).

(ii) The molecules of a given gas are all identical but are different than those of another gas. The molecules of a gas are identical, spherical, rigid and perfectly elastic point masses.

(iii) The volume of molecules is negligible in comparison to the volume of gas. (The volume of molecules is only 0.014% of the volume of the gas).

(iv) The density of gas is constant at all points of the container.

(v) The molecules obey Newton's laws of motion, but as a whole, their motion is isotropic: any molecule can move randomly in all possible directions with all possible velocities.

(vi) The time spent in a collision between two molecules is negligible in comparison to time between two successive collisions. The number of collisions per unit volume in a gas remains constant.

(vii) No attractive or repulsive force acts between gas molecules.

(viii) Molecules constantly collide with the walls of container due to which their momentum changes. The change in momentum is transferred to the walls of the container. Consequently, pressure is exerted by gas molecules on the walls of container.

PRESSURE OF AN IDEAL GAS

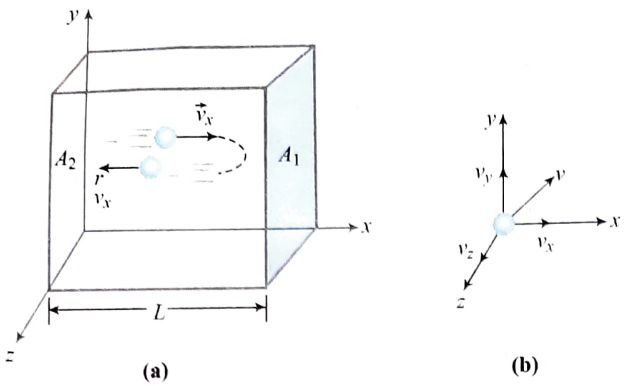
Consider a collection of N molecules of an ideal gas in a cubical container with edges of length L.

Let us consider that one of these molecules of mass m is moving with velocity $\vec{v}$ in any direction.

Then $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, where $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

Due to random motion of molecule, $v_x = v_y = v_z$

$\Rightarrow \quad v^2 = 3v_x^2 = 3v_y^2 = 3v_z^2$



When the molecule collides elastically with any wall, its velocity component perpendicular to the wall is reversed. Let this molecule collide with the shaded wall (A_1) with velocity v_x and rebounds with velocity $-v_x$.

The change in momentum of the molecule

$$\Delta p = (-mv_x) - (mv_x) = -2mv_x$$

As the momentum remains conserved in a collision, the change in momentum of the wall A_1, $\Delta p = 2\,mv_x$.

After rebounding this molecule travels towards opposite wall A_2 with velocity $-v_x$, collides to it and again rebounds with velocity v_x towards wall A_1.

Time between two successive collisions with the wall A_1,

$$\Delta t = \frac{\text{Distance travelled by molecule between two successive collisions}}{\text{Velocity of molecule}} = \frac{2L}{v_x}$$

Therefore, number of collisions per second, or collision frequency, $n = \frac{1}{\Delta t} = \frac{v_x}{2L}$

Force exerted by a single molecule on shaded wall is equal to rate at which the momentum is transferred to the wall by this molecule.

i.e., $$F_{\text{Single molecule}} = \frac{\Delta p}{\Delta t} = \frac{2mv_x}{(2L/v_x)} = \frac{mv_x^2}{L}$$

Total force on the wall A_1 due to all the molecules,

$$F_x = \frac{m}{L}\sum v_x^2 = \frac{m}{L}(v_{x_1}^2 + v_{x_2}^2 + v_{x_3}^2 + \cdots) = \frac{mN}{L}(\overline{v_x^2})$$

$\overline{v_x^2}$ = Mean square of x component of the velocity

Now, pressure is defined as force per unit area, hence pressure on shaded wall, $P_x = \frac{F_x}{A} = \frac{mN}{AL}(\overline{v_x^2}) = \frac{mN}{V}(\overline{v_x^2})$

For any molecule, the mean square velocity, $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2}$;

By symmetry, we have

$$\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$$

$$\Rightarrow \quad \overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = \frac{\overline{v^2}}{3}$$

Total pressure inside the container,

$$P = \frac{1}{3}\frac{mN}{V}\overline{v^2} = \frac{1}{3}\frac{mN}{V}v_{\text{rms}}^2 \qquad \left(\text{where } v_{\text{rms}} = \sqrt{\overline{v^2}}\right)$$

Also, we can write $PV = \frac{1}{3}mN\,v_{\text{rms}}^2$

Effect of mass, volume and temperature on pressure: We hav

$$P = \frac{1}{3}\frac{mN}{V}v_{\text{rms}}^2$$

or $$P \propto \frac{(mN)T}{V} \qquad [\text{As } v_{\text{rms}}^2 \propto T]$$

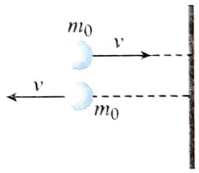
Following conclusions can be drawn

- We have found the pressure on one face of the vessel bu it is the same on all of its faces.
- If volume and temperature of a gas are constant, the $P \propto mN$, i.e., Pressure $\propto$ (Mass of gas), i.e., if mass o gas is increased, number of molecules and hence number of collisions per second increases i.e., pressur increases.
- If mass and temperature of a gas are constant, the $P \propto (1/V)$ i.e., if volume decreases, number of collision per second will increase due to lesser effective distanc between the walls resulting in greater pressure.
- If mass and volume of gas are constant, then $P \propto (1/T$ i.e., if temperature increases, the mean square speed of ga molecules will increase and as gas molecules are movin faster, they will collide with the walls more often wit greater momentum resulting in greater pressure.

ILLUSTRATION 3.2

In a period of 1.00 s, 5.00×10^{23} nitrogen molecules strike a wall with an area of 8.00 cm^2. Assume that the molecules move with a speed of 300 m/s and strike the wall head-on in elastic collisions. What is the pressure exerted on the wall? (The mass of one N_2 molecule is 4.65×10^{-26} kg.)

Sol. To find the pressure exerted by the nitrogen molecule we first calculate the average force exerted by the molecules.



$$\overline{F} = Nm_0\frac{\Delta v}{\Delta t}$$

Here $N = 5.00 \times 10^{23}$, $m_0 = 4.65 \times 10^{-26}$ kg

$\Delta v = [300 - (-300)] = 2 \times 300$ m/s

$$= \frac{(5.00\times10^{23})\left[(4.65\times10^{-26}\text{ kg})\,2(300\text{ m/s})\right]}{1.00\text{ s}} = 14.0\text{ N}$$

Then pressure, $P = \frac{\overline{F}}{A} = \frac{14.0\text{ N}}{8.00\times10^{-4}\text{ m}^2} = 17.4\text{ kPa}$

VARIOUS TYPES OF SPEEDS OF GAS MOLECULES

The motion of molecules in a gas is characterised by any of following three speeds:

Root mean square speed: It is defined as the square root mean of squares of the speed of different molecules.

i.e., $$v_{\text{rms}} = \sqrt{\frac{v_1^2 + v_2^2 + v_3^2 + v_4^2 + \cdots}{N}} = \sqrt{\overline{v^2}}$$

From the expression of pressure, $P = \frac{1}{3}\rho v_{rms}^2$, we get

$$v_{rms} = \sqrt{\frac{3P}{\rho}} = \sqrt{\frac{3PV}{\text{Mass of gas}}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3kT}{m}}$$

Important Points:

- With rise in temperature, rms speed of gas molecules increases as $v_{rms} \propto \sqrt{T}$.
- With increase in molecular weight, rms speed of gas molecule decreases as $v_{rms} \propto \frac{1}{\sqrt{M}}$. e.g., rms speed of hydrogen molecules is four times that of oxygen molecules at the same temperature.
- The rms speed of gas molecules does not depends on the pressure of gas (if temperature remains constant) because $P \propto \rho$ (Boyle's law) if pressure is increased n times then density will also increase by n times but v_{rms} will remain constant.

Most probable speed: The particles of a gas have a range of speeds. This is defined as the speed which is possessed by maximum fraction of total number of molecules of the gas, e.g., if speeds of 10 molecules of a gas are 1, 2, 2, 3, 3, 3, 4, 5, 6, 6 km/s, then the most probable speed is 3 km/s, as maximum fraction of total molecules possess this speed.

Most probable speed, $v_{mp} = \sqrt{\frac{2P}{\rho}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2kT}{m}}$

Average speed: It is the arithmetic mean of the speeds of molecules in a gas at given temperature.

$$v_{av} = \frac{v_1 + v_2 + v_3 + v_4 + \cdots}{N}$$

and according to kinetic theory of gases

Average speed $v_{av} = \sqrt{\frac{8P}{\pi\rho}} = \sqrt{\frac{8}{\pi}\frac{RT}{M}} = \sqrt{\frac{8}{\pi}\frac{kT}{m}}$

ILLUSTRATION 3.3

Two molecules of a gas have speeds of 9×10^6 ms^{-1} and 1×10^6 ms^{-1} respectively. What is the root mean square speed of these molecules?

Sol. For n-molecules, we know that

$$v_{rms} = \sqrt{\frac{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}{n}}$$

[v_{rms} = root mean square velocity]

where $v_1, v_2, v_3, \ldots, v_n$ are individual velocities of n-molecules of the gas.

For two molecules, $v_{rms} = \sqrt{\frac{v_1^2 + v_2^2}{2}}$

Given, $v_1 = 9 \times 10^6$ m/s and $v_2 = 1 \times 10^6$ m/s

$$\therefore \quad v_{rms} = \sqrt{\frac{(9\times10^6)^2 + (1\times10^6)^2}{2}} = \sqrt{41} \times 10^6 \text{ m/s}$$

ILLUSTRATION 3.4

The molecules of a given mass of a gas have root mean square speeds of 100 ms^{-1} at 27°C and 1.00 atmospheric pressure. What will be the root mean square speeds of the molecules of the gas at 127°C and 2.0 atmospheric pressure?

Sol. We know that for a given mass of a gas $v_{rms} = \sqrt{\frac{3RT}{M}}$

where R is gas constant, T is temperature in Kelvin and M is molar mass of the gas.

Clearly for a given gas, $v_{rms} \propto \sqrt{T}$, as R and M are constants.

Hence, $\frac{(v_{rms})_1}{(v_{rms})_2} = \sqrt{\frac{T_1}{T_2}}$

Given, $(v_{rms})_1 = 100$ m/s

$T_1 = 27°C = 27 + 273 = 300$ K

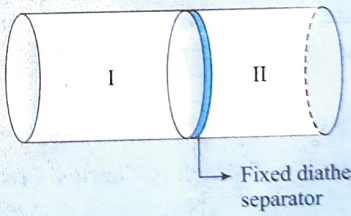
$T_2 = 127°C = 127 + 273 = 400$ K

Thus, from Eq. (i), $\frac{100}{(v_{rms})_2} = \sqrt{\frac{300}{400}} = \frac{\sqrt{3}}{2}$

$$\Rightarrow \quad (v_{rms})_2 = \frac{2\times100}{\sqrt{3}} = \frac{200}{\sqrt{3}} \text{ m/s}$$

ILLUSTRATION 3.5

Figure shows a vessel portioned by a fixed diathermic separator. Different ideal gases are filled in the two parts. The rms speed of the molecules in the left part equal the mean speed of the molecules in the right part. Calculate the ratio of the mass of a molecule in the left part and the mass of a molecule in the right part.



Sol. The separator is diathermal which means that the temperature of gas on both parts of vessel should be equal. Let this common temperature be T.

Hence, rms speed of a molecule of the gas in left part,

$$v_{rms} = \sqrt{\frac{3kT}{m_1}} \qquad \ldots(i)$$

Here, m_1 is the mass of a molecule in left part.

Similarly, the mean speed of a molecule of the gas in right part,

$$v_{mean} = \sqrt{\frac{8kT}{\pi m_2}} \qquad \ldots(ii)$$

Here, m_2 is the mass of a molecule in right part.

As per question, $v_{rms} = v_{mean}$

$$\Rightarrow \quad \sqrt{\frac{3kT}{m_1}} = \sqrt{\frac{8kT}{\pi m_2}}. \quad \text{Hence, } \frac{m_1}{m_2} = \frac{3\pi}{8}$$

DERIVATION OF GAS LAWS FROM KINETIC THEORY OF GASES

BOYLE'S LAW

At a given temperature, the pressure of a given mass of a gas is inversely proportional to its volume. This is known as Boyle's law.

From kinetic theory of gases, we have

$$PV = \frac{1}{3}mN v_{rms}^2 \qquad \ldots(i)$$

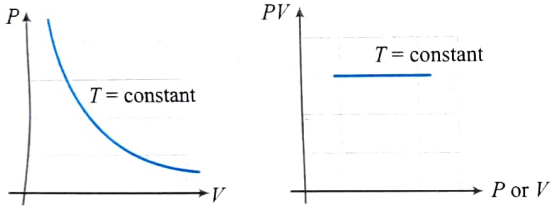
As for a given gas $v_{rms}^2 \propto T$, the value of v_{rms}^2 is constant at a given temperature. Also, for a given mass of the gas, m and N are constants. Thus, from (i), PV = Constant or, $P \propto \frac{1}{V}$ which is Boyle's law.

Also, we can write $PV = P\left(\frac{m}{\rho}\right)$ = constant

m = mass of the gas and ρ is the density of gas

$$\Rightarrow \quad \frac{P}{\rho} = \text{constant} \quad \text{or} \quad \frac{P_1}{\rho_1} = \frac{P_2}{\rho_2}$$

Thus, P–V graph is a rectangular hyperbola. Or PV versus P or V graph is a straight line parallel to P or V axis.



CHARLES'S LAW

At a given pressure, the volume of a given mass of a gas is proportional to its absolute temperature. This is known as Charles' law.

From kinetic theory of gases, if P is constant, $V \propto v_{rms}^2$

As $v_{rms}^2 \propto T$, we get $V \propto T$ which is Charles' law.

$$\Rightarrow \quad \frac{V}{T} = \text{constant} \Rightarrow \frac{V_1}{T_1} = \frac{V_2}{T_2}$$

$$\frac{V}{T} = \frac{m}{\rho T} = \text{constant} \qquad \left(\text{As volume } V = \frac{m}{\rho}\right)$$

or $\quad \rho T = \text{constant}$

$$\Rightarrow \quad \rho_1 T_1 = \rho_2 T_2$$

Thus, V–T graph is a straight line passing through origin. Or V/T versus V or T graph is a straight line parallel to V or T axis.

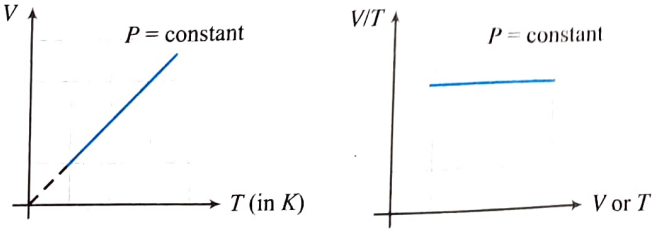


ILLUSTRATION 3.6

The volume of a given mass of a gas at 27°C, 1 atm is 100 cc. What will be its volume at 327°C?

Sol. We have to convert the given temperatures in Kelvin.

If pressure of a given mass of the gas is kept constant, then $V \propto T$

$$\Rightarrow \quad \frac{V}{T} = \text{constant} \qquad \begin{bmatrix} V = \text{Volume of gas} \\ T = \text{Temperature of gas} \end{bmatrix}$$

$$\Rightarrow \quad \frac{V_1}{T_1} = \frac{V_2}{T_2} \Rightarrow V_2 = V_1\left(\frac{T_2}{T_1}\right)$$

Here, we must convert the given temperature in Kelvin.

$T_1 = 273 + 27 = 300$ K

$T_2 = 273 + 327 = 600$ K

But $V_1 = 100$ cc

$$V_2 = V_1\left(\frac{600}{300}\right) \Rightarrow V_2 = 2V_1$$

$$\therefore \quad V_2 = 2 \times 100 = 200 \text{ cc}$$

GAY-LUSSAC'S LAW OR PRESSURE LAW

At a given volume, the pressure of a given mass of gas is proportional to its absolute temperature. It is also known as **Charles's law for pressure**.

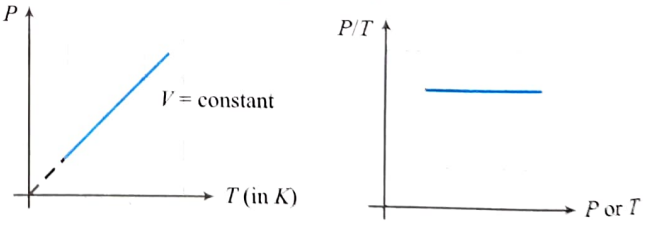
In fact, this is the definition of the absolute temperature T. If one starts from the fact that $v_{rms}^2 \propto T$ and uses the fact that V is constant, one gets from kinetic theory of gases,

$$P \propto v_{rms}^2 \quad \text{or} \quad P \propto T$$

The volume remaining constant, the pressure of a given mass of a gas is directly proportional to its absolute temperature.

$$\text{or} \quad \frac{P}{T} = \text{constant} \quad \Rightarrow \quad \frac{P_1}{T_1} = \frac{P_2}{T_2}$$

Thus, P–T graph is a straight line passing through origin or P/T versus P or T graph is a straight line parallel to P or T axis.



AVOGADRO'S LAW

At the same temperature and pressure, equal volumes of all gases contain equal number of molecules. This is known as Avogadro's law.

Consider equal volumes of two gases kept at the same pressure and temperature. Let,

m_1 = mass of a molecule of the first gas

m_2 = mass of a molecule of the second gas

N_1 = number of molecules of the first gas

N_2 = number of molecules of the second gas

P = common pressure of the two gases

V = common volume of the two gases

From kinetic theory of gases, we have

$$PV = \frac{1}{3} N_1 m_1 v_1^2 \text{ and } PV = \frac{1}{3} N_2 m_2 v_2^2$$

where v_1 and v_2 are rms speeds of the molecules of the first and the second gas respectively. Thus,

$$N_1 m_1 v_1^2 = N_2 m_2 v_2^2 \quad \ldots(i)$$

As the temperatures of the gases are the same, the average kinetic energy of the molecules is same for the two gases, i.e.,

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2 \quad \ldots(ii)$$

Form Eqs. (i) and (ii), $N_1 = N_2$ which proves Avogadro's law.

GRAHAM'S LAW OF DIFFUSION

When two gases at the same pressure and temperature are allowed to diffuse into each other, the rate of diffusion of each gas is inversely proportional to the square root of the density of the gas. This is known as Graham's law of diffusion.

It is reasonable to assume that the rate of diffusion is proportional to the rms speed of the molecules of the gas. Then if r_1 and r_2 be the rates of diffusion of the two gases,

$$\frac{r_1}{r_2} = \frac{v_{1,\text{rms}}}{v_{2,\text{rms}}} \quad \ldots(i)$$

From kinetic theory of gases, we have $v_{\text{rms}} = \sqrt{\frac{3P}{\rho}}$

If the pressure of the two gases are the same, $\frac{v_{1,\text{rms}}}{v_{2,\text{rms}}} = \sqrt{\frac{\rho_2}{\rho_1}}$

so that from (i) $\frac{r_1}{r_2} = \sqrt{\frac{\rho_2}{\rho_1}}$ which is Graham's law.

DALTON'S LAW OF PARTIAL PRESSURE

Dalton's law of partial pressure says that the pressure exerted by a mixture of several gases equals the sum of the pressures exerted by each gas occupying the same volume as that of the mixture.

In kinetic theory, we assume that the pressure exerted by a gas on the walls of a container is due to the collisions of the molecules with the walls. The total force on the wall is the sum of the forces exerted by the individual molecules. Suppose there are N_1 molecules of gas 1, N_2 molecules of gas 2 etc. in the mixture.

Thus, the force on a wall of surface area A,

F = force by N_1 molecules of gas 1 + force by N_2 molecules of gas 2 + $\ldots = F_1 + F_2 + \ldots$

Thus, the pressure, $P = \frac{F_1}{A} + \frac{F_2}{A} + \ldots$

If the first gas alone is kept in the container, its N_1 molecules will exert a force F_1 on the wall. If the pressure in this case is P_1,

Then $P_1 = \frac{F_1}{A}$, Similarly for other gases.

Thus, $P = P_1 + P_2 + P_3 + \ldots$

ILLUSTRATION 3.7

A vessel of volume 1660 cm^3 contains 0.1 mole of oxygen and 0.2 mole of nitrogen. If the temperature of the mixture is 300 K, find its pressure.

Sol. We have $PV = nRT \Rightarrow P = \frac{nRT}{V}$

The partial pressure due to oxygen,

$$P_1 = \frac{(0.1\,\text{mol})(8.3\,\text{J/mol-K})(300\,\text{K})}{(1660 \times 10^{-6}\,\text{m}^3)} = 1.5 \times 10^5\,\text{Pa}$$

Similarly, the partial pressure due to nitrogen, $P_2 = 3.0 \times 10^5$ Pa

The total pressure in the vessel,

$P = P_1 + P_2 = (1.5 + 3.0) \times 10^5\,\text{Pa} = 4.5 \times 10^5\,\text{Pa}$

EQUATION OF STATE OR IDEAL GAS EQUATION

Experimenters have found, though, that if we confine 1 mol samples of various gases in boxes of identical volume and hold the gases at the same temperature, then their measured pressures are almost the same, and at lower densities the differences tend to disappear. Further experiments show that, at low enough densities, all real gases act as ideal gas and tend to obey the following relation.

$$PV = nRT \quad \ldots(i)$$

Here, P is the absolute (not gauge) pressure, n is the number of moles of gas present, and T is the temperature in Kelvins. The symbol R is a constant called the gas constant that has the same value for all gases—namely, R= 8.31 J/mol-K.

Equation (i) is called the **ideal gas law or equation of state**. Provided the gas density is low, this law holds for any single gas or for any mixture of different gases. (For a mixture, n is the total number of moles in the mixture.)

We can rewrite Eq. (i) in an alternative form, in terms of a constant called the **Boltzmann constant** k, which is defined as

$$k = \frac{R}{N_A} = \frac{8.31\,\text{J/mol-K}}{6.02 \times 10^{23}\,\text{mol}^{-1}} = 1.38 \times 10^{-23}\,\text{J/K}$$

This allows us to write $R = kN_A$ and we have $n = \frac{N}{N_A}$

From Eq. (i), $PV = \left(\frac{N}{N_A}\right)(kN_A)T \Rightarrow PV = NkT$

Important Points:

- One mole of an ideal gas at STP has a volume of 22.4 litres (= 22.4 × 10^{-3} m^3). It can be shown by substituting P = 1.013 Pa, R = 8.31 J/mol, T = 273 K and n = 1 in $PV = nRT$.
- If the pressure is expressed in atmospheres and the volume in litres (1 L = 10^{-3} m^3), then

$$R = \frac{PV}{T} = \frac{(1\,\text{atm})(22.4\,\text{L})}{(273\,\text{K})} = 0.0821\,\text{L atm/mol-K}$$

ILLUSTRATION 3.8

We have 0.5 g of hydrogen gas in a cubic chamber of side 3 cm kept at NTP. The gas in the chamber is compressed keeping the temperature constant till a final pressure of 100 atm. Is one justified in assuming the ideal gas law, in the final state? (Hydrogen molecules can be considered as spheres of radius 1 Å).

Sol. For justification of assuming the ideal gas law, the molecular volume and the volume occupied by the ideal gas is compared. The ideal gas law is applicable if the volume of molecules is negligible in comparison to the volume of gas.
It is given hydrogen molecules as spheres of radius 1 Å.

The volume of hydrogen molecules, $V = \frac{4}{3}\pi r^3$

$\Rightarrow \quad V = \frac{4}{3}(3.14)(10^{-10})^3 \simeq 4\times10^{-30}\,\text{m}^3$

The number of moles of Hydrogen,

$$n = \frac{\text{mass of gas}}{\text{molecular mass of gas}} = \frac{0.5}{2} = 0.25$$

Molecules of H_2 present
= Number of moles of H_2 present × Avogadro number
$\Rightarrow \quad N = 0.25 \times 6.023 \times 10^{23}$
$\therefore$ Volume of molecules present, $V_m = N \times V$
$\Rightarrow \quad V_m = 0.25 \times 6.023 \times 10^{23} \times 4 \times 10^{-30} \approx 6 \times 10^{-7}\ \text{m}^3$

Now, using ideal gas equation $P_iV_i = P_fV_f$

$$V_f = \left(\frac{P_i}{P_f}\right)V_i = \left(\frac{1}{100}\right)(3\times10^{-2})^3 = 2.7\times10^{-7}\,\text{m}^3$$

Hence, on compression the volume of the gas (2.7×10^{-7} m^3) is of the order of the molecular volume (6×10^{-7} m^3). The intermolecular forces will play role and the gas will deviate from ideal gas behaviour.

ILLUSTRATION 3.9

A vessel of volume 2×10^{-2} m^3 contains a mixture of hydrogen and helium at 47°C temperature and 4.15×10^5 N/m^2 pressure. The mass of the mixture is 10^{-2} kg. Calculate the masses of hydrogen and helium in the given mixture.

Sol. Let mass of H_2 is m_1 and He is m_2.

$\therefore \quad m_1 + m_2 = 10^{-2}\ \text{kg} = 10\times10^{-3}$ ...(i)

Let P_1, P_2 are partial pressure of H_2 and He.

$$P_1 + P_2 = 4.15\times10^5\ \text{N/m}^2 \text{ for the mixture}$$

$$(P_1 + P_2)V = \left(\frac{m_1}{n_1} + \frac{m_2}{n_2}\right)RT$$

$\Rightarrow \quad 4.15 \times 10^5 \times 2 \times 10^{-2} = \left(\frac{m_1}{2\times10^{-3}} + \frac{m_2}{4\times10^{-3}}\right)8.31\times320$

$\Rightarrow \quad \frac{m_1}{2} + \frac{m_2}{4} = \frac{4.15\times2}{8.31\times320} = 0.00312 = 3.12\times10^{-3}$

$\Rightarrow \quad 2m_1 + m_2 = 12.48 \times 10^{-3}$ kg ...(ii)

Solving Eqs. (i) and (ii) $m_1 = 2.48 \times 10^{-3}$ kg $\simeq 2.5 \times 10^{-3}$ kg
and $m_2 = 7.5 \times 10^{-3}$ kg

RELATION BETWEEN PRESSURE AND KINETIC ENERGY

As we know $P = \frac{1}{3}\frac{mN}{V}v_{rms}^2 = \frac{1}{3}\left(\frac{M}{V}\right)v_{rms}^2$

$\Rightarrow \quad P = \frac{1}{3}\rho v_{rms}^2$ ...(i)

[As $M = mN$ = Total mass of the gas and density, $\rho = \frac{M}{V}$]

Thus, K.E. per unit volume $E = \frac{1}{2}\left(\frac{M}{V}\right)v_{rms}^2 = \frac{1}{2}\rho v_{rms}^2$ ...(ii)

From Eqs. (i) and (ii), we get $P = \frac{2}{3}E$ ...(iii)

i.e., the pressure exerted by an ideal gas is numerically equal to two third of the mean kinetic energy of translation per unit volume of the gas.
If volume of gas is V, then total kinetic energy (or internal energy) of an ideal gas can be related as $PV = \frac{2}{3}E_{total}$.

AVERAGE TRANSLATIONAL KINETIC ENERGY PER MOLECULE OF GAS

In ideal gases, the molecules are considered as point particles. For point particles, there is no internal excitation, no vibration and no rotation. The point particles can have only translational motion and thus only translational energy. For an ideal gas the internal energy can only be translational kinetic energy.

From microscopic consideration based on kinetic theory of gases,

$$P = \frac{1}{3}\left(\frac{Nm}{V}\right)v_{rms}^2 \quad \text{or} \quad PV = \frac{1}{3}Nmv_{rms}^2$$ ...(i)

From macroscopic consideration based on experimental observation of gases,

$$PV = nRT$$ ...(ii)

From Eqs. (i) and (ii), $\frac{1}{3}Nmv_{rms}^2 = nRT$

or $\frac{1}{2}mv_{rms}^2 = \frac{3}{2}\frac{nRT}{N} = \frac{3}{2}\left(\frac{R}{N_A}\right)T$

(N_A = number of molecules per mole = N/n)

While dealing with energies of molecules, we denote the ratio R/N_A by the symbol k_B which is called the Boltzmann constant.

Hence, we can write $\frac{1}{2}mv_{rms}^2 = \frac{3}{2}k_BT$

$\frac{1}{2}mv_{rms}^2$ is the average kinetic energy of translation $\overline{K}$ of a gas molecule.

Thus, the average kinetic energy of translation of a gas molecule $\left(\text{i.e., } \frac{1}{2}mv_{rms}^2 = \overline{K}\right)$ depends on its temperature and is independent of pressure, volume or nature of the gas and is given by $\frac{3}{2}k_BT$.

Obviously, molecules of different gases (say He, H_2, O_2, etc) at the same temperature possess same value of $\overline{K}$ though their rms speeds are different as $v_{rms} \propto \frac{1}{\sqrt{\rho}}$. In fact, for an ideal gas, $\overline{K}$ is directly proportional to the absolute temperature.

Important Points:

- Kinetic energy (or internal energy) of 1 mole ideal gas, $E=\frac{1}{2}Mv_{rms}^2=\frac{1}{2}M\times\frac{3RT}{M}=\frac{3}{2}RT$. Hence, kinetic energy per mole of gas depends only upon the temperature of gas.
- Kinetic energy per molecule of gas does not depend upon the mass of the molecule but only depends upon the temperature of the gas. As $E=\frac{3}{2}kT$ or $E\propto T$, i.e., molecules of different gases, say He, H_2 and O_2 etc., at same temperature, will have same translational kinetic energy though their rms speeds are different.
- Kinetic energy per gram of gas depends upon the temperature as well as molecular weight (or mass of one molecule) of the gas.

$$E_{gram}=\frac{3}{2}\frac{k}{m}T \Rightarrow E_{gram}\propto\frac{T}{m}$$

ILLUSTRATION 3.10

(a) How many atoms of helium gas fill a spherical balloon of diameter 30.0 cm at 20.0°C and 1.00 atm? **(b)** What is the average kinetic energy of the helium atoms? **(c)** What is the rms speed of the helium atoms?

Sol.

(a) The volume of helium gas in the balloon,

$$V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi(0.150\text{ m})^3=1.41\times10^{-2}\text{ m}^3.$$

Number of moles can be obtained from ideal gas equation,

$$PV=nRT \Rightarrow n=\frac{PV}{RT}$$

$$n=\frac{(1.013\times10^5\text{ N/m}^2)(1.41\times10^{-2}\text{ m}^3)}{(8.314\text{ N.m/mol-K})(293\text{ K})}=0.588\text{ mol}$$

The number of molecules of helium gas,

$N = nNA = (0.588\text{ mol})(6.02\times10^{23}\text{ molecules/mol})$

$\Rightarrow \quad N=3.54\times10^{23}$

(b) The average kinetic energy,

$$\overline{K}=\frac{1}{2}m_0v_{rms}^2=\frac{3}{2}k_BT$$

$$\Rightarrow \quad \overline{K}=\frac{3}{2}(1.38\times10^{-23}\text{ J/K})(293\text{ K})=6.07\times10^{-21}\text{ J}$$

(c) The mass of an atom of helium gas,

$$m_0=\frac{M}{N_A}=\frac{4.0026\text{ g/mol}}{6.02\times10^{23}\text{ molecules/mol}}$$

$$=6.65\times10^{-24}\text{ g}=6.65\times10^{-27}\text{ kg}$$

So the root-mean-square speed, $v_{rms}=\sqrt{\frac{2\overline{K}}{m_0}}$

$$\Rightarrow \quad v_{rms}=\sqrt{\frac{2\times6.07\times10^{-27}\text{ J}}{6.65\times10^{-27}\text{ kg}}}=1.35\text{ km/s}$$

ILLUSTRATION 3.11

A cylinder contains a mixture of helium and argon gases in equilibrium at 150°C.

(a) What is the average kinetic energy for each type of gas molecule?

(b) What is the rms speed of each type of molecule?

Sol.

(a) Both kinds of molecules have the same average kinetic energy as the temperature is same for both the gases.

The average kinetic energy of a molecule, $\overline{K}=\frac{3}{2}k_BT$

$$\Rightarrow \quad \overline{K}=\frac{3}{2}(1.38\times10^{-23}\text{ J/K})(423\text{ K})=8.76\times10^{-21}\text{ J}$$

(b) The root-mean-square velocity can be calculated from the kinetic energy:

We know that $\overline{K}=\frac{1}{2}m_0v_{rms}^2$

$$\Rightarrow \quad v_{rms}=\sqrt{\frac{2\overline{K}}{m_0}}$$

So $$v_{rms}=\sqrt{\frac{1.75\times10^{-21}\text{ J}}{m_0}} \qquad \ldots(i)$$

For helium, mass of a molecule,

$$(m_0)_{He}=\frac{4.00\text{ g/mol}}{6.02\times10^{23}\text{ molecules/mol}}=6.64\times10^{-24}\text{ g}$$

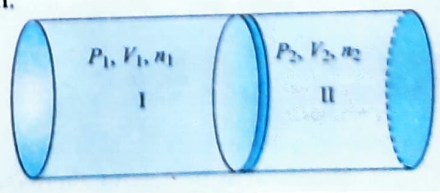
Similarly, mass of a molecule of argon,

$$(m_0)_{Ar}=\frac{39.9\text{ g/mol}}{6.02\times10^{23}\text{ molecule/mol}}=6.63\times10^{-23}\text{ g}$$

Substituting values of masses into Eq. (i), we find root-mean-square velocity for helium, v_{rms} = 1.62 km/s and for argon, v_{rms} = 514 m/s.

ILLUSTRATION 3.12

The container shown in figure has two chambers, separated by a partition, of volumes V_1 = 2.0 L and V_2 =3.0 L. The chambers contain n_1 = 4.0 mol and n_2 = 5.0 mol of an ideal gas at pressures P_1 = 1.00 atm and P_2 = 2.00 atm. Calculate the pressure after the partition is removed and the mixture attains equilibrium.



Sol. Consider the diagram,

For chamber I: $P_1V_1=n_1RT_1$

For chamber II: $P_2V_2=n_2RT_2$

When the partition is removed, the gases get mixed without any loss of energy. The mixture now attains a common equilibrium pressure and the total volume of the system is sum of the volume of individual chambers, V_1 and V_2.

So, $n = n_1 + n_2$ and $V = V_1 + V_2$

From kinetic theory of gases,

The kinetic energy per mole, $E = \frac{3}{2}RT$

For n moles, of gas, $E_{\text{total}} = \frac{3}{2}nRT = \frac{3}{2}PV$

Initially total energy,

$$E_{\text{initial}} = n_1E_1 + n_2E_2 = \frac{3}{2}P_1V_1 + \frac{3}{2}P_2V_2 = \frac{3}{2}(P_1V_1 + P_2V_2)$$

Final total energy, $E_{\text{final}} = \frac{3}{2}P(V_1 + V_2)$

As energy remains constant hence, $E_{\text{initial}} = E_{\text{final}}$

$$\Rightarrow \quad \frac{3}{2}P(V_1 + V_2) = \frac{3}{2}(P_1V_1 + P_2V_2)$$

$$\Rightarrow \quad P = \frac{P_1V_1 + P_2V_2}{V_1 + V_2}$$

$$\Rightarrow \quad P = \left(\frac{1.00 \times 2.0 + 2.00 \times 3.0}{2.0 + 3.0}\right)\text{atm} = \frac{8.0}{5.0} = 1.60\,\text{atm}$$

MAXWELL'S LAW (FOR THE DISTRIBUTION OF MOLECULAR SPEEDS

The molecules of a gas being in a constant state of motion, frequently collide with each other and also with the walls of the container. As a result of this, their velocities continuously change and it can be concluded that molecules of all possible velocities are present in a gas. But at a given temperature, the rms speed (v_{rms}) remains unchanged. James Clerk Maxwell was the first to consider the distribution of molecular speeds and he made the following assumptions:

(i) The gas molecules can have all velocities between 0 to ∞. Of course, we now know that no particle can have speed more than the speed of light, $c = 3 \times 10^8$ m/s.

(ii) Though due to collisions, the speeds of the individual molecules are continuously changing, the fraction of the molecules having their speeds within a certain range remains the same at any instant.

Maxwell gave a mathematical relation based on statistical mechanics for the distribution of molecular speeds which is known as the Maxwell's law of distribution. According to this relation,

$$N_v = 4\pi N\left(\frac{m}{2\pi k_BT}\right)^{3/2} v^2 e^{-mv^2/2k_BT} \qquad \ldots(i)$$

where $N_v = \frac{dN}{dv}$ is called Maxwell–Boltzmann speed distribution function and

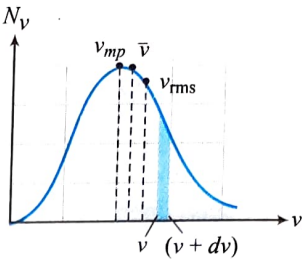
dN = number of molecules with speeds between v and $(v + dv)$

m = mass of gas molecule

k_B = Boltzman constant

N = total number of molecules

The graph representing Maxwell distribution of speed is shown in the figure.



From the graph, following important conclusions emerge:

(i) The area under the graph represents the total number of molecules.

Area under the graph $= \int N_v dv = \int \frac{dN}{dv}dv = N$

(ii) The peak of the curve suggests that there exists a speed corresponding to which the number of molecules is maximum. This speed is called the most probable speed, v_{mp}.

(iii) The shape of the curve is such that area (shown shaded) enclosed by its portion on right side of v_{mp} is more than the area on the left side of v_{mp}. Thus, the number of molecules having speeds less than v_{mp} is less than the number of molecules having speeds more than v_{mp}. Obviously, the average speed ($\bar{v}$) is greater than v_{mp}. It can be shown from Eq. (i) that

$$v_{\text{rms}} = \sqrt{\frac{3k_BT}{m}} = 1.73\sqrt{\frac{k_BT}{m}} \qquad \ldots(ii)$$

$$\bar{v} = \sqrt{\frac{8k_BT}{\pi m}} = 1.60\sqrt{\frac{k_BT}{m}} = 0.92\,v_{\text{rms}} \qquad \ldots(iii)$$

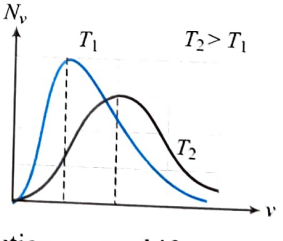
$$v_{\text{mp}} = \sqrt{\frac{2k_BT}{m}} = 1.41\sqrt{\frac{k_BT}{m}} = 0.82\,v_{\text{rms}} \qquad \ldots(iv)$$

Thus, $v_{\text{rms}} > \bar{v} > v_{\text{mp}}$.

(iv) The number of molecules with too low and too high speeds is very small as the function N_v is zero when $v = 0$ and approaches zero as v approaches infinity.

(v) The number of molecules in the range v and $(v + dv)$ is equal to the area of the shaded rectangle.

(vi) As the temperature increases,



(a) the distribution curve shifts to the right indicating that the average speed increases with increasing temperature.

(b) the distribution curve broadens indicating that the range of speed increases.

MEAN FREE PATH

According to kinetic theory of gases, the molecules of a gas are in a state of random motion and collide frequently with each other. The path of an individual molecule is random and resembles that shown in Fig. (a). The path transversed by a molecule between two successive collisions with other molecules is called the free path. Since all the free paths are not of the same length, we define a more useful quantity known as mean free path.

The average distance travelled by a molecule between two successive collisions is called the mean free path.

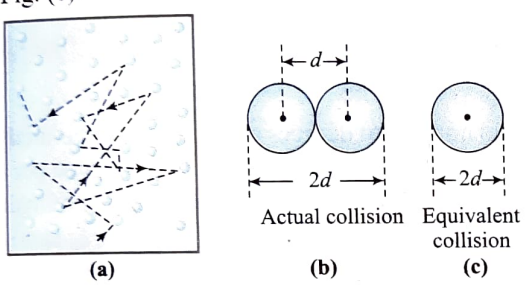
Obviously, mean free path,

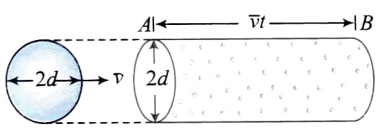
$$\lambda = \frac{\text{Total distance travelled by a gas molecule between successive collisions}}{\text{Total number of collisions}} \quad \ldots(i)$$

To calculate the mean free path, we make the following two assumptions:

(i) Only the particular molecule that we are considering is in motion, all others being at rest.

(ii) The molecules are spheres, each of diameter d. Therefore, a collision will take place when the centers of two molecules are at a distance d apart as shown in Fig. (b). An equivalent description of collisions made by the moving molecule is to regard that this molecule has a diameter $2d$ and all the other molecules at rest are point masses as shown in Fig. (c).





(d)

Let this molecule of diameter $2d$ travel from A to B in time t. Let $\bar{v}$ be the average speed of this molecule. Clearly, it will sweep out a cylinder of diameter $2d$ (i.e., cross-sectional area πd^2) and length $\bar{v}t$ as shown in Fig. (d).

Volume of the cylinder, $AB = \pi d^2(\bar{v}t) = \pi d^2\,\bar{v}t$ ...(ii)

If n is the number of molecules per unit volume, number of molecules in the cylinder $= (\pi d^2 \bar{v}t)n$ ...(iii)

Since the moving molecule meets $(\pi d^2 \bar{v}t)n$ molecules on its way while travelling a distance $\bar{v}t$, it will collide with all these molecules thereby making $(\pi d^2 \bar{v}t)n$ collision in time t.

From Eq. (i), $\lambda = \dfrac{\bar{v}t}{(\pi d^2 \bar{v}t)n} = \dfrac{1}{\pi d^2 n}$ ...(iv)

Equation (iv) has been derived on the assumption that all the molecules in the cylinder are at rest. Actually all these molecules are moving with speeds governed by the Maxwell law of distribution of speeds. Applying this law, it can be shown that

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n} \quad \ldots(v)$$

Collision frequency: Since the number of collisions in time t is $(\pi d^2 \bar{v}t)n$.

Number of collisions per unit time i.e., collision frequency,

$$f = \frac{(\pi d^2 \bar{v}t)n}{t} = \pi d^2 \bar{v} n \quad \ldots(vi)$$

Applying Maxwell's law of distribution of speeds,

$$f = \sqrt{2}\pi d^2 \bar{v} n = \frac{\bar{v}}{\lambda} \quad \ldots(vii)$$

Mean free time: The inverse of collision frequency is the average time between collisions and is called the mean free time.

As $PV = Nk_BT, n = \dfrac{N}{V} = \dfrac{P}{k_BT}$

$$\therefore \quad \lambda = \frac{1}{\sqrt{2}\pi d^2}\frac{k_BT}{P} = \frac{k_BT}{\sqrt{2}\pi d^2 P} \quad \ldots(viii)$$

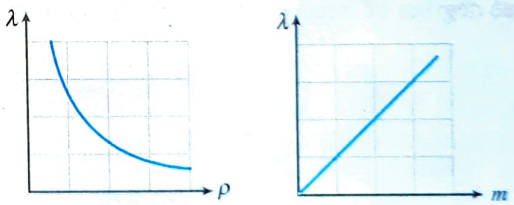
From Eq. (viii) it follows that

- $\lambda \propto \dfrac{1}{d^2}$, i.e., mean free path is inversely proportional to the square of the molecular diameter. Thus, smaller the diameter of the molecule, longer is the mean free path.
- $\lambda \propto T$, i.e., the mean free path varies directly with the absolute temperature of the gas.
- $\lambda \propto \dfrac{1}{P}$, i.e., mean free path varies inversely with the pressure of the gas.

During two successive collisions, a molecule of a gas moves in a straight line with constant velocity.

Important Points:

- As $\lambda \propto \dfrac{1}{\rho}$ and $\lambda \propto m$, i.e., the mean free path is inversely proportional to the density of a gas and directly proportional to the mass of each molecule.



- As $\lambda = \dfrac{1}{\sqrt{2}}\dfrac{kT}{\pi d^2 P}$. For constant volume and hence constant number density n of gas molecules, $\dfrac{P}{T}$ is constant so that λ will not depend on P and T. But if volume of given mass of a gas is allowed to change with P or T then $\lambda \propto T$ at constant pressure and $\lambda \propto \dfrac{1}{P}$ at constant temperature.

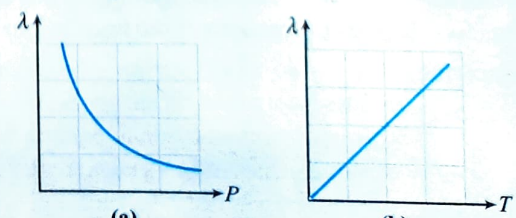


ILLUSTRATION 3.13

Calculate the ratio of the mean free paths of the molecules of two gases having molecular diameters 1Å and 2Å. The gases may be considered under identical conditions of temperature, pressure and volume.

Sol. Mean free path of a molecule is given by $l = \frac{1}{\sqrt{2}d^2 n}$
where, n = number of molecules/volume
d = diameter of the molecule

Now, we can write $l \propto \frac{1}{d^2}$

Given, $d_1 = 1\text{Å}$, $d_2 = 2\text{Å}$

As $l_1 \propto \frac{1}{d_1^2}$ and $l_2 \propto \frac{1}{d_2^2}$ $\Rightarrow \frac{l_1}{l_2} = \left(\frac{d_2}{d_1}\right)^2 = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$

Hence, $l_1 : l_2 = 4 : 1$

DEGREE OF FREEDOM

The term degree of freedom of a system refers to the possible independent motions that system can have.

or

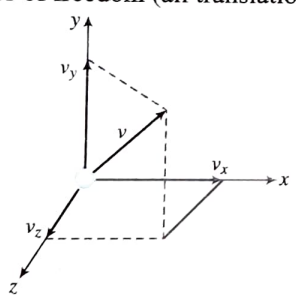
The total number of independent modes (ways) in which a system can possess energy is called the degree of freedom (f).

The independent motions can be translational, rotational or vibrational or any combination of these.

So the degree of freedom are of three types:

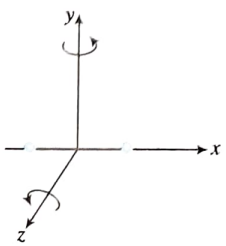
(i) Translational degree of freedom
(ii) Rotational degree of freedom
(iii) Vibrational degree of freedom

Monoatomic gas: Molecule of monoatomic gas can move in any direction in space so it can have three independent motions and hence three degrees of freedom (all translational).

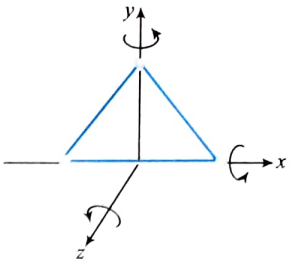


Diatomic gas: Molecules of diatomic gas are made up of two atoms joined rigidly to one another through a bond. This cannot only move bodily, but also rotate about one of the three co-ordinate axes. However its moment of inertia about the axis joining the two atoms is negligible compared to that about the other two axes.

Hence it can have only two rotational motions. Thus a diatomic molecule has five degrees of freedom: three translational and two rotational.



Triatomic gas (non-linear): A non-linear molecule can rotate about any of three co-ordinate axes. Hence it has six degrees of freedom; three translational and three rotational.



The above degrees of freedom are shown at room temperature. Further at high temperature, in case of diatomic or polyatomic molecules, the atoms within the molecule may also vibrate with respect to each other. In such cases, the molecule will have an additional degrees of freedom, due to vibrational motion.

ILLUSTRATION 3.14

Calculate the number of degrees of freedom of molecules of hydrogen in 1 cc of hydrogen gas at NTP.

Sol. Total number of degrees of freedom in a thermodynamical system = Number of degrees of freedom associated per molecule × Number of molecules, As given molecules are of hydrogen,

∴ Volume occupied by 1 mole
= 1 mole of the gas at NTP = 22400 mL = 22400 cc

Any ideal gas has a molar volume of 22400 mL (cc) at NTP.

∴ Number of molecules in 1 cc of hydrogen

$$= \frac{6.023 \times 10^{23}}{22400} = 2.688 \times 10^{19}$$

H_2 is a diatomic gas, having a total of 5 degrees of freedom (3 translational + 2 rotational)

∴ Total degrees of freedom possessed by all the molecules
$= 5 \times 2.688 \times 10^{19} = 1.344 \times 10^{20}$

LAW OF EQUIPARTITION OF ENERGY

We know that whenever the molecules of a gas collide, they share their energies and when thermal equilibrium is reached the average energy of each molecule is the same. The law of equipartition of energy, first deduced by Maxwell in 1869 and later on, improved by Boltzmann, showed as to how much should be the average energy of a gas molecule. This law is remarkable as it is applicable to large collections of atoms, electrons and molecules. We shall, however, confine ourselves to its application to the molecules of a gas.

According to this law, for any system in thermal equilibrium the total energy is equally distributed among its various degrees of freedom. And each degree of freedom is associated with energy $\frac{1}{2}kT$ (where $k = 1.38 \times 10^{-23}$ J/K, T = absolute temperature of the system).

At a given temperature T, all ideal gas molecules no matter what their mass have the same average translational kinetic energy; namely, $\frac{3}{2}kT$. When measure the temperature of a gas we are also measuring the average translational kinetic energy of its molecules.

At same temperature gases with different degrees of freedom (e.g., He and H_2) will have different average energy or internal energy namely $\frac{f}{2}kT$. (f is different for different gases)

Important Points:

Different energies of a system of degree of freedom f are as follows:

- Total energy associated with each molecule = $\frac{f}{2}kT$
- Total energy associated with N molecules = $\frac{f}{2}NkT$
- Total energy associated with one mole = $\frac{f}{2}RT$
- Total energy associated with n moles = $\frac{f}{2}nRT$

ILLUSTRATION 3.15

Calculate

(a) the average kinetic energy of translation of an oxygen molecule at 27°C,

(b) the total kinetic energy of an oxygen molecule at 27°C,

(c) the total kinetic energy in joule of one mole of oxygen at 27°C. Given Avogadro's number = 6.02×10^{23} and Boltzman constant = 1.38×10^{-23} J/(mol-K).

Sol.

(a) An oxygen molecule has three translational degrees of freedom, thus the average translational kinetic energy of an oxygen molecule at 27°C is given as

$$E_T = \frac{3}{2}kT = \frac{3}{2}\times 1.38\times 10^{-23}\times 300$$

$$= 6.21 \times 10^{-21} \text{ J/molecule}$$

(b) An oxygen molecule has total five degrees of freedom, hence its total kinetic energy is given as

$$E_T = \frac{5}{2}kT = \frac{5}{2}\times 1.38\times 10^{-23}\times 300$$

$$= 10.35 \times 10^{-21} \text{ J/molecule}$$

(c) Total kinetic energy of one mole of oxygen is its internal energy, which can be given as

$$U = \frac{f}{2}\mu RT = \frac{5}{2}\times 1\times 8.314\times 300 = 6235.5 \text{ J/mole}$$

ILLUSTRATION 3.16

A gas mixture consists of 2.0 moles of oxygen and 4.0 moles of neon at temperature T. Neglecting all vibrational modes, calculate the total internal energy of the system. (Oxygen has two rotational modes.)

Sol. To find exact value of total energy for a given molecule of a gas, we must know the number of degrees of freedom associated with the molecules of the gas. O_2 is a diatomic gas having 5 degrees of freedom.

Energy (total internal) per mole of the gas = $\frac{5}{2}RT$

For 2 moles of the gas total internal energy

$$= 2\times\frac{5}{2}RT = 5RT \quad \ldots(i)$$

Neon (Ne) is a monoatomic gas having 3 degrees of freedom.

$\therefore$ Energy per mole = $\frac{3}{2}RT$

We have 4 moles of Ne.

Hence, energy $= 4\times\frac{3}{2}RT = 6RT$ ...(ii)

Using Eqs. (i) and (ii), we get

Total energy = $5\,RT + 6\,RT = 11\,RT$

ILLUSTRATION 3.17

A light container having a diatomic gas enclosed within is moving with velocity v. Mass of the gas is M and number of moles is n.

(a) What is the kinetic energy of gas w.r.t. centre of mass of the system?

(b) What is KE of gas w.r.t. ground?

Sol. The kinetic energy of gas w.r.t. centre of mass of the system is the kinetic energy of molecules due to their random motion. This will be the internal energy of the gas.

(a) K.E. = $\frac{5}{2}nRT$

mass of gas = M, temperature T → V

(b) Kinetic energy of gas w.r.t. ground = Kinetic energy of centre of mass w.r.t. ground + Kinetic energy of gas w.r.t. centre of mass.

$$\text{K.E.} = \frac{1}{2}MV^2 + \frac{5}{2}nRT$$

ILLUSTRATION 3.18

An insulated container containing monoatomic gas of molar mass m is moving with a velocity v_0. If the container is suddenly stopped, find the change in temperature.

Sol. As the container is suddenly stopped there is no time for exchange of heat in the process.

According to kinetic interpretation of temperature, absolute temperature of a given sample of a gas is proportional to the total translational kinetic energy of its molecules.

Hence, any change in absolute temperature of a gas will contribute to corresponding change in translational kinetic energy and vice-versa.

Assuming n = number of moles.

Given, m = molar mass of the gas.

When the container stops, its total kinetic energy is transferred to gas molecules in the form of translational kinetic energy, thereby increasing the absolute temperature.

If ΔT = change in absolute temperature.

Then, kinetic energy of molecules due to velocity v_0,

$$\Delta K_{\text{motion}} = \frac{1}{2}(mn)v_0^2 \quad \ldots(i)$$

Increase in translational kinetic energy,

$$\Delta K_{\text{translation}} = n\frac{3}{2}R(\Delta T) \quad \ldots(ii)$$

According to kinetic theory Eqs. (i) and (ii) are equal

$$\Rightarrow \quad \frac{1}{2}(mn)v_0^2 = n\frac{3}{2}R(\Delta T) \quad \Rightarrow \quad (mn)v_0^2 = 3nR(\Delta T)$$

$$\Rightarrow \quad \Delta T = \frac{(mn)v_0^2}{3nR} \quad \Rightarrow \quad \Delta T = \frac{mv_0^2}{3R}$$

ILLUSTRATION 3.19

A balloon has 5.0 mole of helium at 7°C. Calculate:
(a) the number of atoms of helium in the balloon.
(b) the total internal energy of the system.

Sol. Energy associated with a monoatomic molecule is $\frac{3}{2}kT$.
Given, number of moles of helium = 5

$T = 7°C = 7 + 273 = 280$ K

(a) Hence, number of atoms (He is monoatomic)
= Number of moles × Avogadro's number
$= 5 \times 6.023 \times 10^{23} \simeq 3.0 \times 10^{24}$ atoms

(b) Now, average kinetic energy per molecule $\bar{K} = \frac{3}{2}k_BT$
Here, k_B = Boltzmann constant.
(It has only 3 degrees of freedom.)
∴ Total energy of all the atoms = Total internal energy (E)

$$E = \frac{3}{2}k_BT \times \text{number of atoms}$$

$$= \frac{3}{2} \times 1.38 \times 10^{-23} \times 280 \times 3.0 \times 10^{24} = 1.74 \times 10^4 \text{ J}$$

ILLUSTRATION 3.20

Nitrogen of mass 15 g is enclosed in a vessel at temperature T = 300K so that it does not loose any energy to external system. Find the amount of heat required to double the root mean square velocity of these molecules.

Sol. The root mean square velocity of gas molecules is given by, $v_{rms} = \sqrt{\frac{3RT}{M}} \Rightarrow v_{rms} = \sqrt{T}$

The v_{rms} is directly proportional to $\sqrt{T}$, to double rms speed, temperature must be raised four times, hence T_f = 1200 K.
The initial internal energy is given as

$$U_i = \frac{f}{2}nRT = \frac{5}{2}nRT \qquad \text{[As for a diatomic gas } f = 5\text{]}$$

$$\Rightarrow \quad U_i = \frac{5}{2} \times \frac{15}{28} \times 8.314 \times 300 = 3340.45 \text{ Joule}$$

Finally, when gas temperature is raised to 1200 K, its internal energy is given as

$$U_f = \frac{5}{2}nRT_f = \frac{5}{2} \times \frac{15}{28} \times 8.314 \times 1200 = 13361.6 \text{ Joule}$$

Increase in internal energy or amount of heat supplied,

$$\Delta Q = U_f - U_i = 10021.35 \text{ joule}$$

EQUIVALENT DEGREES OF FREEDOM FOR A GASEOUS MIXTURE

We know if two substances at same temperature are connected or mixed, they do not exchange any thermal energy and the temperature of mixture remains same.

If A' gases with degrees of freedom $f_1, f_2, f_3 \ldots f_N$ are mixed with $n_1, n_2, n_3, \ldots n_N$ moles at same temperature T then their total internal energy before mixing can be given as

$$U_i = \frac{f_1}{2}n_1RT + \frac{f_2}{2}n_2RT + \cdots + \frac{f_N}{2}n_NRT \quad \ldots(i)$$

If after homogeneous mixing, for analytical purpose we assume f_{eq} are the number of degrees of freedom for the mixture then after mixing the internal energy of this mixture can be given as

$$U_f = \frac{f_{eq}}{2}(n_1 + n_2 + \cdots n_N)RT \quad \ldots(ii)$$

As no energy loss is taking place during mixing of gases, we have from Eqs. (i) and (ii)

$$U_i = U_f$$

$$\frac{f_1}{2}n_1RT + \frac{f_2}{2}n_2RT + \cdots + \frac{f_N}{2}n_NRT = \frac{f_{eq}}{2}(n_1 + n_2 + \cdots n_N)RT$$

or $$F_{eq} = \frac{f_1n_1 + f_2n_2 + \cdots f_Nn_N}{n_1 + n_2 + \cdots n_N} \quad \ldots(iii)$$

MIXING OF GASES AT CONSTANT VOLUME

When some gases at different temperature are mixed at constant volume in a thermally insulated vessel, the total internal energy of all the gases remains constant. For example if N gases with $n_1, n_2, \ldots n_N$ moles at temperature $T_1, T_2 \ldots T_N$ are mixed in a container and if the gases have degrees of freedom $f_1, f_2 \cdots f_N$ then the total internal energy of gases before mixing is

$$U_i = \frac{f_1}{2}n_1RT + \frac{f_2}{2}n_2RT + \cdots + \frac{f_N}{2}n_NRT_N \quad \ldots(i)$$

If after mixing temperature of mixture become T_f then the total internal energy of gas after mixing of gases is

$$U_f = \frac{f_1}{2}n_1RT_f + \frac{f_2}{2}n_2RT_f + \cdots + \frac{f_N}{2}n_NRT_f \quad \ldots(ii)$$

As total internal energy of the gaseous mixture must remains constant thus we have from Eqs. (i) and (ii),

$$U_i = U_f$$

$$\frac{f_1}{2}n_1RT_f + \frac{f_2}{2}n_2RT_f + \cdots + \frac{f_N}{2}n_NRT_f$$

$$= \frac{f_1}{2}n_1RT_1 + \frac{f_2}{2}n_2RT_2 + \cdots + \frac{f_N}{2}n_NRT_N$$

or $$T_f = \frac{f_1n_1T_1 + f_2n_2T_2 + \cdots f_Nn_NT_N}{f_1n_1 + f_2n_2 + \cdots f_Nn_N} \quad \ldots(iii)$$

Here Eq. (iii) can be used to find the final temperature of a gaseous mixture of different gases after mixing in a thermally insulated container.

CONCEPT APPLICATION EXERCISE 3.1

1. The highest vacuum attained so far is of the order of 10^{-11} mm of mercury. How many molecules are there in 1 cc of a vessel under such a high vacuum at 0°C?
2. A thermally insulated vessel with gaseous nitrogen at a temperature of 27°C moves with velocity 100 m/s^{-1}. How much (in percentage) and in what way will the gas pressure change if the vessel is brought to rest suddenly?
3. At 127°C and 1.00×10^{-2} atm pressure, the density of a gas is 1.24×10^{-2} kg m^{-3}.
   (a) Find v_{rms} for the gas molecules.
   (b) Find the molecular weight of the gas and identify it.
4. The temperature of a gas consisting of rigid diatomic molecules is $T = 300$ K. Calculate the angular root-mean-square velocity of a rotating molecule if its moment of inertia is $I = 2.0 \times 10^{-40}$ kg m^2.
5. A closed container of volume 0.02 m^3 contains a mixture of neon and argon gases at 27°C temperature and 1.0×10^5 N/m^2 pressure. The gram-molecular weights of neon and argon are 20 and 40, respectively. Find the masses of the individual gases in the container. Assuming them to be ideal (R = 8.314 J/mol-K). Total mass of the mixture is 28 g.
6. A glass container encloses a gas at a pressure of 8×10^5 Pa and 300 K temperature. The container walls can bear a maximum pressure of 10^6 Pa. If the temperature of container is gradually increased, find the temperature at which the container will break.
7. Calculate the number of molecules in 1 cc of an ideal gas at 27°C and a pressure of 10 mm of mercury. Mean kinetic energy of molecule at 27°C is 4×10^{-14} erg; the density of mercury is 13.6 g/cc.
8. A spherical balloon of volume V contains helium at a pressure P. How many moles of helium are in the balloon if the average kinetic energy of the helium atoms is $\bar{K}$?
9. A 5.00 L vessel contains nitrogen gas at 27.0°C and 3.00 atm. Find
   (a) the total translational kinetic energy of the gas molecules and
   (b) the average kinetic energy per molecule. [Take atm. pressure = 1.0×10^5 Pa and $N_A = 6.0 \times 10^{23}$]
10. A 7.00 L vessel contains 3.5 moles of gas at a pressure of 1.66×10^6 Pa. Find
    (a) the temperature of the gas;
    (b) the average kinetic energy of the gas molecules in the vessel.
    (c) What additional information would you need if you were asked to find the average speed of the gas molecules? (R = 8.3 J/mol-K, $\bar{K} = 1.38 \times 10^{-23}$ J/K)
11. A spherical balloon of volume 4.00×10^3 cm^3 contains helium at a pressure of 1.20×10^5 Pa. How many moles of helium are in the balloon if the average kinetic energy of the helium atoms is 3.60×10^{-22} J?

ANSWERS

1. 3.5×10^{14} **2.** 2.25%
3. (a) 495 m/s (b) M = 40.6 g/mol Argon
4. 4.5×10^9 rad/s
5. Mass of neon gas = 4.07 g, Mass of argon gas = 23.93 g
6. 375 K **7.** 5×10^{17} molecules **8.** $\frac{3}{2}\frac{pV}{kN_A}$
9. (a) 2.285 kJ (b) 6.22×10^{-21} J
10. (a) 400 K (b) 8.28×10^{-21} J **11.** 3.32 mol

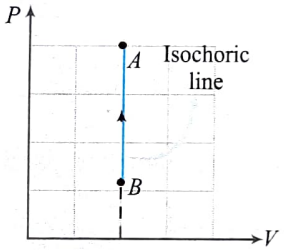
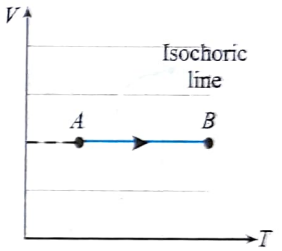
DIFFERENT PROCESSES IN IDEAL GAS

ISOCHORIC PROCESS

If the state of system changes in such a way that its volume remains constant, the process is called isochoric.
Ideal gas equation: $PV = nRT$
In case of isochoric process V = constant.

Hence $P = \left(\frac{nR}{V}\right)T$ or $P \propto T$ ...(i)

Different Graphs for Isochoric Process

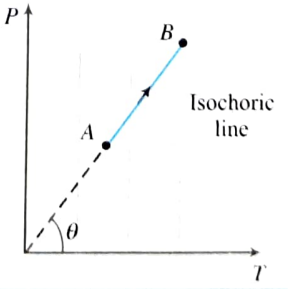
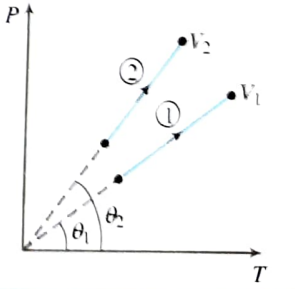
P–V graph	V–T graph
	
As $P \propto T$ Here, $T_A > T_B$	As $P \propto T$ Here, $P_B > P_A$

P–T graph

For isochoric process, we have $P = \left(\frac{nR}{V}\right)T$ or $P \propto T$.

It is clear that P–T graph should be straight line passing through origin having slope equal to $\left(\frac{nR}{V}\right)$.

Its equation will be $P = (\tan\theta)T$ [as $y = mx$]

	
As $P = \left(\frac{nR}{V}\right)T$ Slope of isochoric line, $\tan\theta = \frac{nR}{V}$ or $\tan\theta \propto \frac{1}{V}$	Slope of isochoric line is inversely proportional to volume. In this case, $\theta_2 > \theta_1$ or $\tan\theta_2 > \tan\theta_1$ Hence, $V_2 < V_1$

ISOBARIC PROCESS

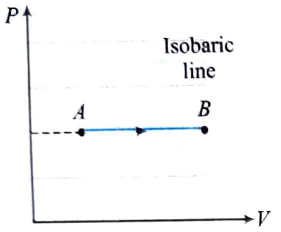
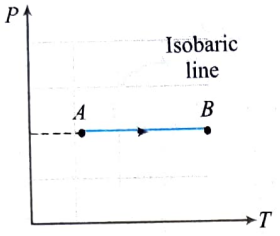
If the state of a system changes in such a way that pressure remains constant, the process is called isobaric.

Ideal gas equation: $PV = nRT$

or $V = \left(\frac{nR}{P}\right)T$ ...(i)

Here, P = constant, hence $V \propto T$

Different Graphs for Isobaric Process

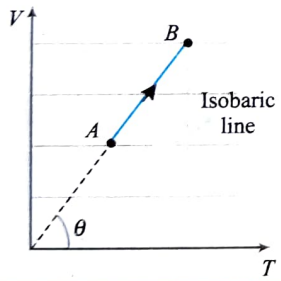
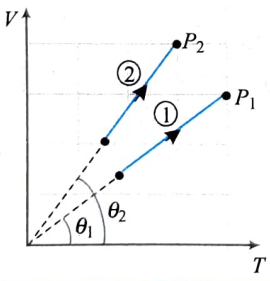
P–V graph	V–T graph
	
As $V \propto T$ Here, $T_B > T_A$	As $V \propto T$ Here, $V_B > V_A$

V–T graph

For isobaric process, we have $V = \left(\frac{nR}{P}\right)T$ or $V \propto T$.

It is clear that V–T graph should be straight line passing through origin having slope equal to $\left(\frac{nR}{P}\right)$.

Its equation will be $V = (\tan\theta)\,T$ [as $y = mx$]

	
As $V = \left(\frac{nR}{P}\right)T$ Slope of isochoric line, $\tan\theta = \frac{nR}{P}$ or $\tan\theta \propto \frac{1}{P}$	Slope of isobaric line is inversely proportional to pressure. In this case, $\theta_2 > \theta_1$ or $\tan\theta_2 > \tan\theta_1$ Hence, $P_2 < P_1$

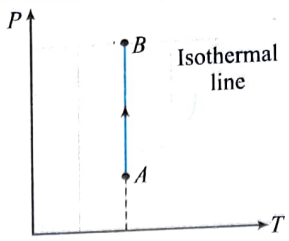
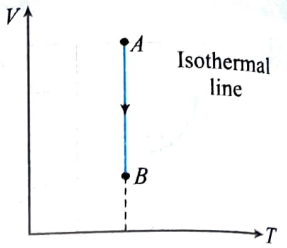
ISOTHERMAL PROCESS

If the state of a system changes in such a way that temperature remains constant, the process is called isothermal.

Ideal gas equation: $PV = nRT$

Here, T = constant

Hence, PV = constant

P–T graph	V–T graph
	
As PV = constant or $V \propto \frac{1}{P}$ Hence, $V_A > V_B$	As $V \propto \frac{1}{P}$ Hence, $P_A < P_B$

Isothermal curve on P–V diagram

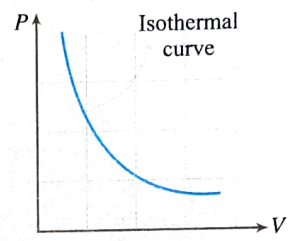
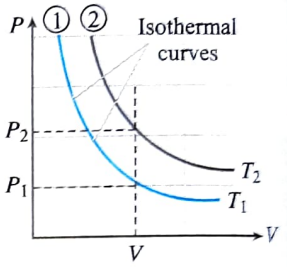
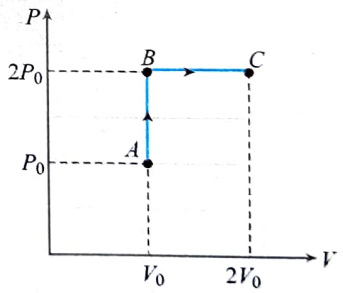
	
As PV is constant, the P-V graph is rectangular hyperbola (xy = constant)	For a given volume of V, $P_2 > P_1$ Hence, $T_2 > T_1$

ILLUSTRATION 3.21

In given P-V diagram, if the temperature of point A is T_0, find the temperature at point B and C. Also, draw P-T and V-T diagrams of given process.



Sol. Considering the process $A \to B$.

The volume is constant, hence the process is isochoric.

For isochoric process: $P = \left(\frac{nR}{V}\right)T$ or $P \propto T$

$$\frac{P_A}{T_A} = \frac{P_B}{T_B} \Rightarrow \frac{P_0}{T_0} = \frac{2P_B}{T_B} \text{ or } T_B = 2T_0$$

Now, consider process $B \to C$.

The pressure is constant, hence process is isobaric.

For isobaric process: $V = \left(\frac{nR}{V}\right)T$ or $V \propto T$

$$\frac{V_B}{T_B} = \frac{V_C}{T_C} \Rightarrow T_C = \left(\frac{V_C}{V_B}\right)T_B \text{ or } T_C = \left(\frac{2V_0}{V_0}\right)(2T_0) = 4T_0$$

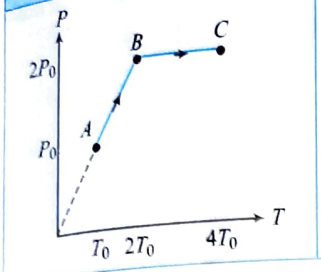
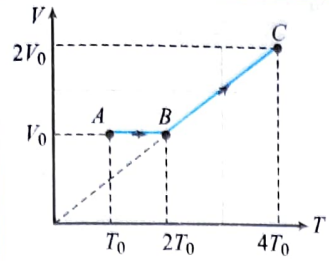
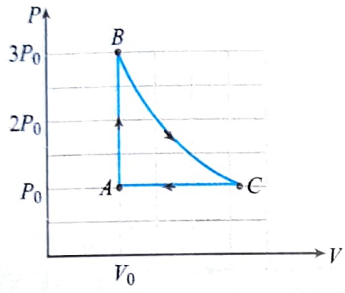
P–T graph	V–T graph
	

ILLUSTRATION 3.22

In given P–V diagram, the temperature at point A is T_0, draw P–T and V–T diagram of given process.



Sol. In process $A \rightarrow B$, the volume is constant, hence the process is isochoric.

For isochoric process: $P = \left(\frac{nR}{V}\right)T$ or $P \propto T$

$$\frac{P_A}{T_A} = \frac{P_B}{T_B} \Rightarrow T_B = \left(\frac{P_B}{T_B}\right)T_A \text{ or } T_B = \left(\frac{3P_0}{P_0}\right)T_0 = 3T_0$$

The process $B \rightarrow C$ is isothermal process.

Hence, $P_BV_B = P_CV_C \Rightarrow V_C = V_B\left(\frac{P_B}{P_C}\right)$

Or $V_C = V_0\left(\frac{3P_0}{P_0}\right) = 3V_0$

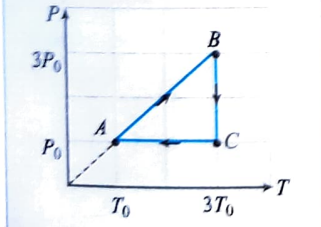
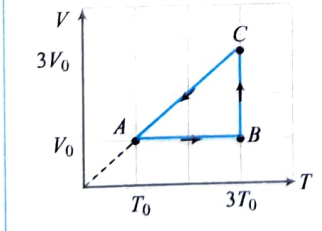
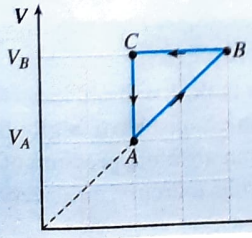
P-T graph	V-T graph
	

ILLUSTRATION 3.23

A cyclic process $ABCA$, shown in V–T diagram, is performed with a constant mass of an ideal gas. Show the same process on a P–V diagram.



Sol. **In process $A \rightarrow B$:** For line AB, we have equation of line $V = aT$ and both V and T increase.

From ideal gas equation $PV = nRT$,

$$P(aT) = nRT \Rightarrow P = \left(\frac{nR}{a}\right) = \text{constant}$$

Hence, it is an isobaric process with V increasing. So in P–V diagram, line AB will be a straight line parallel to V axis (with V increasing)

In process $B \rightarrow C$: The volume is constant, hence the process is isochoric. For line BC, V is constant and T is decreasing, so from the gas equation $PV = nRT$.

$$P = \frac{nRT}{\text{constant}}, \quad \text{or} \quad P \propto T \text{ (with } T \text{ decreasing)}$$

i.e, along line BC, P decreases with V is constant. So, in P–V diagram, line BC will be a straight line parallel to P axis (with P decreasing).

In process $C \rightarrow A$: In this process, temperature is constant, hence it is isothermal process.

From ideal gas equation $PV = nRT$ = constant (with V decreasing).

So, in P–V curve, line CA will be a hyperbola (with P increasing).

The complete cycle on P–V diagram is shown in figure.

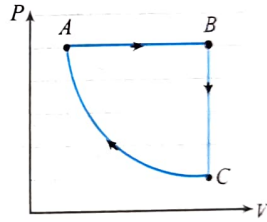
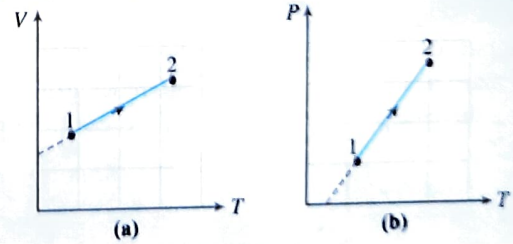


ILLUSTRATION 3.24

The diagrams given depict two different processes for a given mass of an ideal gas. What inference can be drawn regarding the change of pressure and volume of the gas from Figs. (a) and (b) respectively?



Sol.

(a) **Method 1:** As the given process is a straight line with positive slope and positive intercept with V-axis, its equation will be

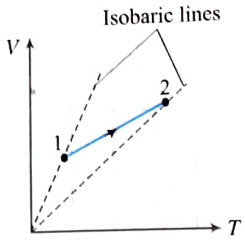
$$V = aT + b \qquad [\text{as } y = mx + c]$$

The ideal gas equation $PV = nRT$ in the light of above yields

$$P = \frac{nRT}{V} = \frac{nRT}{aT + b} = \frac{nR}{a + (b/T)}$$

Now from the given Fig. (a), $T_2 > T_1$ so $P_2 < P_1$, i.e., when the gas in heated its pressure decreases.

Method 2: The slope of the isochoric line is given by (nR/P). The slope is small for larger pressure. Here, point 1 lies on the constant pressure line which forms a smaller angle with the x-axis than the constant pressure line passing through point 2. Hence, pressure of the gas decreases while heating.



(b) Method 1: As the given process is a straight line with positive slope and negative intercept with P-axis, its equation will be

$$P = aT - b \quad [\text{as } y = mx + c]$$

The ideal gas equation $PV = nRT$

$$\text{Hence } V = \frac{nRT}{P} = \frac{nRT}{aT - b} = \frac{nR}{a - (b/T)}$$

Now from the given Fig. (b), $T_2 > T_1$ so $V_2 < V_1$, i.e., when the gas in heated its volume increases.

Method 2: The slope of the isobaric line is given by (nR/V). The slope is small for larger pressure. Here point 1 lies on the constant pressure line which forms a smaller angle with the x-axis than the constant pressure line passing through point 2. Hence pressure of the gas decreases while heating.

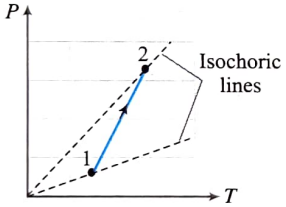
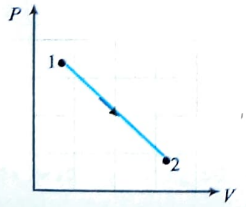


ILLUSTRATION 3.25

The diagram depicts variation of volume with pressure for a given mass of an ideal gas. What inference can be drawn regarding the change of temperature of the gas from point 1 to point 2?



Sol. The given process is a straight line with negative slope and positive intercept with P-axis, its equation will be

$$P = -aV + b \qquad [\text{as } y = mx + c]$$

So the gas equation $PV = nRT$ in the light of above yields

$$(-aV + b)\,V = nRT \Rightarrow -aV^2 + bV = nRT$$

$$\text{or} \quad T = \frac{-a}{nR}V^2 + \frac{b}{nR}V$$

This is the equation of a parabola as shown in figure. So during expansion, the temperature of gas first increases, reaches a maximum and then decreases.

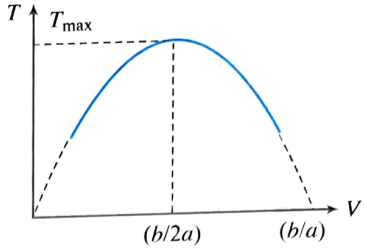
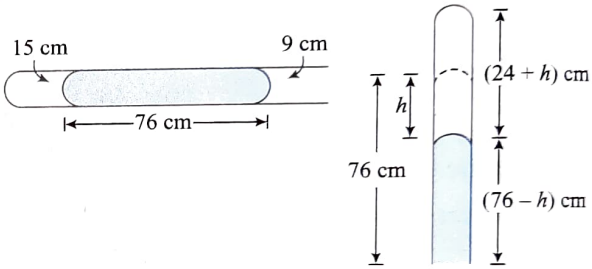


ILLUSTRATION 3.26

A 1 metre long narrow bore tube held horizontally (and closed at one end) contains a 76 cm long mercury thread which traps a 15 cm column of air. What happens if the tube is held vertically with the open end at the bottom?

Sol. If a is the area of cross-section of the tube (as shown in figure).



Initial volume of air, $V_1 = 15a$

Initial pressure of air, $P_1 = 76$ cm (of Hg column)

On holding this tube of narrow bore vertically, let h be the vertical distance through which the mercury in the tube gets depressed to balance the atmospheric pressure. Obviously, h denotes the final pressure of air, i.e., $P_2 = h$.

It is to be noted that the length of the air column left = length of the tube – length of Hg column $= 100 - (76 - h) = (24 + h)$

Final volume, $V_2 = (24 + h)a$

Applying Boyle's law, $P_1V_1 = P_2V_2$

$$\text{or} \quad 76 \times 15\, a = h\,(24 + h)a$$

$$\text{or} \quad 1140 = 24\,h + h^2$$

$$\text{or} \quad h^2 + 24\,h - 1140 = 0$$

$$\text{or} \quad h = \frac{-24 + \sqrt{(24)^2 + 4 \times 1 \times 1140}}{2}$$

$$= \frac{-24 \pm 72}{2} = 24\,\text{cm} \text{ or } -48\,\text{cm}$$

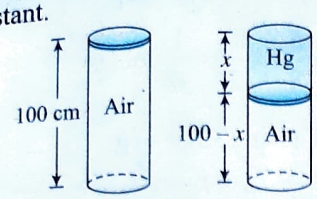
Neglecting $h = -48$ cm as mercury cannot flow into tube, $h = 24$ cm.

Thus, 24 cm of mercury flows out and the remaining 52 cm of mercury along with 48 cm of air column are in equilibrium with the outside atmospheric pressure.

ILLUSTRATION 3.27

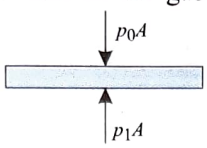
A vertical cylindrical tank, 1 m high, has its top end closed by a tightly fitted frictionless piston of negligible weight. The air inside the cylinder is an absolute pressure of 1 atm. The piston is depressed by pouring mercury on it very slowly. How far

will the piston descend before mercury spills over the top of cylinder? The temperature of the air inside the cylinder is maintained constant.



Sol. Let A be the area of cross-section of cylinder. The pressure above the piston before pouring mercury will be equal to atmospheric pressure (p_0). Let p_1 be the pressure of the gas before pouring mercury.

From FBD of piston before pouring mercury,



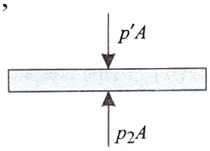
$$p_1A = p_0A$$

$$\Rightarrow \quad p_1 = p_0 = 1 \text{ atm} = 76 \text{ cm Hg}$$

Let p_2 be the pressure of the gas after pouring mercury. When mercury is poured, the pressure above the piston will be

p' = atmospheric pressure + pressure due to mercury
$= (76 + x)$ cm of mercury.

From FBD of the piston after pouring mercury,



$$p_2A = p'A$$

$$\Rightarrow \quad p_2 = p' = (76 + x) \text{ cm Hg}$$

As the temperature is constant,

use $p_1V_1 = p_2V_2$ for air inside the cylinder.

p_1 = initial air pressure = 1 atm = 76 cm Hg

p_2 = final air pressure = $(76 + x)$ cm Hg

V_1 = initial volume = $100\,A$

V_2 = final volume = $(100 - x)\,A$

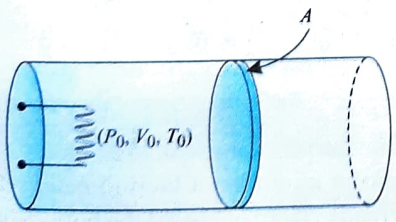
Now, $76(100\,A) = (76 + x)(100 - x)\,A$

$$\Rightarrow \quad 7600 = 7600 - 76x + 100x - x^2 \quad \Rightarrow \quad x = 24 \text{ cm}$$

Hence, The piston descends by 24 cm.

ILLUSTRATION 3.28

An ideal gas is filled in cylinder which is fitted with a movable non-conducting piston as shown in the figure. The gas in the cylinder is heated slowly. The gas expands and piston moves towards right. If the temperature of the gas increased slowly from T_0 to $2T_0$, find the displacement of the piston. (Given atmospheric pressure = P_0 and area of cross-section of the cylinder = A).



Sol. As piston is movable and the gas in the cylinder is heated slowly, the pressure of the gas inside the cylinder should be same as pressure outside (atmospheric pressure). In this case the expansion of the gas is isobaric.

Let final temperature of the gas be T_f

Final volume of the gas, $V_f = V_0 + Ax$

Hence for isobaric process

$$\frac{V_0}{T_0} = \frac{V_f}{T_f} \Rightarrow \frac{V_0}{T_0} = \frac{(V_0 + Ax)}{2T_0}$$

$$\Rightarrow \quad V_0 + A_x = 2V_0 \quad \text{or} \quad x = \frac{V_0}{A}$$

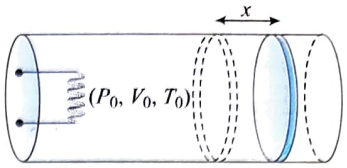
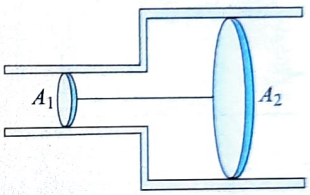


ILLUSTRATION 3.29

The figure shows a horizontal cylinder with two sections of unequal cross-section in which two points of area A_1 and A_2 can move freely. The pistons are joined together with an inextensible string and the space between the pistons is filled with an ideal gas. The whole system is in equilibrium.

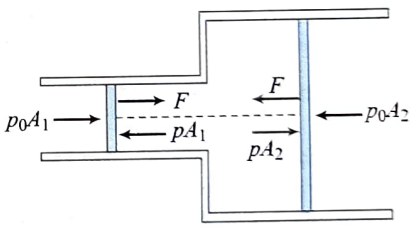
(a) If the atmospheric pressure is p_0, then determine the pressure of the gas.

(b) If the gas is heated, in which direction will the pistons move?



Sol.

(a) Let p be the pressure of the gas and F be the tension in the string.



Then, considering the equilibrium of each piston,

For piston I, $pA_1 = p_0A_1 + F$

$$\Rightarrow \quad p = p_0 + \frac{F}{A_1} \qquad \text{...(i)}$$

For piston II, $pA_2 = p_0A_2 + F$

$$p = p_0 + \frac{F}{A_2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), if $A_1 \neq A_2$ then the gas may have unique value of pressure only if tension in the string is zero, i.e., $F = 0$.

Thus, pressure of the gas is also atmospheric, $p = p_0$.

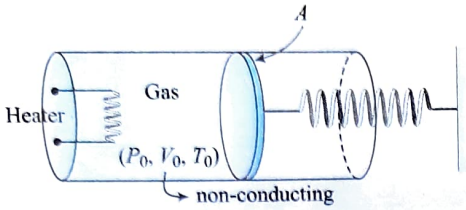
(b) When the gas is heated its volume increases at constant pressure, therefore, both the pistons shift towards right to accommodate the increased volume of the gas.

ILLUSTRATION 3.30

An ideal gas is filled in an insulated cylinder fitted with a movable piston connected with an ideal spring (of force constant K) as shown in figure. Initially spring is relaxed

and gas in the piston has volume V_0 and temperature T_0. Now the gas in the cylinder is heated slowly so that the piston slide towards right to a distance x. Find final pressure and temperature of the gas.

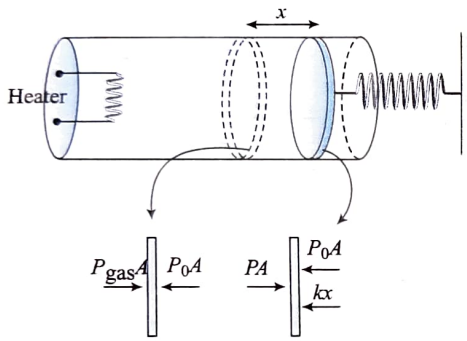
(Given atmospheric pressure = P_0 and area of cross-section of the cylinder = A).



Sol. As initially the spring is relaxed, from FBD of the piston, initially the pressure of the gas in the cylinder should be P_0.

Now the gas in the cylinder is heated, let the pressure of the gas in the cylinder is P.

From FBD of the piston



FBD of piston before heating | FBD of piston after heating

From FBD of the piston after heating,

$$PA = P_0A + kx \Rightarrow P = P_0 + \frac{kx}{A} \quad \text{...(i)}$$

The number of the moles in the cylinder should be constant, hence

$$n = \frac{P_0V_0}{RT_0} = \frac{P(V_0 + \Delta V)}{RT}$$

Here, $\Delta V = Ax \Delta V = Ax$

$$\Rightarrow \quad T = \frac{PT_0}{P_0V_0}(V_0 + Ax) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we have $T = \frac{T_0}{P_0V_0}\left(P_0 + \frac{Kx}{A}\right)(V_0 + Ax)$

ILLUSTRATION 3.31

A vessel containing 1 g of oxygen at a pressure of 10 atm and a temperature of 47°C. It is found that because of a leak, the pressure drops to 5/8th of its original value and the temperature falls 27°C. Find the volume of the vessel and the mass of oxygen that is leaked out.

Sol. The pressure, temperature and the number of moles of oxygen in the vessel change due to leak while the volume remains fixed. Hence using $PV = nRT$, we have

$$\frac{P_1}{n_1T_1} = \frac{P_2}{n_2T_2} \Rightarrow n_2 = \frac{T_1}{T_2} \times \frac{P_2}{P_1} \times n_1$$

Hence, $n_2 = \frac{320}{300} \times \frac{5}{8} \times \frac{1}{32} = \frac{1}{48}$ moles

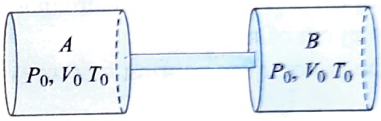
Mass leaking out = $1 - n_2(32) = \frac{1}{3}$ g

Volume of vessel, $V = \frac{n_1RT_1}{P_1}$

$$\Rightarrow \quad V = \left(\frac{1}{32}\right)\frac{8.3 \times 320}{10 \times 1 \times 10^5} = 8.3 \times 10^{-5}\,\text{m}^3 = 0.083\,\text{L}$$

ILLUSTRATION 3.32

Two identical vessel A and B each of volume V_0 joined together with a small tube (as shown in the figure) initially containing the same gas at pressure P_0 and absolute temperature T_0. The vessel A is now heated to maintain its temperature at $2T_0$ while the vessel B is maintained at the same temperature T_0.



(a) Find the number of moles of the gas in each vessel.

(b) Find the common pressure of the gas.

Sol.

(a) The total number of moles of the gas,

$$n = \frac{P_0(2V_0)}{RT_0} = \frac{2P_0V_0}{RT_0}$$

Initially both the vessels contain equal number of moles. When the vessel A is heated, a few moles of the gas move from vessel A to B. Let n_A and n_B be the final number of moles present in A and B, respectively, then

$$n_ART_A = n_BRT_B = PV = \text{constant}$$

$$\frac{n_A}{n_B} = \frac{T_B}{T_A} = \frac{T_0}{2T_0} = \frac{1}{2}$$

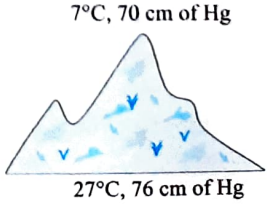
Thus $n_A = \frac{n}{3} = \frac{2P_0V_0}{3RT_0}$ and $n_B = \frac{2}{3}n = \frac{4P_0V_0}{3RT_0}$

(b) The final common pressure of the gas,

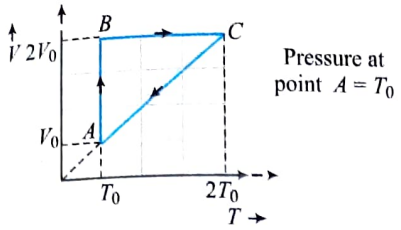
$$P = \frac{n_BRT_B}{V_0} = \left(\frac{4}{3}\frac{P_0V_0}{RT_0}\right)\frac{RT_0}{V_0} = \frac{4}{3}P_0$$

CONCEPT APPLICATION EXERCISE 3.2

1. At the top of a mountain a thermometer reads 7°C and a barometer reads 70 cm of Hg. At the bottom of the mountain these read, 27°C and 76 cm of Hg, respectively. Find the ratio of density of air at the top with that of bottom.



2. Find out the values of co-ordinates at point A, B, C in terms of pressure, volume and temperature and draw curve.

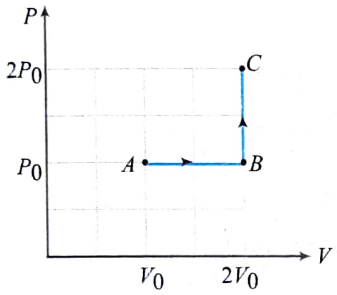


Pressure at point $A = T_0$

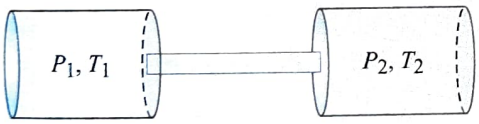
3. Draw the P–T and V–T diagrams of an isochoric process of n moles of an ideal gas from pressure P_0, volume V_0 to pressure $4P_0$, indicating the pressures, and temperatures, of the gas, in the initial and the final states.

4. Draw the P–T and V–T diagrams for an isobaric process of expansion, corresponding to n moles of an ideal gas at a pressure P_0, from V_0 to $2V_0$.

5. In given P–V diagram, the temperature at A is T_0. Find the temperature at point B and C. Also draw P–T and V–T diagrams for given process.

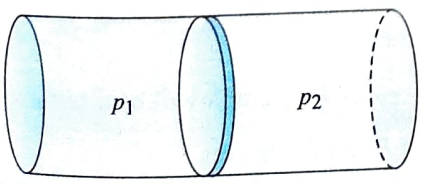


6. Two identical vessels contain the same gas at pressures P_1 and P_2 at absolute temperatures T_1 and T_2, respectively. On joining the vessels with a small tube as shown in the figure, the gas reaches a common temperature T and a common pressure P. Determine the ratio P/T.



7. A flask of volume 2 litres, provided with a stopcock contains oxygen at 300 K and atmospheric pressure. The system is heated to a temperature of 400 K, with the stopcock open to the atmosphere. The stopcock is then closed and the flask is then cooled to its original temperature.
   (a) What is the final pressure of oxygen in the flask?
   (b) How many grams of oxygen remains in the flask?

8. Figure shows a cylindrical tube of volume V_0 divided in two parts by a frictionless separator. The walls of the tube are non-conducting but the separator is diathermic. Ideal gases are filled in the two parts. When the separator is kept in the middle, the pressures are p_1 and p_2 in the left part and the right part respectively. The separator is slowly slide and is released at a position where it can stay in equilibrium. Find the volumes of the two parts.



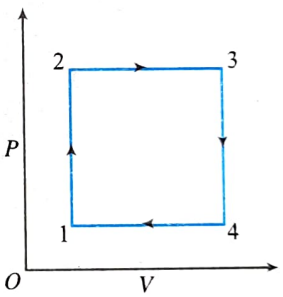
9. An ideal gas is taken in a cylinder at pressure p_0, volume V_0 and temperature T_0. It is taken an isochoric process till its pressure becomes $4p_0$. It is now isothermally expanded to get the original pressure. Finally, the gas is isobarically compressed to its original volume V_0.
   (a) Show the process on a P–V diagram.
   (b) What is the temperature in the isothermal part of the process?
   (c) What is the volume at the end of the isothermal part of the process?

10. During an experiment, an ideal gas is found to obey an additional law VP^2 = constant. The gas is initially at temperature T and volume V. What will be the temperature of the gas when it expands to a volume $2V$?

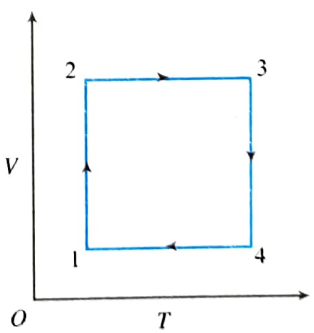
11. A mixture of hydrogen and oxygen has volume 1660 cm^3, temperature 400 K, pressure 200 kPa and mass 2.0 g. Calculate the masses of hydrogen and oxygen in the mixture.

12. A container contains 14 g of hydrogen and 96 g of oxygen at ATP (pressure at 1atm = 1.0×10^5 Pa and temperature 273 K). Chemical reaction is induced by passing electric spark in the vessel till one of the gases is consumed. The temperature is brought back to its starting value 273 K. Find the pressure in the container.

13. A gas has been subjected to isochoric–isobaric processes 1–2–3–4–1 (see figure). Plot this process on T–V and P–T diagrams.



14. A gas has been subjected to an isothermal–isochoric cycle 1–2–3–4–1 (see figure). Represent the same cycle on P–V and P–T diagrams.



ANSWERS

1. $\frac{75}{76}$ 6. $\frac{1}{2}\left[\frac{P_1}{T_1}+\frac{P_2}{T_2}\right]$ 7. (a) $\frac{3}{4}$atm (b) 0.6504 g

8. $\frac{p_1V_0}{p_1+P_2}, \frac{p_2V_0}{p_1+p_2}$ 9. (b) $T_B = 4\,T_0$ (c) $V_C = 2\,V_0$

10. $T' = (\sqrt{2})T$ 11. 0.12 g Hydrogen, 1.28 g Oxygen

12. 0.10 atm

Solved Examples

EXAMPLE 3.1

N molecules each of mass m of gas A and $2N$ molecules each of mass $2m$ of gas B are contained in the same vessel which is maintained at a temperature T. The mean square of the velocity of the molecules of B type is denoted by v_B^2 and the mean square of the x-component of the velocity of A type is denoted by v_A^2. What is the ratio of v_B^2/v_A^2 = ?

Sol. The mean square velocity of gas molecule is given by

$$v^2 = \frac{3kT}{m}$$

For gas A, $v_A^2 = \frac{3kT}{m}$ ...(i)

For a gas molecule, $v^2 = v_x^2 + v_y^2 + v_z^2 = 3v_x^2$ $(v_x^2 = v_y^2 = v_z^2)$

or $v_x^2 = \frac{v^2}{3}$

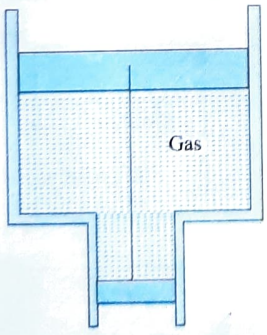
From Eq. (i), we get $v_x^2 = \frac{v_A^2}{3} = \left[\frac{(3kT/m)}{3}\right] = \frac{kT}{m}$ ...(ii)

For gas B, $v_B^2 = v^2 = \frac{3kT}{m_B} = \frac{3kT}{m}$ ...(iii)

Dividing Eq. (ii) by Eq. (iii), we get $\frac{v_B^2}{v_A^2} = \frac{kT/m}{3kT/2m} = \frac{2}{3}$

EXAMPLE 3.2

A smooth vertical tube having two different cross sections is open from both the ends but closed by two sliding pistons as shown in the figure and tied with an inextensible string. One mole of an ideal gas is enclosed between the piston. The difference in cross-sectional areas of the two pistons is given ΔS. The masses of piston are m_1 and m_2 for larger and smaller ones, respectively. Find the temperature by which tube is raised so that the pistons will be displaced by a distance l. Take atmospheric pressure equal to P_0.



Sol. If initial pressure of gas is P and let S_1 and S_2 be the cross-sectional area of the larger and the smaller piston, respectively, then for equilibrium of the two pistons, we have

For larger piston

$$P_0S_1 + m_1g + T = PS_1 \quad \text{(if } T \text{ is the tension in string)} \quad ...(i)$$

For smaller piston $PS_2 + m_2g = T + P_0S_2$ ...(ii)

Adding Eqs. (i) and (ii), we get

$$P_0(S_1 - S_2) + m_1g + m_2g = P(S_1 - S_2) + m_1g + m_2g = P(S_1 - S_2)$$

or $P_0 + \left(\frac{m_1 + m_2}{\Delta S}\right)g = P$ ...(iii)

If gas temperature is increased from T_1 to T_2, the volume of gas increase from V to $V + l\Delta S$ as l is the displacement of pistons, then from gas law we must have

$$PV = RT_1 \quad \text{(for initial state)} \quad ...(iv)$$

$$P(V + l\Delta S) = RT_2 \quad \text{(for final state)} \quad ...(v)$$

According to Eq. (iii), we find that pressure of the gas does not change, as it does not depend on temperature.

From Eqs. (iv) and (v), if we subtract these equation, we get

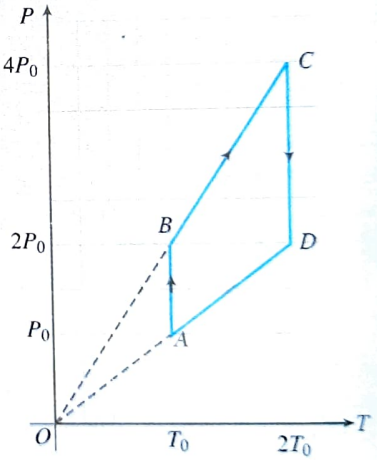
$$Pl\Delta S = R(T_2 - T_1)$$

or $T_2 - T_1 = \frac{Pl\Delta S}{R}$

$$= \left(P_0 + \frac{(m_1 + m_2)}{\Delta S}g\right)\frac{l\Delta S}{R} = [P_0\Delta S + (m_1 + m_2)g]\frac{l}{R}$$

EXAMPLE 3.3

Pressure–temperature (P–T) graph of n moles of an ideal gas is given in figure. Plot the corresponding graphs:



(a) density–volume (ρ–V) graph
(b) pressure–volume (P–V) graph and
(c) density–pressure (ρ–P) graph.

Sol. In this given graph, we are having two isothermal (AB and CD) and two isochoric lines (BC and DA).

We have ideal gas equation

$$PV = nRT = \left(\frac{m}{M}\right)RT \quad \Rightarrow \quad P = \frac{m}{V}\left(\frac{RT}{M}\right) = \rho\left(\frac{RT}{M}\right)$$

or $\rho = P\left(\frac{M}{RT}\right)$

Hence we can say that $\rho \propto \frac{1}{V}$ and $\rho \propto P$.

Considering the process $A \rightarrow B$: It is an isothermal process. Here, T = constant

Hence, $P \propto \frac{1}{V}$ or P–V graph will be a rectangular hyperbola with increasing P and decreasing V.

Since density and volume are related as $\rho \propto \frac{1}{V}$, ρ–V graph is also a rectangular hyperbola with decreasing V and hence increasing ρ.

As $P \propto \frac{1}{V}$, $\rho \propto P$

Hence ρ–V graph will be a straight line passing through origin, with increasing ρ and P.

Considering the process $B \rightarrow C$: It is an isochoric process, because P–T graph is a straight line passing through origin.

In this process V = constant

Hence P–V graph will be a straight line parallel to P-axis with increasing P.

As $\rho \propto \frac{1}{V}$, V = constant hence ρ will also be constant

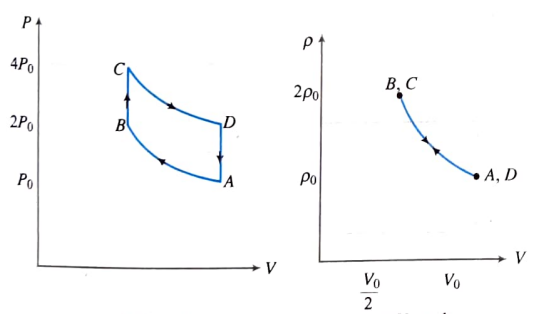
Hence ρ–V graph will be a point (dot).

As $\rho \propto \frac{1}{V}$ and V = constant, ρ–P graph will be a straight line parallel to P-axis with increasing P, because ρ = constant

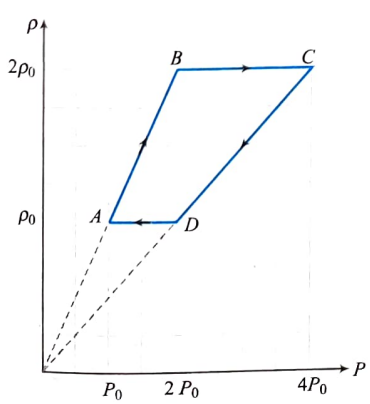
Considering the processes $C \rightarrow D$ and $D \rightarrow A$: These processes are inverse of $A \rightarrow B$ and $B \rightarrow C$, respectively.

Different values of P, V, T and ρ in tabular form are shown below

	A	B	C	D
P	P_0	$2P_0$	$4P_0$	$2P_0$
V	V_0	$\frac{V_0}{2}$	$\frac{V_0}{2}$	V_0
T	T_0	T_0	$2T_0$	$2T_0$
ρ	ρ_0	$2\rho_0$	$2\rho_0$	ρ_0



P–V graph ρ–V graph



ρ–P graph

EXAMPLE 3.4

Ten small planes are flying at a speed of 150 km/h in total darkness in an air space that is $20 \times 20 \times 1.5$ km^3 in volume. You are in one of the planes, flying at random within this space with no way of knowing where the other planes are. On an average about how much time will elapse when the nearby plane collides with your plane? Assume for this rough computation that a safety region around the plane can be approximated by a sphere of radius 10 m.

Sol. To solve this problem, we have to consider the relaxation time as well as mean free path. The situation can be considered as the time of relaxation, based on kinetic theory of gases. Mean free path is the distance between two successive collisions, which we will consider here as the distance travelled by the plane before it just avoids the collision safe radius is equivalent to radius of the atom.

Hence, the required time, $t = \frac{\lambda}{v}$

λ = mean free path $= \frac{1}{\sqrt{2}\pi d^2 n}$,

Here, v is the average velocity of the molecule and d is the diameter of molecule.

n = number density $= \frac{N}{V}$

$$n = \frac{\text{Number of aeroplanes}(N)}{\text{Volume}(V)} = \frac{10}{20 \times 20 \times 1.5} = 0.0167 \text{ km}^{-3}$$

$t = \frac{1}{\sqrt{2}\pi d^2 n} \times \frac{1}{v}$, here v is the velocity of aeroplane

By putting the given data,

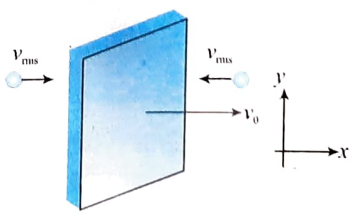
$$t = \frac{1}{\sqrt{2} \times 3.14 \times (20)^2 \times 0.0167 \times 10^{-6} \times 150}$$

$$= 224.74 \text{ h} \approx 225 \text{ h}$$

EXAMPLE 3.5

Consider a rectangular block of wood moving with a velocity v_0 in a gas at temperature T and mass density ρ. Assume the velocity is along x-axis and the area of cross-section of the block perpendicular to v_0 is A. Show that the drag force on the block is $4\rho A v_0 \sqrt{\frac{kT}{m}}$, where, m is the mass of the gas molecule.

Sol. Consider the diagram



Let n = number of molecules per unit volume

m = mass of the gas molecule

ρ = density of the gas

v_{rms} = rms speed of the gas molecules

v = velocity of a gas molecule moving along x-axis.

As $v_x = v_y = v_z$

and $v_{rms}^2 = v_x^2 + v_y^2 + v_z^2 = 3v^2 \Rightarrow v^2 = \frac{v_{rms}^2}{3}$

When block is moving with speed v_0, relative speed of molecules w.r.t. front face $= v + v_0$. The linear momentum transferred to block per collision by molecules colliding normal to the block $= 2m(v + v_0)$.

If n is the number of the molecules per unit volume. All the molecules at a distance $(v + v_0)\,\Delta t$ will collide the block.

Number of collisions in time Δt, $N = \frac{1}{2}(v+v_0)\,n\Delta t A$, where, A = area of cross-section of block and factor of 1/2 appears due to particles moving towards block and n = number of the molecules per unit volume.

$\therefore$ Momentum transferred from front surface in time Δt,

$$\Delta p_{front} = m(v + v_0)^2\, nA\,\Delta t$$

Similarly, linear momentum transferred to block from back surface (in time Δt), $\Delta p_{back} = m(v - v_0)^2\, nA\,\Delta t$

$\therefore$ Net force (drag force) due to molecular collision

$$F_{drag} = mnA\left[(v+v_0)^2 - (v-v_0)^2\right] \quad \text{[from front]}$$

$$\Rightarrow \quad F_{drag} = mnA(4vv_0) = (4mnAv)v_0$$

$$\Rightarrow \quad F_{drag} = (4\rho Av)v_0 \qquad \ldots(i)$$

where we have assumed that $\rho = \frac{mn}{V} = \frac{M}{V}$

We know that $v_{rms} = \sqrt{\frac{3kT}{m}}$

$$\Rightarrow \quad v_{rms}^2 = \frac{3kT}{m} \Rightarrow 3v^2 = \frac{3kT}{m} \text{ or } v = \sqrt{\frac{kT}{m}}$$

Thus, from Eq. (i),

Drag force, $F_{drag} = (4\rho Av)v_0 = 4\rho A\sqrt{\frac{kT}{m}}v_0$.

EXAMPLE 3.6

(a) If it has enough kinetic energy, a molecule at the surface of the Earth can escape the Earth's gravitation which means that it can continue to move away from the Earth forever. Using the principle of conservation of energy, show that the minimum kinetic energy needed to escape is $m_0 gR_E$, where m_0 is the mass of the molecule, g is the free-fall acceleration at the surface, and R_E is the radius of the Earth.

(b) Calculate the temperature for which the minimum escape kinetic energy is ten times the average kinetic energy of an oxygen molecule.

Sol.

(a) Consider the molecule–Earth system to be isolated. Treat an escaping molecule as going from

$r = R_g$, $v = v_0$ to $r = \infty$, $v = 0$

Using conservation of mechanical energy, $\Delta K + \Delta U = 0$

$$\Rightarrow \quad \left(0 - \frac{1}{2}m_0 v_0^2\right) + \left[0 - \left(-\frac{Gm_0 M}{R_E}\right)\right] = 0$$

$$\Rightarrow \quad \frac{1}{2}m_0 v_0^2 = \frac{Gm_0 M}{R_E}$$

Since the free-fall acceleration at the surface is $g = \frac{GM}{R_E^2}$, this can also be written as:

$$\frac{1}{2}m_0 v_0^2 = \frac{Gm_0 M}{R_E} = m_0 gR_E$$

(b) For O_2, the mass of one molecule,

$$m_0 = \frac{0.0320 \text{ kg/mol}}{6.02\times10^{23} \text{ molecules/mol}}$$

$$= 5.32 \times 10^{-26} \text{ kg/molecule}$$

If $m_0 gR_E = 10\left(\frac{3k_B T}{2}\right)$, then the temperature

$$T = \frac{m_0 gR_E}{15k_B} = \frac{(5.32\times10^{-26} \text{ kg})(9.80 \text{ m/s}^2)(6.37\times10^{6} \text{ m})}{15(1.38\times10^{-23} \text{ J/mol-K})}$$

$$= 1.60 \times 10^4 \text{ K}$$

EXAMPLE 3.7

An ideal gas of molar mass M is filled in a horizontal cylinder closed at one end. The cylinder rotates with a constant angular velocity ω about a vertical axis passing through the open end of the cylinder. The pressure at the axis of the cylinder is P_0 and temperature is T.

Find the air pressure as a function of the distance r from the rotation axis, in isothermal condition.

Sol. We consider a differential layer of thickness dr, at a distance r from axis of rotation. The centripetal force to rotate this layer should be provided by pressure difference on both sides of layer.

$$F_{centripetal} = (P + dP)A - PA = dmr\,\omega^2$$

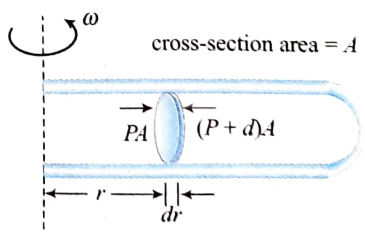
The mass of the gas in layer, $dm = \rho\, adr$

$\therefore \quad dPA = (\rho A\, dr)\, r\, \omega^2$

$$\Rightarrow \quad dP = \rho\omega^2 r\, dr \qquad \ldots(ii)$$

From ideal gas equation, we have $PV = \frac{m}{M}RT$

$$\Rightarrow \quad PM = \left(\frac{m}{V}\right)RT = \rho RT \Rightarrow \rho = \frac{PM}{RT} \qquad \ldots(iii)$$



From Eqs. (i) and (ii), we get $dP = \frac{PM}{RT}\omega^2 r\, dr$

$$\Rightarrow \quad \int_{P_0}^{P}\frac{dP}{P} = \frac{M\omega^2}{RT}\int_0^r r\, dr \quad \Rightarrow \quad \ln\frac{P}{P_0} = \frac{M\omega^2}{RT}\frac{r^2}{2}$$

$$\Rightarrow \quad \frac{P}{P_0} = e^{\frac{M\omega^2 r^2}{2RT}} \quad \Rightarrow \quad P = P_0 e^{\frac{M\omega^2 r^2}{2RT}}$$

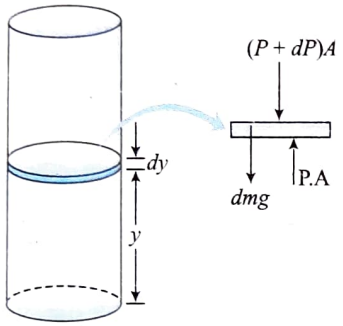
EXAMPLE 3.8

A tall vertical cylinder of height h and cross-sectional area A is in uniform gravitational field. The cylinder contains an ideal gas at uniform temperature and is closed at both ends. If density of gas at bottom and the top of the cylinder is ρ_1 and ρ_2, respectively, calculate molecular weight of the gas contained in the cylinder.

Sol. Here we are given the density of gas is ρ_1 at bottom and ρ_2 at top or density of gas is not uniform. Therefore, to calculate mass of gas contained in the cylinder, first density ρ is to be calculated as function of height y from bottom.

Let temperature of gas be T.

Considering a gas layer of elemental thickness dy at height y from bottom as shown in figure. Let pressure at lower face of this layer be P and r, respectively, and at upper face be $(P + dP)$.



From FBD of gas layer $(P + dP)A + dmg = PA$

$\Rightarrow \quad (P + dP)A + (\rho dy)g = PA$

Hence, $dP = -\rho g \cdot dy$ ...(i)

If molar mass of gas is M, then pressure, $P = \dfrac{\rho RT}{M}$

Differentiating above equation, $dP = \dfrac{RT}{M} . d\rho$ ...(ii)

From Eqs. (i) and (ii), $\dfrac{RT}{M} . d\rho = -\rho g\, dy$

or $\dfrac{d\rho}{\rho} = -\dfrac{Mg}{RT} \cdot dy$

But at bottom $(y = 0)$, $\rho = \rho_1$ and at top $(y = h)$, $\rho = \rho_2$.

Now integrating with proper limits, $\int_{\rho_1}^{\rho_2} \dfrac{d\rho}{\rho} = -\dfrac{Mg}{RT}\int_0^h . dy$

$\Rightarrow \quad \log \dfrac{\rho_2}{\rho_1} = \dfrac{Mgh}{RT}$

$\Rightarrow \quad \dfrac{Mg}{RT} = \dfrac{1}{h} \cdot \log \dfrac{\rho_1}{\rho_2} \Rightarrow M = \dfrac{RT}{gh} \log \dfrac{\rho_1}{\rho_2}$

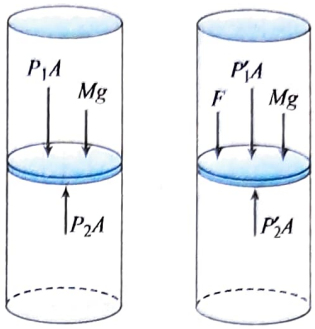
EXAMPLE 3.9

A vertical cylinder of cross-sectional area 0.1 m^2 closed at both ends is fitted with a frictionless piston of mass M dividing the cylinder into two parts. Each part contains one mole of an ideal gas in equilibrium at 300 K. The volume of the upper part is 0.1 m^3 and that of the lower part is 0.05 m^3.

What force must be applied to the piston so that the volume of the two parts remain unchanged when the temperature is increased to 500 K ?

Sol. Let P_1 and P_2 be the pressures of gas in upper and lower part of the cylinder, respectively, then

$$P_1 + \frac{Mg}{A} = P_2 \text{ or } P_2 - P_1 = \frac{Mg}{A} \quad \text{...(i)}$$



At temperature T_1 At temperature T_2

On heating, if F is the force applied to the piston. Also, if P'_1 and P'_2 are the pressures of gas in two sections, then

$$P'_1 + \frac{Mg}{A} + \frac{F}{A} = P'_2 \Rightarrow P'_2 - P'_1 = \frac{Mg}{A} + \frac{F}{A}$$

$$\Rightarrow \quad \frac{F}{A} = P'_2 - P'_1 - \frac{Mg}{A}$$

Substituting for Mg/A from Eq. (i), we get

$$\frac{F}{A} = P'_2 - P'_1 - P_2 + P_1 \quad \text{...(iii)}$$

As each part of the cylinder contains 1 g mole of gas,

$P_1 V_1 = RT_1$ and $P_2 V_2 = RT_2$

$P'_1 V_1 = RT_2$ and $P'_2 V_2 = RT_2$

Substituting for P_1, P_2, P'_1, P'_2 in Eq. (ii), we get

$$\frac{F}{A} = R\left[\frac{T_2}{V_2} - \frac{T_2}{V_1} - \frac{T_1}{V_2} + \frac{T_1}{V_1}\right]$$

$$\Rightarrow \quad \frac{F}{A} = R\left[\frac{500}{0.05} - \frac{500}{0.1} - \frac{300}{0.05} + \frac{300}{0.1}\right]$$

$$\Rightarrow \quad F = 0.1 \times 2000\, R = 200\, R$$

EXAMPLE 3.10

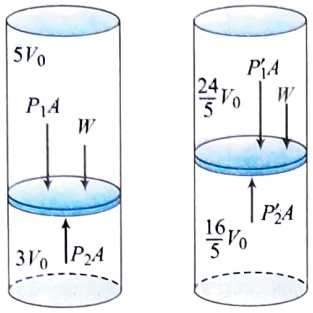
A vertical cylinder contains a movable frictionless piston separating the former into two portions. The two portions are filled with ideal gases, and at equilibrium, the ratio of the volume of the upper portion to that of the lower portion is 5 : 3. When the temperature of the system is doubled, the ratio becomes 3 : 2. Determine the ratio of number of molecules in the upper portion to that of the lower portion.

Sol. Let the weight of the piston be W and cross-sectional area of the cylinder be A. Let the initial and the final pressures of the upper and the lower regions be P_1, P'_1 and P_2, P'_2 respectively. Further, if initial temperature be taken to be T_0, then final temperature will be $2T_0$.

Let the volume of upper part is $5V_0$ and lower part is $3V_0$, hence total volume of the cylinder is $(5V_0 + 3V_0 = 8V_0)$. When the cylinder is heated to $2T_0$, the ratio of volume of upper and lower part becomes 3 : 2. Hence, the volume of

upper part $V_1' = \frac{3}{5} \times 8V_0 = \frac{24}{5}V_0$ and volume of lower part $V_2' = \frac{2}{5} \times 8V_0 = \frac{16}{5}V_0$.

From the FBD of the piston, initially,



At temperature T_0 At temperature $2T_0$

$$P_1A + W = P_2A$$

or $$\frac{n_1RT_0}{5V_0} + \frac{W}{A} = \frac{n_2RT_0}{3V_0} \quad ...(i)$$

where n_1 and n_2 are the numbers of moles in the two portions.

Finally, from the FBD of the piston,

$$P_1'A + W = P_2'A$$

or, $$\frac{n_1R(2T_0)}{3(8V_0/5)} + \frac{W}{A} = \frac{n_2R(2T_0)}{2(8V_0/5)} \quad ...(ii)$$

Subtracting equation (ii) from (i), we have

$$n_1\left(\frac{1}{5} - \frac{5}{12}\right) = n_2\left(\frac{1}{3} - \frac{5}{8}\right)$$

or, $n_1 = n_2 = 35 : 26$

EXAMPLE 3.11

A cubical vessel of side 1m contains one mole of nitrogen at a temperature of 300K. If the molecules are assuming to move with the rms velocity **(a)** find the number of collisions per second which the molecules may make with the wall of the vessel, **(b)** further if the vessel now thermally insulated moved with a constant speed v and then suddenly stopped and this results in rise of temperature by 2°C, find v.

Sol. Given, Volume of container = $1m^2$
Temperature of gas = 300 K
Amount of gas = 1mole

(a) If n_0 be the number of molecules per unit volume, we have

$$n_0 = 1 \text{ mole} = 6.023 \times 10^{23} \text{ m}^{-3}$$

The rms velocity of molecules at 300 K temperature is

$$v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3 \times 3.314 \times 300}{28 \times 10^{-3}}} = 516.75 \text{ m/s}$$

We know that number of collisions per second per square meter of wall is given by

$$N = \frac{n_0 v}{6} = \frac{6.023 \times 10^{23} \times 516.75}{6} = 5.187 \times 10^{25} \text{ s}^{-1}\text{m}^{-1}$$

(b) When container is moving at speed v, the kinetic energy in the nitrogen molecules is

$$E_k = \frac{1}{2}Mv^2 = \frac{1}{2} \times 28 \times 10^{-3} \times v^2$$

[mass of 1 mole of gas = 28×10^{-3} kg]

When the container is suddenly stopped, this kinetic energy is transformed into thermal energy and increases the internal energy of gas as container insulated. If temperature increment of gas ΔT, rise in its internal energy is

$$\Delta U = \frac{f}{2}nR\Delta T$$

or $$\Delta U = -\frac{5}{2}nR\Delta T = \frac{5}{2} \times 1 \times 8.314 \times 2 = 4157 J$$

From energy conservation, we have

$$\frac{1}{2} \times 28 \times 10^{-3} \times v^2 = 41.57$$

or $$v = \sqrt{\frac{41.57 \times 2}{28 \times 10^{-3}}} = 54.5 \text{ m/s}$$

EXAMPLE 3.12

Five grams of helium having rms speed of molecules 1000 m/s and 24 g of oxygen having rms speed of 1000 m/s are introduced into a thermally isolated vessel. Find the rms speeds of helium and oxygen individually when thermal equilibrium is attained. Neglect the heat capacity of the vessel.

Sol. For helium, $100 = \sqrt{\frac{3RT}{M}}$

$$1000 = \sqrt{\frac{3RT_1}{4 \times 10^{-3}}} \Rightarrow 3RT_1 = 4000$$

For oxygen, $1000 = \sqrt{\frac{3RT_2}{32 \times 10^{-3}}} \Rightarrow 3RT_2 = 32000$

By internal energy conservation $U_1 + U_2 = U_1' + U_2'$

As we know $U = \frac{f}{2}nRT$;

$$\frac{3}{2} \times \frac{5}{4}RT_1 + \frac{5}{2} \times \frac{24}{32}RT_2 = \left(\frac{3}{2} \times \frac{5}{4} + \frac{5}{2} \times \frac{24}{32}\right)RT$$

$$T = \frac{T_1 + T_2}{2}$$

After mixing rms speed of helium,

$$v_1' = \sqrt{\frac{3RT}{mw}} = \sqrt{\frac{3R}{2}\frac{(T_1 + T_2)}{mw}} = \sqrt{\frac{4000 + 32000}{2 \times 4 \times 10^{-3}}}$$

$$v_1' = 1500\sqrt{2} \text{ m/s}$$

For oxygen,

$$v_2' = \sqrt{\frac{3RT(T_1 + T_2)}{3mw}} = \sqrt{\frac{4000 + 32000}{2 \times 32 \times 10^{-3}}}$$

$$v_2' = 750 \text{ m/s}$$

EXAMPLE 3.13

Find the minimum attainable pressure of an ideal gas in the process $T = T_0 + \alpha V^2$, where T_0 and α are positive constants and V is the volume of one mole of gas. Draw the approximate T-V plot of this process.

Sol. Given, $T = T_0 + \alpha V^2$ ...(i)

For 1 mol of a gas, $PV = RT$

or $V = \dfrac{RT}{P}$

Substituting this value in Eq. (i), we get

$$T = T_0 + \alpha\left(\frac{RT}{P}\right)^2 = T_0 + \alpha\frac{R^2T^2}{P^2}$$

or $TP^2 = T_0P^2 + \alpha R^2T^2$

or $P = \sqrt{\alpha}\, RT(T - T_0)^{-1/2}$ ...(ii)

After differentiating, we get

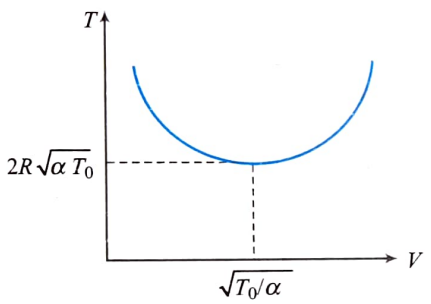
$$\frac{dP}{dT} = \sqrt{\alpha}\, R\left[(T - T_0)^{-1/2} - \frac{1}{2}T(T - T_0)^{-3/2}\right]$$

For minimum pressure, $\dfrac{dP}{dT} = 0$

$$\therefore \quad 0 = \sqrt{\alpha}\, R\left[(T - T_0)^{-1/2} - \frac{1}{2}T(T - T_0)^{-3/2}\right]$$

After solving, $T = 2T_0$

From Eq. (ii), $P_{\min} = \sqrt{\alpha}R2T_0(2T_0 - T_0)^{-3/2} = 2R\sqrt{\alpha T_0}$



EXAMPLE 3.14

An adiabatic vessel contains $n_1 = 3$ mole of diatomic gas. Moment of inertia of each molecule is $I = 2.76 \times 10^{-46}$ kg m^2 and root-mean-square angular velocity is $\omega_0 = 5 \times 10^{12}$ rad/s. Another adiabatic vessel contains $n_2 = 5$ mole of a monatomic gas at a temperature 470 K. Assume gases to be ideal, calculate root-mean-square angular velocity of diatomic molecules when the two vessels are connected by a thin tube of negligible volume. Boltzmann constant $k = 1.38 \times 10^{-23}$ J/molecule.

Sol. We know according to law of equipartition of energy, each gas molecule has $1/2\ kT$ energy associated with each of its degrees of freedom. As a diatomic gas molecule has two rotational degrees of freedom, final temperature of the system is

$$T_f = \frac{f_1n_1T_1 + f_2n_2T_2}{f_1n_1 + f_2n_2} \qquad \text{(i)}$$

Here for diatomic gas, $f_1 = 5$, $n = 3$ and $T_1 = 250$ K

For monatomic gas, $f_2 = 3$, $n_2 = 5$ and $T_2 = 470$ K

Thus from Eq. (i), $T_f = \dfrac{5\times3\times250 + 3\times5\times470}{5\times3 + 3\times5} = 360$

Thus final mixture of the two gases is at temperature 360 K. If final rms angular velocity of diatomic gas molecules is $\omega_{rms\,f}$, according to law of equipartition of energy, we have

$$\frac{1}{2}I\omega_{rms\,f}^2 = kT$$

or $$\omega_{rms\,f} = \sqrt{\frac{2kT}{I}} = \sqrt{\frac{2\times1.38\times10^{-23}\times360}{2.76\times10^{-46}}}$$

$$\omega_{rms\,f} = 6 \times 10^{12} \text{ rad/s}$$

EXAMPLE 3.15

A cylinder containing 2 litres of air at 2 atmospheric pressure, and 200 K, is subjected to the following cyclic process in four different stages. It is **(a)** heated at constant pressure to 600 K, **(b)** cooled at constant volume to 300 K, **(c)** cooled at constant pressure to 100 K and **(d)** heated at constant volume to 200 K. Depict the complete cycle on a P-V diagram showing the values of P and V at the end of each stage.

Sol. Let initially, volume $V_0 = 2$ litres, $P_0 = 2$ atm., and temperature $T_0 = 200$ K.

(a) Since pressure is constant, $P_1 = 2$ atm and $T_1 = 600$ K.

Applying Charles's law, $V \propto T \Rightarrow \dfrac{V_1}{V_0} = \dfrac{T_1}{T_0}$

or $$\frac{V_1}{2\text{ litres}} = \frac{600\text{ K}}{200\text{ K}} \Rightarrow V_1 = 6\text{ litres}$$

(b) Since volume is constant, $V_2 = 6$ litres and $T_2 = 300$ K.

Applying pressure law, $P \propto T \Rightarrow \dfrac{P_2}{P_1} = \dfrac{T_2}{T_1}$

or $$\frac{P_2}{2\text{ atm.}} = \frac{300\text{ K}}{600\text{ K}} \Rightarrow P_2 = 1\text{ atm}$$

(c) Since, pressure is constant, so $P_3 = 1$ atm and $T_3 = 100$ K.

Applying Charles's law, $V \propto T \Rightarrow \dfrac{V_3}{V_2} = \dfrac{T_3}{T_2}$

or $$\frac{V_3}{6\text{ litres}} = \frac{100\text{ K}}{300\text{ K}} \Rightarrow V_3 = 2\text{ litres}$$

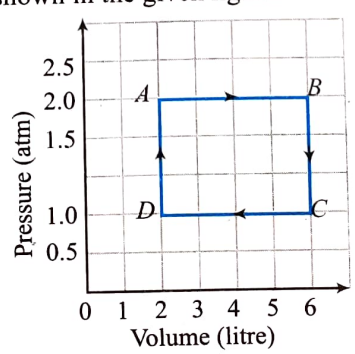
(d) Finally, since volume remains constant,

$V_4 = 2$ litres and $T_4 = 200$ K

Applying pressure law, $P \propto T \Rightarrow \dfrac{P_4}{P_3} = \dfrac{T_4}{T_3}$

or $$\frac{P_4}{1\text{ atm.}} = \frac{200\text{ K}}{100\text{K}} \Rightarrow P_4 = 2\text{ atm}$$

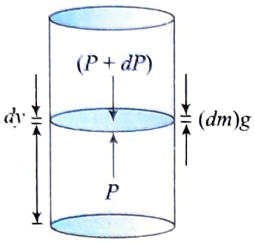
Thus, comparing P_0, V_0 and T_0 with P_4, V_4 and T_4, it is evident that, the air is brought back to its initial state. All these are shown in the given figure.



EXAMPLE 3.16

An ideal gas of molar mass M is contained in a very tall vertical cylindrical vessel whose cross-sectional area is A and it is kept on earth's surface. Assuming the temperature of the gas as T and the pressure on the bottom as P_0, determine the mass of the gas, up to a height h. Ignore the variation of acceleration due to gravity with height.

Sol. Consider a gas layer of elemental thickness dy at a height y from the base of the cylinder.



If P and $P + dP$ are the pressures at the lower and upper surfaces, of the layer respectively, then for the vertical equilibrium of the layer,

$$(P + dP)\,A + (dm)g = PA$$

Where dm is the mass of the layer.

or, $dP = -\rho g dy$ $\quad [\because dm = \rho(Ady)]$

Now, from equation of state for an ideal gas,

$$PV = nRT \text{ or } PV = \frac{m}{M}RT$$

or $\quad \frac{m}{V} = \rho = \frac{PM}{RT} \quad \therefore \quad dP = \frac{-PMg}{RT}dy$

or $\quad \int_{P_0}^{P}\frac{dP}{P} = \frac{-Mg}{RT}\int_0^h dy \quad$ or $\quad P = P_0\exp\left(\frac{-Mgh}{RT}\right)$

Now, mass of the layer, $dm = \rho A dh$

$$= \frac{PMAdh}{RT} = \frac{P_0MA}{RT}\exp\left(\frac{-Mgh}{RT}\right)dh$$

The mass upto height h from the bottom can be obtained as the sum (integral) of the masses dm of each elemental layers upto height h.

$$\therefore \quad m = \int dm = \int_0^h \frac{P_0MA}{RT}\exp\left(\frac{-Mgh}{RT}\right)dh$$

or $\quad m = \frac{P_0A}{g}\left[1 - \exp\left(\frac{-Mgh}{RT}\right)\right]$

EXAMPLE 3.17

Helium and oxygen gases are contained in two identical vessels at temperatures T_0 and $2T_0$, respectively. Their respective pressures are P_0 and $4P_0$. They are connected by a thin pipe and are allowed to diffuse without any loss of energy. Find the equilibrium pressure and temperature of the gaseous mixture.

Sol. Let n_1 and n_2 be the number of moles of helium and oxygen gases, respectively, and V_0 be the volume of each vessel then from equation of state

$$n_1 = \frac{P_0V_0}{RT_0} \text{ and } n_2 = \frac{4P_0V_0}{R(2T_0)} = \frac{2P_0V_0}{RT_0}$$

$$\therefore \quad n_2 = 2n_1 \qquad \ldots(i)$$

Now, if T is the final temperature, then the total internal energy (kinetic) energy of the two gases, initially should be same as that of the mixture,

i.e, $\quad U_i = n_1\left(\frac{3}{2}RT_0\right) + n_2\left(\frac{5}{2}R.2T_0\right)$

$$U_f = n_1\left(\frac{3RT}{2}\right) + n_2\left(\frac{5RT}{2}\right)$$

$$U_i = U_f$$

or, $\quad T = T_0\left(\frac{3n_1 + 10n_2}{3n_1 + 5n_2}\right) = T_0\left(\frac{3n_1 + 20n_1}{3n_1 + 10n_1}\right) \qquad \ldots(ii)$

[From Eq. (i)

$$\Rightarrow \quad T = \frac{23}{13}T_0$$

Now, the final pressure P will be, (from equation of state given by

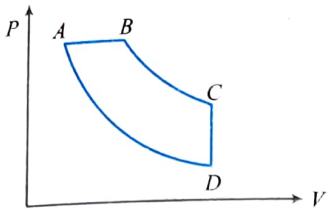
$$P = \frac{(n_1 + n_2)RT}{2V_0}$$

or, $\quad P = \frac{3n_1RT}{2V_0} = \frac{3R}{2V_0}\left(\frac{P_0V_0}{RT_0}\right)\left(\frac{23}{13}T_0\right) = \frac{69}{26}P_0$

Exercises

Single Correct Answer Type

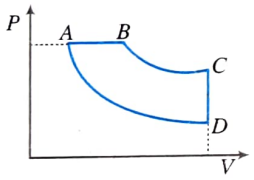
1. In the following pressure–volume diagram, the isochoric, isothermal and isobaric parts, respectively, are

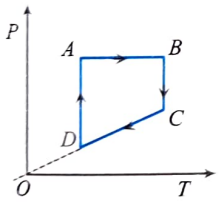


(1) BA, AD, DC (2) DC, CB, AD
(3) AB, BC, CD (4) CD, DA, AB

2. An ideal gas expands in such a manner that its pressure and volume can be related by equation PV^2 = constant. During this process, the gas is
(1) heated
(2) cooled
(3) neither heated nor cooled
(4) first heated and then cooled

3. A cyclic process $ABCD$ is shown in the following P–V diagram. Which of the following curves represents the same process?



(1)


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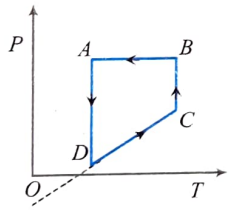

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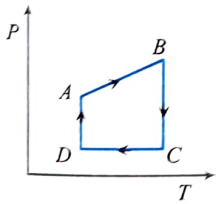

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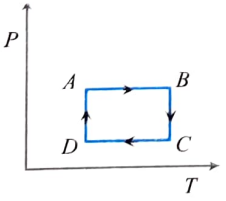
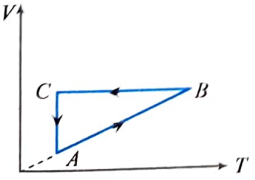


4. A cyclic process $ABCA$ is shown in the V–T diagram. Process on the P–V diagram is



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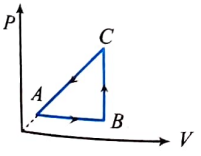

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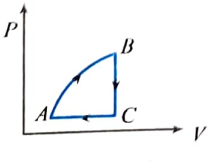

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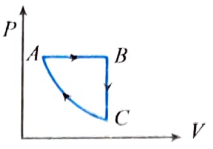

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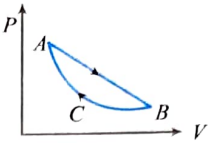
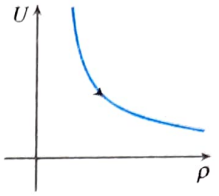
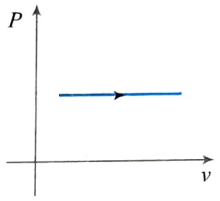


5. Variation of internal energy with density of 1 mole of monatomic gas is depicted in given figure. Corresponding variation of pressure with volume can be depicted as (assume the curve is rectangular hyperbola)



(1)


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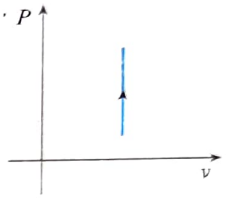

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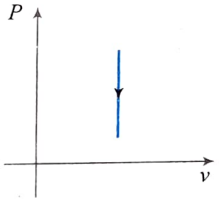

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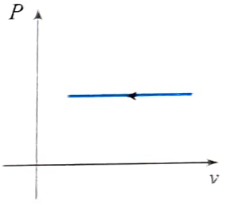
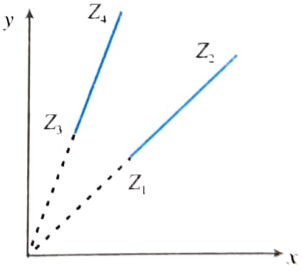


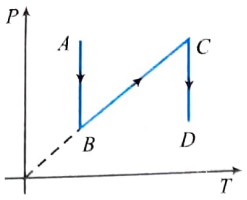
6. One mole of a diatomic gas undergoes a process $P = P_0/[1 + (V/V_0)^3]$ where P_0 and V_0 are constants. The translational kinetic energy of the gas when $V = V_0$ is given by
(1) $5P_0V_0/4$ (2) $3P_0V_0/4$
(3) $3P_0V_0/2$ (4) $5P_0V_0/2$

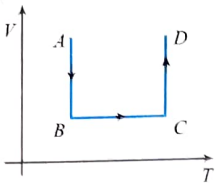
7. In the diagram as shown, find parameters representing x and y axes and also parameter Z, if $Z_1 = Z_2$, $Z_3 = Z_4$ and $Z_2 > Z_3$:



(1) Volume, pressure and temperature
(2) Temperature, volume and pressure, it is only possible option
(3) Temperature, pressure and volume, it is only possible option.
(4) Both (2) and (3)

8. P–T diagram is shown in given figure. Choose the corresponding V–T diagram.



(1)


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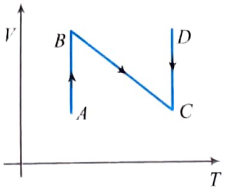

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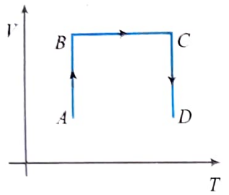

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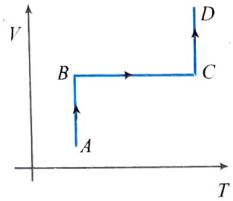
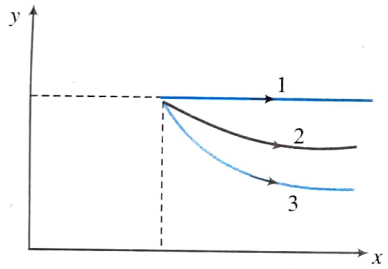


9. If the mean free path of atoms is doubled then the pressure of gas will become

(1) $P/4$ (2) $P/2$

(3) $P/8$ (4) P

10. For process 1, ΔU is positive, for process 2, ΔU is zero and for process 3, ΔU is negative. Find the parameters indicating x and y axes.



(1) Temperature and volume

(2) Temperature and pressure

(3) No. of moles and pressure

(4) Volume and pressure

11. At room temperature, the rms speed of the molecules of certain diatomic gas is found to be 1930 m/s. The gas is

(1) H_2 (2) F_2

(3) O_2 (4) Cl_2

12. A cubic vessel (with face horizontal + vertical) contains an ideal gas at NTP. The vessel is being carried by a rocket which is moving at a speed of 500 ms^{-1} in vertical direction. The pressure of the gas inside the vessel as observed by us on the ground

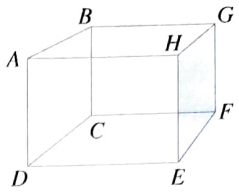
(1) remains the same because 500 ms^{-1} is very much smaller than v_{rms} of the gas

(2) remains the same because motion of the vessel as a whole does not affect the relative motion of the gas molecules and the walls

(3) will increase by a factor equal to $(v_{rms}^2 + (500)^2)/v_{rms}^2$ where v_{rms} was the original mean square velocity of the gas

(4) will be different on the top wall and bottom wall of the vessel

13. 1 mole of an ideal gas is contained in a cubical box of volume V, ABCDEFGH at 300K (figure). One face of the box (EFGH) is made up of a material which totally absorbs any gas molecule incident on it.



At any given time,

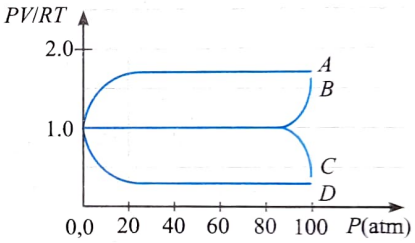
(1) the pressure on EFGH would be zero

(2) the pressure on all the faces will be equal

(3) the pressure on EFGH. would be double the pressure on ABCD

(4) the pressure on EFGH would be half that on ABCD

14. An experiment is carried on a fixed amount of gas at different temperatures and at high pressure such that it deviates from the ideal gas behaviour. The variation of $\frac{PV}{RT}$ with P is shown in the diagram. The correct variation will correspond to



(1) Curve A (2) Curve B

(3) Curve C (4) Curve D

15. Let A and B be the two gases such that $\frac{T_A}{M_A} = 4\frac{T_B}{M_B}$, where T is the temperature and M is the molecular mass.

If v_A and v_B are the rms speeds, then the ratio $\frac{v_A}{v_B}$ will be equal to

(1) 2 (2) 4

(3) 1 (4) 0.5

16. The temperature, pressure and volume of two gases X and Y are T, P and V, respectively. When the gases are mixed then the volume and temperature of mixture remains V and T, respectively. The pressure and mass of the mixture will be

(1) $2P$ and $2M$ (2) P and M

(3) P and $2M$ (4) $2P$ and M

17. The rms speed of the molecules of a gas in a vessel is 400 ms^{-1}. If half of the gas leaks out at constant temperature, the rms speed of the remaining molecules will be

(1) 800 ms^{-1} (2) $400\sqrt{2}$ ms^{-1}

(3) 400 ms^{-1} (4) 200 ms^{-1}

18. Gas at a pressure P_0 is contained in a vessel. If the masses of all the molecules are halved and their speeds are doubled, the resulting pressure P will be equal to

(1) $4P_0$ (2) $2P_0$

(3) P_0 (4) $P_0/2$

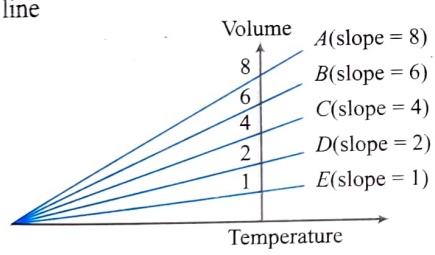
19. The root mean square speed of the molecules of a diatomic gas is v. When the temperature is doubled, the molecules dissociate into two atoms. The new root mean square speed of the atom is
(1) $\sqrt{2}v$ (2) v
(3) $2v$ (4) $4v$

20. A cylinder contains 10 kg of gas at pressure 10^7 N/m^2. The quantity of gas taken out of cylinder, if final pressure is 2.5×10^6 N/m, is (assume the temperature of gas as constant)
(1) Zero (2) 7.5 kg
(3) 2.5 kg (4) 5 kg

21. A flask is filled with 13 g of an ideal gas at 27°C and its temperature is raised to 52°C. The mass of the gas that has to be released to maintain the temperature of the gas in the flask at 52°C, the pressure remaining the same is
(1) 2.5 g (2) 2.0 g
(3) 1.5 g (4) 1.0 g

22. Air is filled at 60°C in a vessel of open mouth. The vessel is heated to a temperature T so that 1/4th part of air escapes. Assuming the volume of vessel remaining constant, the value of T is
(1) 80°C (2) 444°C
(3) 333°C (4) 171°C

23. The expansion of an ideal gas of mass m at a constant pressure P is given by the straight line D. Then the expansion of the same ideal gas of mass $2m$ at a pressure $P/2$ is given by the straight line



(1) E (2) C
(3) B (4) A

24. Two gases occupy two containers A and B; the gas in A, of volume 0.10 m^3, exerts a pressure of 1.40 MPa and that in B, of volume 0.15 m^3, exerts a pressure 0.7 MPa. The two containers are joined by a tube of negligible volume and the gases are allowed to intermingle. Then if the temperature remains constant, the final pressure in the container will be (in MPa)
(1) 0.70 (2) 0.98
(3) 1.40 (4) 2.10

25. A closed vessel contains 8 g of oxygen and 7 g of nitrogen. The total pressure is 10 atm at a given temperature. If now oxygen is absorbed by introducing a suitable absorbent, the pressure of the remaining gas in atm will be
(1) 2 (2) 10
(3) 4 (4) 5

26. Energy of all molecules of a monatomic gas having a volume V and pressure P is 3/2 PV. The total translational kinetic energy of all molecules of a diatomic gas at the same volume and pressure is
(1) 1/2 PV (2) 3/2 PV
(3) 5/2 PV (4) 3 PV

27. A gas mixture consists of 2 mol of oxygen and 4 mol of argon at temperature T. Neglecting all vibrational modes, the total internal energy of the system is
(1) $4RT$ (2) $15RT$
(3) $9RT$ (4) $11RT$

28. A stationary cylinder of oxygen used in a hospital has the following characteristics at room temperature 300 K, gauge pressure 1.38×10^7 Pa, volume 16 L. If the flow area, measured at atmospheric pressure, is constant at 2.4 L/min, the cylinder will last for nearly
(1) 5 h (2) 10 h
(3) 15 h (4) 20 h

29. Internal energy of n_1 mol of hydrogen of temperature T is equal to the internal energy of n_2 mol of helium at temperature $2T$. The ratio n_1/n_2 is
(1) $\frac{3}{5}$ (2) $\frac{2}{3}$
(3) $\frac{6}{5}$ (4) $\frac{3}{7}$

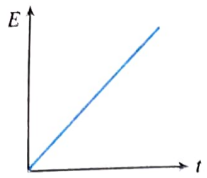
30. If 2 mol of an ideal monatomic gas at temperature T_0 are mixed with 4 mol of another ideal monatomic gas at temperature $2T_0$, then the temperature of the mixture is
(1) $\frac{5}{3}T_0$ (2) $\frac{3}{2}T_0$
(3) $\frac{4}{3}T_0$ (4) $\frac{5}{4}T_0$

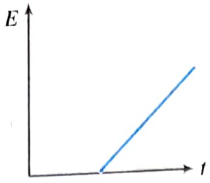
31. If P is the atmospheric pressure in the previous problem, find the percentage increase in tension of the string after heating:
(1) $\frac{25}{(1-P/P_0)}$ (2) $\frac{25}{(1-P_0/P)}$
(3) $25(1-P/P_0)$ (4) $25\,(1-P_0/P)$

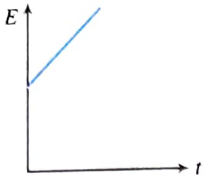
32. A gas mixture consists of molecules of type 1, 2 and 3 with molar masses $m_1 > m_2 > m_3$. v_{rms} and $\bar{K}$ are the rms speed and average kinetic energy of the gases, respectively. Which of the following is true?
(1) $(v_{rms})_1 < (v_{rms})_2 < (v_{rms})_3$ and $(\bar{K})_1 = (\bar{K})_2 = (\bar{K})_3$
(2) $(v_{rms})_1 = (v_{rms})_2 = (v_{rms})_3$ and $(\bar{K})_1 = (\bar{K})_2 > (\bar{K})_3$
(3) $(v_{rms})_1 > (v_{rms})_2 > (v_{rms})_3$ and $(\bar{K})_1 < (\bar{K})_2 > (\bar{K})_3$
(4) $(v_{rms})_1 > (v_{rms})_2 > (v_{rms})_3$ and $(\bar{K})_1 < (\bar{K})_2 < (\bar{K})_3$

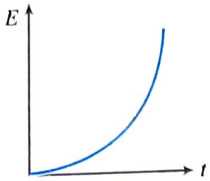
33. A box containing N molecules of an ideal gas at temperature T_1 and pressure P_1. The number of molecules in the box is doubled keeping the total kinetic energy of the gas same as before. If the new pressure is P_2 and temperature is T_2, then
(1) $P_2 = P_1, T_2 = T_1$
(2) $P_2 = P_1, T_2 = \frac{T_1}{2}$
(3) $P_2 = 2P_1, T_2 = T_1$
(4) $P_2 = 2P_1, T_2 = \frac{T_1}{2}$

34. The graph which represents the variation of mean kinetic energy of molecules with temperature t°C is

(1) 

(2) 

(3) 

(4) 

35. The average kinetic energy per molecule of helium gas at temperature T is E and the molar gas constant is R. Then Avogadro's number is equal to

(1) $\frac{RT}{2E}$ (2) $\frac{3RT}{E}$

(3) $\frac{E}{2RT}$ (4) $\frac{3RT}{2E}$

36. One mole of a gas is filled in a container at N.T.P.. The number of molecules in 1 cm^3 of volume will be

(1) $6.02 \times 10^{23}/22400$

(2) 6.02×10^{23}

(3) $1/22400$

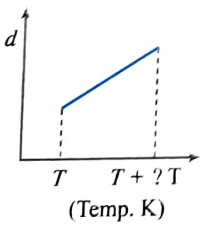
(4) $6.02 \times 10^{23}/76$

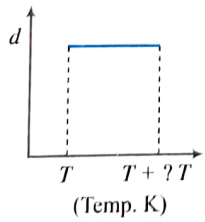
37. Two containers of equal volume contain the same gas at pressures P_1 and P_2 and absolute temperatures T_1 and T_2, respectively. On joining the vessels, the gas reaches a common pressure P and common temperature T. The ratio P/T is equal to

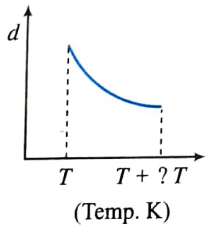
(1) $\frac{P_1}{T_1}+\frac{P_2}{T_2}$ (2) $\frac{P_1T_1+P_2T_2}{(T_1+T_2)^2}$

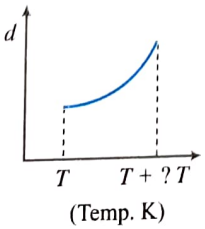
(3) $\frac{P_1T_2+P_2T_1}{(T_1+T_2)^2}$ (4) $\frac{P_1}{2T_1}+\frac{P_2}{2T_2}$

38. An ideal gas is initially at a temperature T and volume V. Its volume is increased by ΔV due to an increase in temperature ΔT, pressure remaining constant. The quantity $\delta = \Delta V/V\Delta T$ varies with temperature as

(1) 

(2) 

(3) 

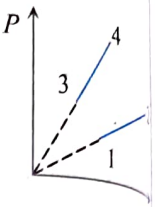
(4) 

39. If pressure of a gas contained in a closed vessel is increased by 0.4% when heated by 1°C, the initial temperature must be

(1) 250 K (2) 250°C

(3) 2500 K (4) 25°C

40. Pressure versus temperature graph of an ideal gas of equal number of moles of different volumes is plotted as shown in the given figure. Choose the correct alternative.



(1) $V_1 = V_2, V_3 = V_4$ and $V_2 > V_3$

(2) $V_1 = V_2, V_3 = V_4$ and $V_2 < V_3$

(3) $V_1 = V_2 = V_3 = V_4$

(4) $V_4 > V_3 > V_2 > V_1$

41. The capacity of a vessel is 3 L. It contains 6 g oxygen, 8 g nitrogen and 5 g CO_2 mixture at 27°C. If R = 8.31 J/mol K then the pressure in the vessel in N/m^2 will be (approx.)

(1) 5×10^5 (2) 5×10^4

(3) 10^6 (4) 10^5

42. A vessel of volume 20 L contains a mixture of hydrogen and helium at temperature of 27°C and pressure 2.0 atm. The mass of the mixture is 5 g. Assuming the gases to be ideal the ratio of the mass of hydrogen to that of helium in the given mixture will be

(1) 1:2 (2) 2:3

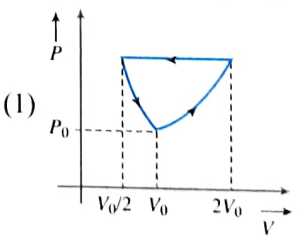
(3) 2:1 (4) 2:5

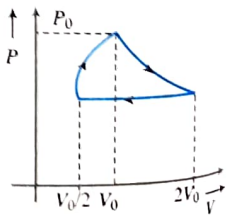
43. Two identical containers A and B have frictionless pistons. They contain the same volume of an ideal gas at the same temperature. The mass of the gas in A is m_A and that in B is m_B. The gas in each cylinder is now allowed to expand isothermally to double the initial volume. The change in the pressure in A and B, respectively, is Δp and 1.5 Δp. Then

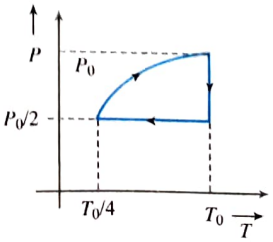
(1) $4m_A = 9m_B$ (2) $2m_A = 3m_B$

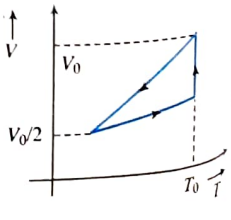
(3) $3m_A = 2m_B$ (4) $9m_A = 4m_B$

44. One mole of an ideal gas at pressure P_0 and temperature T_0 is expanded isothermally to twice its volume and then compressed at constant pressure to $(V_0/2)$ and the gas is brought back to original state by a process in which $P \propto V$ (pressure is directly proportional to volume). The correct temperature of the process is

(1) 

(2) 

(3) 

(4) 

45. A vessel contains a mixture of 7 g of nitrogen and 11 g of carbon dioxide at temperature $T = 300$ K. If the pressure of the mixture is 1 atm (1×10^5 N/m^2), its density is (gas constant $R = 25/3$ J/mol K)
(1) 0.72 kg/m^3
(2) 1.44 kg/m^3
(3) 2.88 kg/m^3
(4) 5.16 kg/m^3

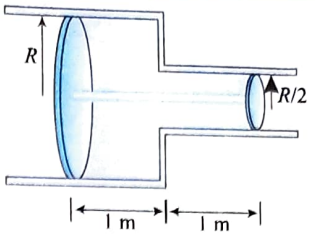
46. A spherical balloon contains air at temperature T_0 and pressure P_0. The balloon material is such that the instantaneous pressure inside is proportional to the square of the diameter. When the volume of the balloon doubles as a result of heat transfer, the expansion follows the law
(1) PV = constant
(2) $PV^{2/5}$ = constant
(3) PV^{-1} = constant
(4) $PV^{-2/3}$ = constant

47. The ratio of pressure of the same gas in two containers is $\frac{n_1T_1}{n_2T_2}$ where n_1 and n_2 are the number of moles and T_1 and T_2 are respective temperatures. If the two containers are now joined find the ratio of pressure to the temperature:
(1) $\frac{P_1T_2 + P_2T_1}{2T_1T_2}$ (2) $\frac{P_1T_1 + P_2T_1}{T_1T_2}$
(3) $\frac{P_1T_1 + P_2T_2}{T_1T_2}$ (4) None of these

48. On an isothermal process, there are two points A and B at which pressures and volumes are $(2P_0, V_0)$ and $(P_0, 2V_0)$ respectively. If A and B are connected by a straight line, find the pressure at a point on this straight line at which temperature is maximum:
(1) $\frac{4P_0}{3}$ (2) $\frac{5}{4}P_0$
(3) $\frac{3}{2}P_0$ (4) $\frac{7}{5}P_0$

49. In the previous problem, the temperature of the isothermal process is how much lower the maximum temperature:
(1) $\frac{P_0V_0}{nR}$ (2) $\frac{P_0V_0}{2nR}$
(3) $\frac{3P_0V_0}{2nR}$ (4) $\frac{P_0V_0}{4nR}$

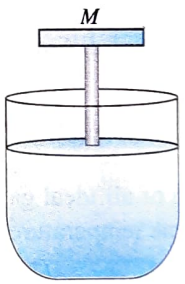
50. Two cylinders fitted with pistons and placed as shown, connected with string through a small tube of negligible volume, are filled with a gas at pressure P_0 and temperature T_0. The radius of smaller cylinder is half of the other. If the temperature is increased to $2T_0$, find the pressure, if the piston of bigger cylinder moves towards left by 1 metre?



(1) $\frac{4}{5}P_0$ (2) $\frac{3}{5}P_0$
(3) $\frac{2}{5}P_0$ (4) $\frac{5}{4}P_0$

51. Two identical containers A and B having same volume of an ideal gas at the same temperature have mass of the gas as m_A and m_B respectively. $2m_A = 3m_B$. The gas in each cylinder expand isothermally to double its volume. If the change in pressure in A is Δp, find the change in pressure in B:
(1) $2\Delta p$ (2) $3\Delta p$
(3) $\frac{2}{3}\Delta p$ (4) $\frac{4}{3}\Delta p$

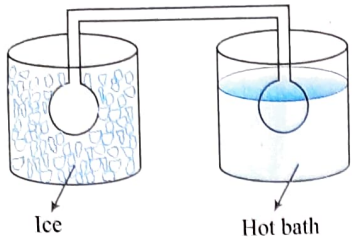
52. A cylinder containing an ideal gas is in vertical position and has a piston of mass M that is able to move up or down without friction (figure).



If the temperature is increased
(1) both p and V of the gas will change
(2) only p will increase according to Charles's law
(3) V will change but not p
(4) p will change but not V

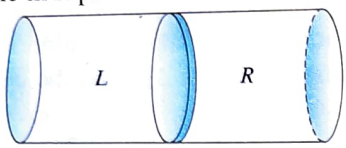
53. 1 mole of H_2 gas is contained in a box of volume $V = 1.00$ m^3 at $T_1 = 300$ K. The gas is heated to a temperature of $T_2 = 3000$ K and the gas gets converted into a gas of hydrogen atoms. The final pressure would be (considering all gases to be ideal)
(1) same as the pressure initially
(2) 2 times the pressure initially
(3) 10 times the pressure initially
(4) 20 times the pressure initially

54. Two identical glass bulbs are interconnected by a thin glass tube. A gas is filled in these bulbs at N.T.P. If one bulb is placed in ice and another bulb is placed in hot bath, then the pressure of the gas becomes 1.5 times. The temperature of hot bath will be



(1) 100°C (2) 182°C
(3) 256°C (4) 546°C

55. A vessel is partitioned in two equal halves by a fixed diathermic separator. Two different ideal gases are filled in left (L) and right (R) halves. The rms speed of the molecules in L part is equal to the mean speed of molecules in the R part. Then the ratio of the mass of a molecule in L part to that of a molecule in R part is



(1) $\sqrt{3/2}$ (2) $\sqrt{\pi/4}$
(3) $\sqrt{2/3}$ (4) $3\pi/8$

56. Read the given statements and decide which is/are correct on the basis of kinetic theory of gases

(I) Energy of one molecule at absolute temperature is zero.

(II) The rms speeds of different gases are same at same temperature.

(III) For one gram of all ideal gases kinetic energy is same at same temperature.

(IV) For one mole of all ideal gases' mean kinetic energy is same at same temperature.

(1) All are correct (2) I and IV are correct
(3) IV is correct (4) None of these

57. An air bubble of volume V_0 is released by a fish at a depth h in a lake. The bubble rises to the surface. Assume constant temperature and standard atmospheric pressure P above the lake. The volume of the bubble just before touching the surface will be (density of water is ρ)

(1) V_0 (2) $V_0(\rho gh/P)$
(3) $\dfrac{V_0}{\left(1+\dfrac{\rho gh}{P}\right)}$ (4) $V_0\left(1+\dfrac{\rho gh}{P}\right)$

58. The mean free path of nitrogen molecules at a pressure of 1.0 atm and temperature 0°C is 0.8×10^{-7} m. If the number of density of molecules is 2.7×10^{25} per m^3, then the molecular diameter is

(1) 3.2 nm (2) 3.2 Å
(3) 3.2 μm (4) 2.3 mm

59. The specific heat at constant volume for the monoatomic argon is 0.075 kcal/kg-K whereas its gram molecular specific heat C_v = 2.98 cal/mole/K. The mass of the argon atom is (Avogadro's number = 6.02×10^{23} molecules/mole)

(1) 6.60×10^{-23} g (2) 3.30×10^{-23} g
(3) 2.20×10^{-23} g (4) 13.20×10^{-23} g

60. 22 g of CO_2 at 27°C is mixed with 16 g of O_2 at 37°C. The temperature of the mixture is

(1) 32°C (2) 27°C
(3) 37°C (4) 30.5°C

61. When an ideal gas is compressed adiabatically, its temperature rises. The molecules, on an average, have more kinetic energy than before. The kinetic energy increases

(1) because of collisions with moving parts of the wall only

(2) because of collisions with the entire wall

(3) because the molecules gets accelerated in their motion inside the volume

(4) because of redistribution of energy amongst the molecules

Multiple Correct Answers Type

1. From the following statements, concerning ideal gas at any given temperature T, select the correct one(s):

(1) The coefficient of volume expansion at constant pressure is same for all ideal gases

(2) The average translational kinetic energy per molecule of oxygen gas is $3KT$ (K being Boltzmann constant)

(3) In a gaseous mixture, the average translational kinetic energy of the molecules of each component is same

(4) The mean free path of molecules increases with the decrease in pressure

2. A container holds 10^{26} molecule/m^3, each of mass 3×10^{-27} kg. Assume that $\frac{1}{6}$ of the molecules move with velocity 2000 ms^{-1} directly toward one wall of the container while the remaining $\frac{5}{6}$ of the molecules move either away from the wall or in perpendicular direction, and all collisions of the molecules with the wall are elastic:

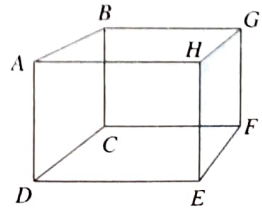
(1) Number of molecules hitting 1 m^2 of the wall every second is $\frac{1}{3}\times10^{29}$

(2) Number of molecules hitting 1 m^2 of the wall every second is 2×10^{29}

(3) Pressure exerted on the wall by molecules is 24×10^5 Nm^{-2}

(4) Pressure exerted on the wall by molecules is 4×10^5 Nm^{-2}

3. $ABCDEFGH$ is a hollow cube made of an insulator (see figure). Face $EFGH$ has positive charge on it. Inside the cube, we have ionised hydrogen.

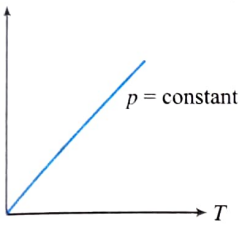


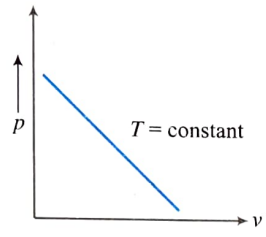
The usual kinetic theory expression for pressure

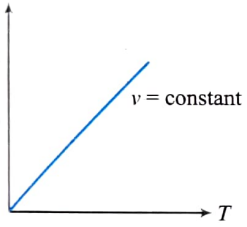
(1) will be valid
(2) will not be valid, since the ions would experience forces other than forces due to collisions with the walls
(3) will not be valid, since collisions with walls would not be elastic
(4) will not be valid because isotropy is lost

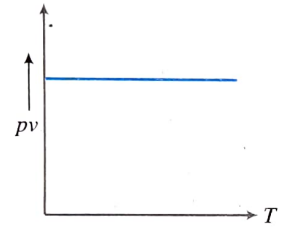
4. In a diatomic molecule, the rotational energy at a given temperature
(1) obeys Maxwell's distribution
(2) have the same value for all molecules
(3) equals the translational kinetic energy for each molecule
(4) is (2/3)rd the translational kinetic energy for each molecule

5. Which of the following diagrams (figure) depicts ideal gas behaviour?

(1) 

(2) 

(3) 

(4) 

6. A gas in container A is in thermal equilibrium with another gas in container B. Both contain equal masses of the two gases. Which of the following can be true?

(1) $P_A V_A = P_B V_B$ (2) $P_A = P_B, V_A \neq V_B$

(3) $P_A \neq P_B, V_A = V_B$ (4) $\frac{P_A}{V_A} = \frac{P_B}{V_B}$

7. Pick the correct statement(s):
(1) The rms translational speed for all ideal gas molecules at the same temperature is not the same but it depends on the mass.
(2) Each particle in a gas has average translational kinetic energy and the equation $1/2\ mv^2_{max} = 3/2\ kT$ establishes the relationship between the average translational kinetic energy per particle and temperature of an ideal gas.
(3) If the temperature of an ideal gas is doubled from 100°C to 200°C, the average kinetic energy of each particle is also doubled.
(4) It is possible for both pressure and volume of a monatomic ideal gas to change simultaneously without causing the internal energy of the gas to change.

8. Two identical containers each of volume V_0 are joined by a small pipe. The containers contain identical gases at temperature T_0 and pressure P_0. One container is heated to temperature $2T_0$ while maintaining the other at the same temperature. The common pressure of the gas is P and n is the number of moles of gas in container at temperature $2T_0$. Then

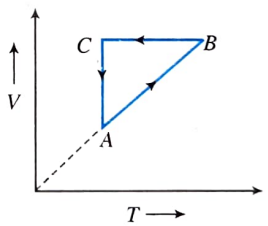
(1) $P = 2P_0$ (2) $P = \frac{4}{3}P_0$

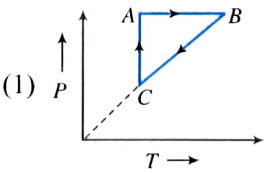
(3) $n = \frac{2}{3}\frac{P_0 V_0}{RT_0}$ (4) $n = \frac{3}{2}\frac{P_0 V_0}{RT_0}$

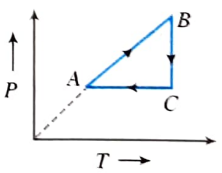
9. During an experiment, an ideal gas is found to obey a condition P^2/ρ = constant (ρ = density of the gas). The gas is initially at temperature T, pressure P and density ρ. The gas expands such that density changes to $\rho/2$.
(1) The pressure of the gas changes to $\sqrt{2}\,P$.
(2) The temperature of the gas changes to $\sqrt{2}\,T$.
(3) The graph of the above process on the P–T diagram is parabola.
(4) The graph of the above process on the P–T diagram is hyperbola.

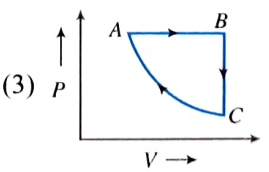
10. A closed vessel contains a mixture of two diatomic gases A and B. Molar mass of A is 16 times that of B and mass of gas A contained in the vessel is 2 times that of B. Which of the following statements are correct?
(1) Average kinetic energy per molecule of A is equal to that of B.
(2) Root-mean-square value of translational velocity of B is four times that of A.
(3) Pressure exerted by B is eight times of that exerted by A.
(4) Number of molecules of B, in the cylinder, is eight times that of A.

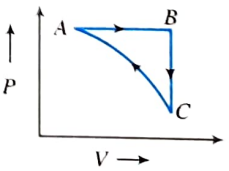
11. An ideal gas undergoes a thermodynamic cycle as shown in the given figure. Which of the following graphs represent the same cycle?



(1) 

(2) 

(3) 

(4) 

12. Which of the following statements are correct?
(1) Two bodies at different temperatures T_1 and T_2 are brought in thermal contact. When thermal equilibrium is attained, the temperature of each body is $(T_1 + T_2)/2$.
(2) The coolant used in a car or a chemical or nuclear plant should have high specific heat.
(3) Vapour in equilibrium with its liquid at a constant temperature does not obey Boyle's law.

(4) Two vessels A and B of equal capacity are connected to each other by a stop cock. Vessel A contains a gas at 0° C and 1 atm pressure. Vessel B is completely evacuated. When the stop cock is opened, the final pressure of gas in each vessel will be 0.5 atm.

13. An ideal gas is taken from state A (pressure P, volume V) to state B (pressure $P/2$, volume $2V$) along a straight line path in the P–V diagram. Select the correct statements from the following:

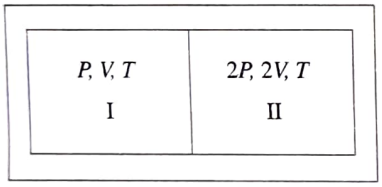
(1) The work done by the gas in the process A to B exceeds the work done that would be done by it if the system were taken from A to B along an isotherm.

(2) In the T–V diagram, the path AB becomes a part of a parabola.

(3) In the P–T diagram, the path AB becomes a part of hyperbola.

(4) In going from A to B, the temperature T of the gas first increases to a maximum value and then decreases.

14. A partition divides a container having insulated walls into two compartments I and II. The same gas fills the two compartments whose initial parameters are given. The partition is a conducting wall which can move freely without friction. Which of the following statements is/are correct, with reference to the final equilibrium position?



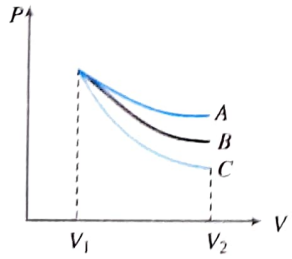
(1) The pressures in the two compartments are equal.

(2) Volume of compartment I is $3V/5$.

(3) Volume of compartment II is $12V/5$.

(4) Final pressure in compartment I is $5P/3$.

15. An ideal gas undergoes an expansion from a state with temperature T_1 and volume V_1 through three different polytropic processes A, B and C as shown in the P–V diagram. If $|\Delta E_A|$, $|\Delta E_B|$ and $|\Delta E_C|$ be the magnitude of changes in internal energy along the three paths respectively, then:



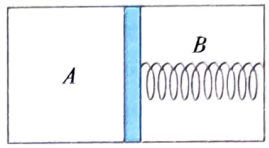
(1) $|\Delta E_A| < |\Delta E_B| < |\Delta E_C|$ if temperature in every process decreases

(2) $|\Delta E_A| > |\Delta E_B| > |\Delta E_C|$ if temperature in every process decreases

(3) $|\Delta E_A| > |\Delta E_B| > |\Delta E_C|$ if temperature in every process increases

(4) $|\Delta E_B| < |\Delta E_A| < |\Delta E_C|$ if temperature in every process increases

16. A thermally insulated chamber of volume $2V_0$ is divided by a frictionless piston of area S into two equal parts A and B. Part A has an ideal gas at pressure P_0 and temperature T and part B is vacuum. A massless spring of force constant K is connected with the piston and the wall of the container as shown. Initially the spring is unstretched. The gas inside chamber A is allowed to expand. Let in equilibrium the spring be compressed by x_0. Then



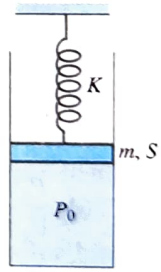
(1) final pressure of the gas is $\dfrac{Kx_0}{S}$

(2) work done by the gas is $\dfrac{1}{2}Kx_0^2$

(3) change in internal energy of the gas is $\dfrac{1}{2}Kx_0^2$

(4) temperature of the gas is decreased

17. In the arrangement shown in the given figure, gas is thermally insulated. An ideal gas is filled in the cylinder having pressure P_0 (> atmospheric pressure P_a). The spring of force constant K is initially unstretched. The piston of mass m and area S is frictionless. In equilibrium, the piston rises up by distance x_0, then



(1) final pressure of the gas is $P_0 + \dfrac{Kx_0}{S} + \dfrac{mg}{S}$

(2) work done by the gas is $\dfrac{1}{2}Kx_0^2 + mgx_0$

(3) decrease in internal energy of the gas is

$$\frac{1}{2}Kx_0^2 + mgx_0 + P_0Sx_0$$

(4) all of the above

Linked Comprehension Type

For Problems 1–2

Two identical vessels A and B are connected by a valve, which operates (opens up) at not less than a pressure difference of 0.25×10^5 Pa on its two sides. Initially, the vessel A contains an ideal gas at a pressure of 1×10^5 Pa, while vessel B contains an ideal gas at a pressure of 1.5×10^5 Pa. The two vessels are

initially at temperature of 27°C. If the two vessels be maintained at this temperature (27°C), then

1. The ratio of number of moles in vessel A and vessel B is

(1) 3/2 (2) 2/3

(3) 5/4 (4) 4/5

2. Which of the following statements are true about the fractional increase or decrease in the number of molecules in the two vessels?

(1) The fraction of molecules escaped from vessel B is 1/12.

(2) The fraction of molecules entered vessel A is 1/8.

(3) The fraction of molecules escaped from vessel B is 1/6.

(4) The fraction of molecules entered vessel A is 1/4.

For Problems 3–5

The Earth's atmosphere consists primarily of oxygen (21%) and nitrogen (78%). The rms speed of oxygen molecules (O_2) in the atmosphere at a certain location is 535 m/s.

[Given 1 amu = 1.66×10^{-27} kg, molecular mass of O_2 = 32, molecular mass of N_2 = 28]

3. What is the temperature of the atmosphere at this location?

(1) 367 K (2) 437 K

(3) 567 K (4) 737 K

4. Would the rms speed of nitrogen molecules (N_2) at this location

(1) be higher than 535 m/s

(2) be equal to 535 m/s

(3) be lower than 535 m/s

(4) cannot be calculated from given data

5. The rms speed of N_2 at this location will be

(1) 437 m/s (2) 572 m/s

(3) 835 m/s (4) 715 m/s

For Problems 6–7

A volume V_0 of helium is confined within a chamber. The initial pressure is P_A (atmospheric pressure) and the initial temperature is T_0. Helium is monatomic gas, (ideal). Each atom of helium has a mass of 4 u, where 1 u = 1.66×10^{-27} kg.

6. What is the number of atoms of helium in the sample? (k = Boltzmann constant, R = gas constant).

(1) $\frac{P_A V_0}{RT_0}$ (2) $\frac{2P_A V_0}{3RT_0}$

(3) $\frac{P_A V_0}{kT_0}$ (4) $\frac{2P_A V_0}{3kT_0}$

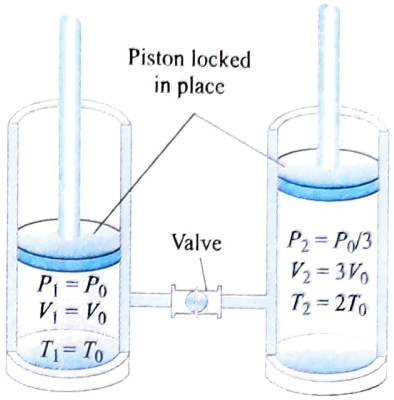
7. What is total internal energy u_0 of the gas?

(1) $\frac{1}{2}P_A V_0$ (2) $\frac{3}{2}P_A V_0$

(3) $\frac{5}{2}P_A V_0$ (4) $\frac{2}{3}P_A V_0$

For Problems 8–9

Two thermally insulated vessels are connected by a narrow tube fitted with a valve that is initially closed as shown in figure. One vessel of volume V_0 contains oxygen at a temperature of T_0 and a pressure of P_0. The other vessel of volume $3V_0$ contains oxygen at a temperature of $2T_0$ K and a pressure of $\frac{P_0}{3}$. When the valve is opened, the gases in the two vessels mix and the temperature and pressure become uniform throughout.



8. What is the final temperature?

(1) $\frac{5}{3}T_0$ (2) $\frac{10}{3}T_0$

(3) $\frac{20}{3}T_0$ (4) $5\,T_0$

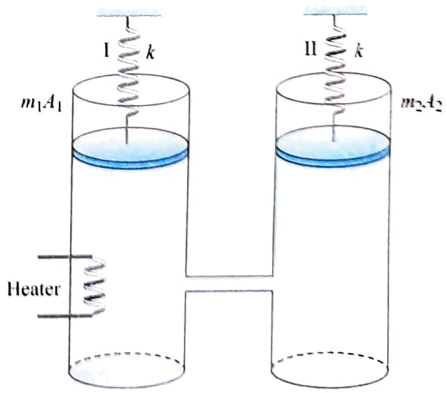
9. What is the final pressure?

(1) P_0 (2) $\frac{2P_0}{3}$

(3) $\frac{P_0}{2}$ (4) $\frac{P_0}{3}$

For Problems 10–12

Two vertical cylinders of different areas of cross-sections ($A_2 < A_1$) shown in the figures have pistons of same mass ($m_1 = m_2$) fitted on the top. The cylinders enclose N moles of monatomic gas. The initial pressure and initial volume of the gas are P and V, respectively. It is given that spring I is relaxed. Now through the electric heater, heat is slowly supplied. Finally spring II becomes relaxed. It is given that the cylinders and the pistons are thermally insulated.



10. What is the initial extension of spring II?

(1) $\frac{mg}{2K}\left(1-\frac{A_2}{A_1}\right)$ (2) $\frac{mg}{K}\left(1+\frac{A_2}{A_1}\right)$

(3) $\frac{mg}{K}\left(1-\frac{A_2}{A_1}\right)$ (4) $\frac{mg}{2K}\left(1+\frac{A_2}{A_1}\right)$

11. What is the final pressure?

(1) $P+mg\left(\frac{A_1+A_2}{A_1A_2}\right)$ (2) $P+mg\left(\frac{A_1-A_2}{A_1A_2}\right)$

(3) $P+2mg\left(\frac{A_1-A_2}{A_1A_2}\right)$ (4) $P-2mg\left(\frac{A_1+A_2}{A_1A_2}\right)$

12. What is the final volume?

(1) $V+\frac{mg}{k}\frac{(A_1+A_2)(A_1^2+A_2^2)}{A_1A_2}$

(2) $V+\frac{mg}{k}\frac{(A_1-A_2)(A_1^2+A_2^2)}{A_1A_2}$

(3) $V+\frac{mg}{2k}\frac{(A_1-A_2)(A_1^2+A_2^2)}{A_1A_2}$

(4) $V+\frac{2mg}{k}\frac{(A_1-A_2)(A_1^2+A_2^2)}{A_1A_2}$

Numerical Value Type

1. A vessel of volume $V = 20$ litre contains a mixture of H_2 and He at 20°C and pressure $P = 2$ atm. The mass of the mixture is $m = 5$ g. Find the ratio of masses $\frac{m_1}{m_2}$ where, m_1 = mass of H_2 and m_2 = mass of He.

2. The temperature of a gas contained in a closed vessel increases by 1°C when pressure of the gas is increased by 1 %. Find the initial temperature of the gas (in K).

3. A vessel contains 1 mole O_2 gas at a temperature T. An identical vessel contains one mole of He gas at a temperature $2T$. Find the ratio of their pressure (He to O_2).

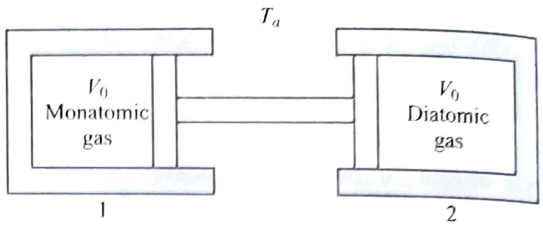
4. A vessel contains a mixture of 1 mole of oxygen and 2 moles of nitrogen at 300 K. Find the ratio of average rotational kinetic energies per O_2 molecule to per N_2 molecule.

5. Calculate the pressure exerted by a mixture of 8 g of oxygen, 14 g of nitrogen and 22 g of carbon dioxide in a container of 10 L at a temperature of 27°C [in 10^5 N/m^2 and to the nearest integer].

6. A cylinder of capacity 20 litres is filled with H_2 gas. The total average kinetic energy of translatory motion of its molecules is 1.5×10^5 J. The pressure of hydrogen in the cylinder is $N \times 10^6$ N/m^2. Find the value of N?

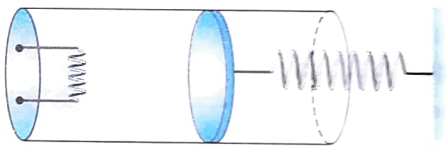
7. A vessel of volume $3V$ contains a gas at pressure $4P_0$ and another vessel of volume $2V$ contains the same gas at pressure $1.5P_0$. Both vessels have same temperature. When both vessels are connected by a tube of negligible volume, the equilibrium is IP_0, where I is an integer. Find the value of I.

8. The two conducting cylinder–piston systems shown below are linked. Cylinder 1 is filled with a certain molar quantity of a monatomic ideal gas, and cylinder 2 is filled with an equal molar quantity of a diatomic ideal gas. The entire apparatus is situated inside an oven whose temperature is $T_a = 27$°C. The cylinder volumes have the same initial value $V_0 = 100$ cc. When the oven temperature is slowly raised to $T_b = 127$°C. What is the volume change ΔV (in cc) of cylinder 1?

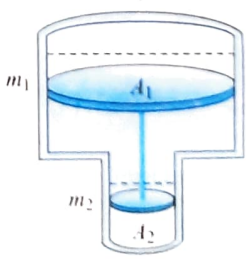


9. A vessel of volume 33.2×10^{-3} m^3 contains an ideal gas at 300 K and 200 kPa. The gas is allowed to leak till the pressure falls to 125 kPa. Calculate the amount of the gas (in moles) leaked assuming that the temperature remains constant? (Given $R = 8.3$ J/mol-K)

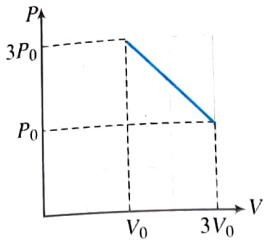
10. An ideal monatomic gas is confined in a cylinder by a spring-loaded piston of cross-section 8.0×10^{-3} m^2. Initially the gas is at 300 K and occupies a volume of 2.4×10^{-3} m^3 and the spring is in its relaxed state. The gas is heated by a small heater until the piston moves out slowly by 0.1 m. The final temperature of the gas is $N \times 100$ K. Find the value of N. The force constant of the spring is 8000 N/m, and the atmospheric pressure is 1.0×10^5 N/m^2. The cylinder and the piston are thermally insulated. The piston and the spring are massless and there is no friction between the piston and the cylinder. Neglect any heat-loss through the lead wires of the heater. The heat capacity of the heater coil is negligible.



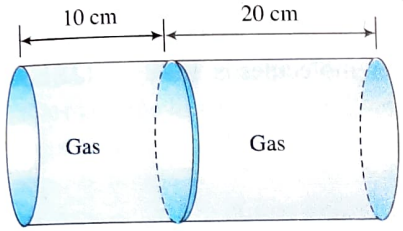
11. Consider a vertical tube open at both ends. The tube consists of two parts, each of different cross-sections and each part having a piston which can move smoothly in respective tubes. The two pistons are joined together by an inextensible wire. The combined mass of the two pistons is 5 kg and area of cross-section of the upper piston is 10 cm^2 greater than that of the lower piston. Amount of gas enclosed by the pistons is one mole. When the gas is heated slowly, pistons move by 50 cm. If the rise in the temperature of the gas is $\frac{15x}{R}$, where R is universal gas constant. Find the value of x. Use $g = 10$ m/s^2 and outside pressure = 10^5 N m^{-2}).



12. n moles of a gas in a cylinder under a piston are transferred to infinitely slowly from a state with a volume of V_0 and a pressure $3P_0$ atoms to state with $3V_0$ and a pressure P_0 as shown in figure. If the maximum temperature that the gas will reach in this process is $\frac{x\,P_0V_0}{nR}$, what is the value of x?



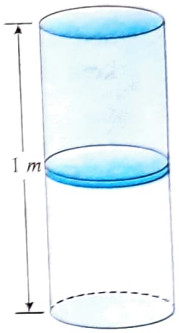
13. Given figure shows a horizontal cylindrical container of length 30 cm, which is partitioned by a tight-fitting separator. The separator is diathermic but conducts heat very slowly. Initially the separator is in the state shown in the figure. The temperature of left part of cylinder is 100 K and that on right part is 400 K. Initially the separator is in equilibrium. As heat is conducted from right to left part, separator displaces to the right. Find the displacement of separator (in cm) after a long when gases on the two parts of cylinder are in thermal equilibrium.



14. An ideal gas has a volume 0.3 m^3 at 150 kPa. It is confined by a spring-loaded piston in a vertical cylinder. Initially, the spring is in relaxed state. If the gas is heated to a final state of 0.5 m^3 and pressure 600 kPa, find the work done (in kJ) on the spring (atmospheric pressure, $P_0 = 1 \times 10^5$ N/m^2).

15. Oxygen is filled in a closed metal jar of volume 1.0×10^{-3} m^3 at a pressure of 1.5×10^5 Pa and temperature 400 K. The jar has a small leak in it. The atmospheric pressure is 1.0×10^5 Pa and the atmospheric temperature is 300 K. Find the mass (in g) of the gas that leaks out by the time the pressure and the temperature inside the jar equalise with the surrounding.

16. Figure shows a vertical cylindrical vessel separated in two parts by a frictionless piston free to move along the length of the vessel. The length of the cylinder is 1.0 m and the piston divides the cylinder in the ratio of 3 : 2. Each of the two parts of the vessel contains 0.2 mole of an ideal gas. The temperature of the gas is 300 K in each part. Calculate the mass of the piston. (in kg)



Archives

JEE MAIN

Single Correct Answer Type

1. One kilogram of a diatomic gas is at a pressure of 8×10^4 N/m^2. The density of the gas is 4 kg/m^3. What is the energy of the gas due to its thermal motion?

(1) 3×10^4 J (2) 5×10^4 J
(3) 6×10^4 J (4) 7×10^4 J

(AIEEE 2009)

2. Three perfect gases at absolute temperatures T_1, T_2 and T_3 are mixed. The masses of molecules are m_1, m_2 and m_3 and the number of molecules are n_1, n_2 and n_3 respectively. Assuming no loss of energy, the final temperature of the mixture is

(1) $\frac{(T_1+T_2+T_3)}{3}$ (2) $\frac{n_1T_1+n_2T_2+n_3T_3}{n_1+n_2+n_3}$

(3) $\frac{n_1T_1^2+n_2T_2^2+n_3T_3^2}{n_1T_1+n_2T_2+n_3T_3}$ (4) $\frac{n_1^2T_1^2+n_2^2T_2^2+n_3^2T_3^2}{n_1T_1+n_2T_2+n_3T_3}$

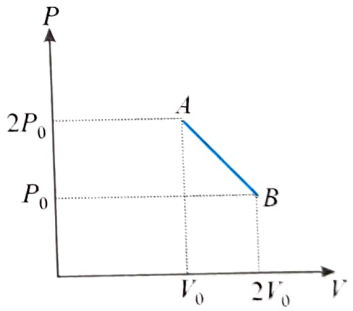
(AIEEE 2011)

3. Consider a spherical shell of radius R at temperature T. The black body radiation inside it can be considered as an ideal gas of photons with internal energy per unit volume $u = \frac{U}{V} \propto T^4$ and pressure $P = \frac{1}{3}\left(\frac{U}{V}\right)$. If the shell now undergoes an adiabatic expansion the relation between T and R is

(1) $T \propto e^{-R}$ (2) $T \propto e^{-3R}$
(3) $T \propto \frac{1}{R}$ (4) $T \propto \frac{1}{R^3}$

(JEE Main 2015)

4. 'n' moles of an ideal gas undergoes a process $A \rightarrow B$ as shown in the figure. The maximum temperature of the gas during the process will be



(1) $\frac{9P_0V_0}{4nR}$ (2) $\frac{P_0V_0}{nR}$
(3) $\frac{9P_0V_0}{2nR}$ (4) $\frac{9P_0V_0}{nR}$

(JEE Main 2016)

5. The temperature of an open room of volume 30 m^3 increases from 17°C to 27°C due to sunshine. The atmospheric pressure in the room remains 1×10^5 Pa. If n_i and n_f are the number of molecules in the room before and after heating, then $n_f - n_i$ will be

(1) 2.5×10^{25} (2) -2.5×10^{25}
(3) -1.61×10^{23} (4) 1.38×10^{23}

(JEE Main 2017)

JEE ADVANCED

Single Correct Answer Type

1. A real gas behaves like an ideal gas if its
(1) pressure and temperature are both high
(2) pressure and temperature are both low
(3) pressure is high and temperature is low
(4) pressure is low and temperature is high

(IIT-JEE 2010)

2. A mixture of 2 moles of helium gas (atomic mass = 4 amu), and 1 mole of argon gas (atomic mass = 40 amu) is kept at 300 K in a container. The ratio of the rms speeds $\left(\frac{v_{rms}(\text{helium})}{v_{rms}(\text{argon})}\right)$ is

(1) 0.32 (2) 0.45
(3) 2.24 (4) 3.16

(IIT-JEE 2012)

3. Two non-reactive monoatomic ideal gases have their atomic masses in the ratio 2 : 3. The ratio of their partial pressures, when enclosed in a vessel kept at a constant temperature, is 4 : 3. The ratio of their densities is
(1) 1 : 4 (2) 1 : 2
(3) 6 : 9 (4) 8 : 9

(JEE Advanced 2013)

Multiple Correct Answers Type

1. A container of fixed volume has a mixture of one mole of hydrogen and one mole of helium in equilibrium at temperature T. Assuming the gases are ideal, the correct statement(s) is(are):
(1) The average energy per mole of the gas mixture is $2RT$
(2) The ratio of speed of sound in the gas mixture to that in helium gas is $\sqrt{6/5}$
(3) The ratio of the rms speed of helium atoms to that of hydrogen molecules is 1/2
(4) The ratio of the rms speed of helium atoms to that of hydrogen molecules is $1/\sqrt{2}$ **(JEE Advanced 2015)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	2. (2)	3. (1)	4. (3)	5. (4)
6. (2)	7. (4)	8. (4)	9. (2)	10. (4)
11. (1)	12. (2)	13. (4)	14. (2)	15. (1)
16. (1)	17. (3)	18. (2)	19. (3)	20. (2)
21. (4)	22. (4)	23. (4)	24. (2)	25. (4)
26. (2)	27. (4)	28. (3)	29. (3)	30. (1)
31. (1)	32. (1)	33. (2)	34. (3)	35. (4)
36. (1)	37. (4)	38. (3)	39. (1)	40. (1)
41. (1)	42. (4)	43. (3)	44. (3)	45. (2)
46. (4)	47. (1)	48. (3)	49. (4)	50. (4)
51. (4)	52. (3)	53. (4)	54. (4)	55. (4)
56. (3)	57. (4)	58. (2)	59. (1)	60. (1)
61. (1)				

Multiple Correct Answers Type

1. (1),(3),(4)	2. (1),(4)	3. (2),(4)
4. (1),(4)	5. (1),(3)	6. (2),(3)
7. (1),(4)	8. (2),(3)	9. (2),(3)
10. (1),(2),(3),(4)	11. (1),(3)	12. (2),(3),(4)
13. (1),(2),(4)	14. (1),(2),(3),(4)	15. (1),(3)
16. (1),(2),(3),(4)	17. (1),(3)	

Linked Comprehension Type

1. (2)	2. (1),(2)	3. (1)	4. (1)	5. (2)
6. (3)	7. (2)	8. (2)	9. (3)	10. (3)
11. (2)	12. (2)			

Numerical Value Type

1. (0.5)	2. (100)	3. (2)	4. (1)	5. (3)
6. (5.0)	7. (3)	8. (0)	9. (1)	10. (8)
11. (5)	12. (4)	13. (10)	14. (45)	15. (0.16)
16. (41.5)				

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)	2. (2)	3. (3)	4. (1)	5. (2)

JEE Advanced

Single Correct Answer Type

1. (4)	2. (4)	3. (4)

Multiple Correct Answers Type

1. (1),(2),(4)

4 Thermodynamics

INTRODUCTION

Thermodynamics is a branch of science which deals with exchange of heat energy between bodies and conversion of the heat energy into mechanical energy and vice-versa.

In thermodynamics, we describe the state of a system using variables such as pressure, volume, temperature, and internal energy. As a result, these quantities belong to a category called **state variables**. For any given configuration of the system, we can identify values of the state variables. For mechanical systems, the state variables include kinetic energy K and potential energy U. A state of a system can be specified only if the system is in thermal equilibrium internally. In the case of a gas in a container, internal thermal equilibrium requires every part of the gas be at the same pressure and temperature.

THERMODYNAMIC SYSTEM

A definite quantity of matter bounded by some closed surface is called a system and a collection of very large number of molecules of matter (solid, liquid or gas) which are so arranged that these possess certain values of pressure, volume and temperature forms a thermodynamics system.

The parameters (pressure, volume, temperature, internal energy etc.) which determine the state or condition of a system are called **thermodynamic state variables** or **thermodynamic coordinates**.

Anything outside the system which can exchange energy with it and has direct effect on its behavior is called surrounding.

Here, we look at some details about how energy can be transferred as heat and work between a system and its environment. If we take a gas confined to a cylinder with a movable piston, we call the gas as system. Every system is confined within certain boundaries. Anything outside the thermodynamic system to which energy or matter is exchanged is called its surroundings.

Thermodynamic system may be of three types:

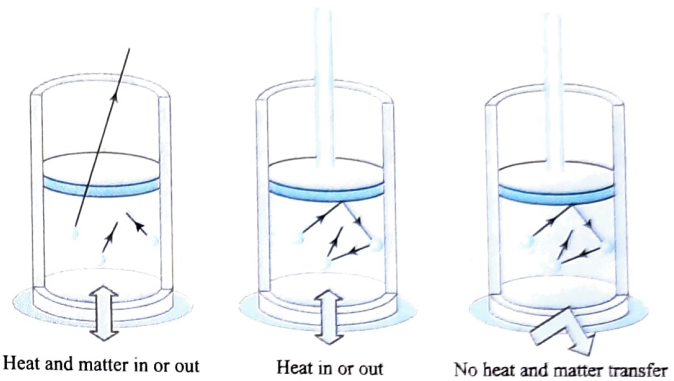
(i) Open system: It exchanges both energy and matter with the surroundings.

(ii) Closed system: It exchanges only energy (not matter) with the surroundings.

(iii) Isolated system: It exchanges neither energy nor matter with the surroundings.

A thermodynamic system can be described by specifying its pressure, volume, temperature, internal energy and the number of moles. These parameters are called thermodynamic variables. The relation between the thermodynamic variables (P, V, T) of the system is called equation of state.

For n moles of an ideal gas, equation of state is $PV = nRT$.



Heat and matter in or out — **(a) Open system**

Heat in or out — **(b) Closed system**

No heat and matter transfer — **(c) Isolated system**

THERMODYNAMIC PROCESS

Thermodynamics basically deals with the exchange of heat between a body and the surrounding along with the other processes accompanying it, such as the work involved and the changes in the internal energy, taking place in the body. Such studies are primarily carried out in terms of a few macroscopic properties, such as pressure, volume, temperature, etc. Any alteration in one or more of the macroscopic properties of a body renders specific changes in other related properties, and the body is said to have been subjected to a thermodynamic process (or is said to have undergone a thermodynamic process). The body which is subjected to a thermodynamic process is known as a thermodynamic system. The macroscopic parameters in terms of which the study of a thermodynamic process is carried out are known as thermodynamic variables. For example, gas enclosed in a cylinder fitted with a piston forms the thermodynamic system but the atmospheric air around the cylinder, movable piston, burner, etc., is the surrounding.

The process of change of state of a system involves change of thermodynamic variables such as pressure P, volume V and temperature T of the system. During such a process, energy may be transferred into the system from the thermal reservoir (positive heat) or vice versa (negative heat). The process is known as thermodynamic process. Some important processes are:

- **Isothermal process:** Temperature remain constant
- **Adiabatic process:** No transfer of heat
- **Isobaric process:** Pressure remains constant
- **Isochoric (isovolumetric process):** Volume remains constant
- **Cyclic and non-cyclic processes:** In cyclic process, initial and final states are same while in non-cyclic process these states are different.

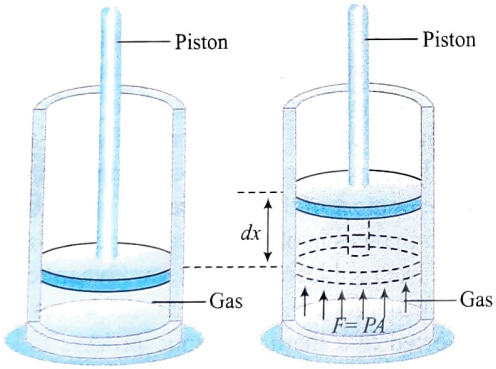
QUASI-STATIC PROCESS

When we perform a process on a given system, its state is, in general, changed. Suppose the initial state of the system is described by the values P_1, V_1, T_1 and the final state by P_2, V_2, T_2. If the process is performed in such a way that at any instant during the process, the system is very nearly in thermodynamic equilibrium, the process is called quasi-static. This means, we can specify the parameters P, V, T uniquely at any instant during such a process.

Actual processes are not quasi-static. To change the pressure of a gas, we can move a piston inside the enclosure. The gas near the piston is acted upon by piston. The pressure of the gas may not be uniform everywhere while the piston is moving. However, we can move the piston very slowly to make the process as close to quasi-static as we wish. Thus, a quasi-static process is an idealised process in which all changes take place infinitely slowly.

WORK DONE BY AN IDEAL GAS

Let us take a system; a gas confined to a cylinder with a movable piston, as shown in the given figure. The walls of the cylinder are made of insulating material that does not allow any transfer of energy as heat. The bottom of the cylinder rests on a reservoir for thermal energy, a thermal reservoir (perhaps a hot plate) whose temperature T you can control by turning a knob.



The system (the gas) starts from an initial state i, described by a pressure P_i, a volume V_i, and a temperature T_i. You want to change the system to a final state f, described by a pressure P_f, a volume V_f, and a temperature T_f.

If P be the pressure of the gas in the cylinder at any time, then force exerted by the gas on the piston of the cylinder $F = PA$ (A = Area of cross-section of piston)

When the piston is pushed outward an infinitesimal distance dx, the work done by the gas

$$dW = F \cdot dx = P(A\,dx) = P\,dV$$

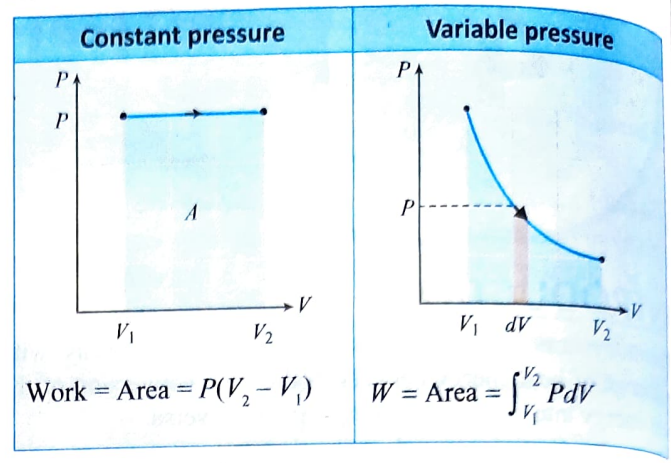
For a finite change in volume from V_i to V_f,

Total amount of work done, $W = \int_{V_i}^{V_f} P\,dV$ ...(i)

During the volume change, the pressure and temperature may also change. To evaluate Eq. (i) directly, we would need to know how pressure varies with volume for the actual process by which the system changes from state i to state f.

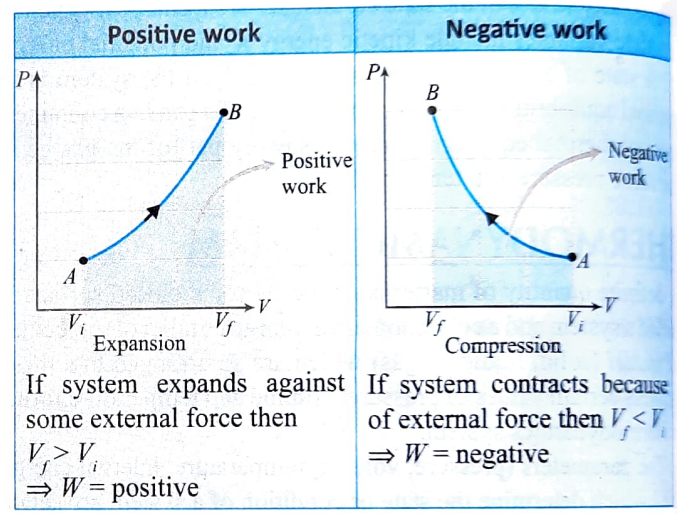
The graphical representation of the change in state of a gas by a thermodynamic process is called *indicator diagram*. Indicator diagram is plotted generally in pressure and volume of gas. If we draw indicator diagram, the area bounded by P–V graph and volume axis represents the work done.

Calculation of work done from P–V diagrams

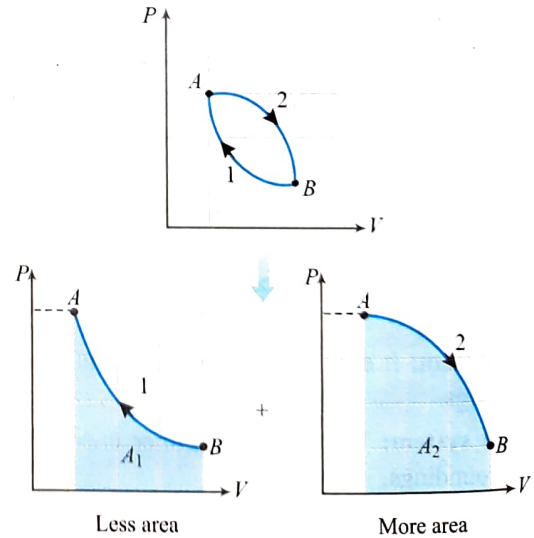


(ii) From $W = \int_{V_i}^{V_f} P\,dV$

Positive and negative work



Work done depends upon initial and final state of the system and path adopted for the process.



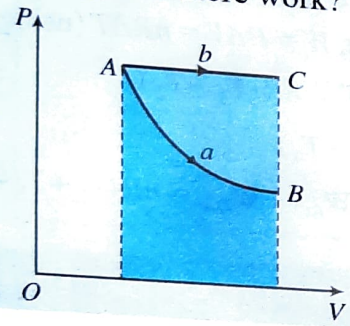
$\therefore \quad A_1 < A_2 \Rightarrow W_1 < W_2$

W_1 is negative and W_2 is positive.

Net work done, $W_{net} = W_1 + W_2$

Here, net work done will be positive.

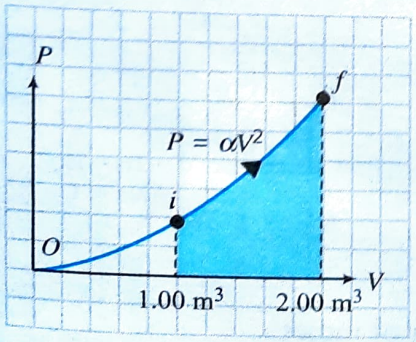
A quantity of gas occupies an initial volume V_0 at pressure P_0 and temperature T_0. It expands to a volume V via path: **(a)** at constant temperature and **(b)** at constant pressure. In which case does the gas do more work?



Sol. The work done is given by the area between the P–V plot and the V-axis. Here, the area between the P–V plot and the V-axis is less in the case (a) than in the case (b). Hence, the work done at constant pressure is greater than the work done at constant temperature.

ILLUSTRATION 4.2

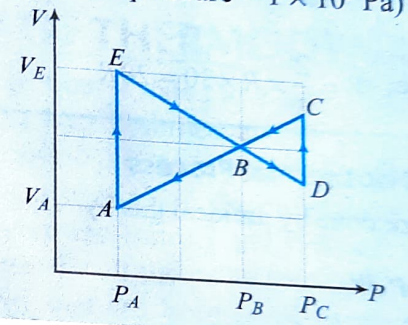
A sample of ideal gas is expanded to twice its original volume of 1.00 m^3 in a quasi-static process for which $P = \alpha V^2$, with $\alpha = 5.00$ atm/m^6, as shown in the given figure. How much work is done on the expanding gas?



Sol. The final pressure is $5 \times 4 = 20$ atm $= 2$ MPa.

The work done on the gas is the negative of the area under the curve $P = \alpha V^2$ from V_i to V_f. The work on the gas is negative,

Figure shows a cyclic process $ABCDBEA$ performed on an ideal cycle. If $P_A = 2$ atm, $P_B = 5$ atm and $P_C = 6$ atm and $V_E - V_A = 20$ litres, find the work done by the gas in the complete process (1 atm pressure $= 1 \times 10^5$ Pa)



Sol. The complete cyclic process can be visualised as made up of two cycles, i.e., cycle $AEBA$ (clockwise) and cycle $BDCB$ (counter-clockwise).

Work done by the gas during the cycle $AEBA$ should be positive.

$$\therefore \quad W_1 = \text{area of the loop in the } P\text{–}V \text{ diagram}$$

$$= \frac{1}{2}(\text{base}) \times (\text{altitude}) = \frac{1}{2}(V_E - V_A)(P_B - P_A)$$

$$= \frac{1}{2}(20 \times 10^{-3}\,\text{m}^3)(5-2) \times 10^5\,\text{Nm}^{-2} = 3\ \text{kJ}$$

Work done by the gas during the cycle $BDCB$ should be negative.

$$\therefore \quad W_2 = -(\text{area of loop } BDCB)$$

Now evidently, triangles ABE and BCD are similar, so the corresponding angles being equal.

$$\therefore \quad \angle EBA = \angle DBC,\ \angle EAB = \angle BCD$$

$$\frac{(V_E - V_A)}{(P_B - P_A)} = \frac{(V_C - V_D)}{(P_C - P_B)}$$

$$\Rightarrow \quad \frac{20}{(5-2)} = \frac{(V_C - V_D)}{(6-5)} \quad \Rightarrow \quad (V_C - V_D) = \frac{20}{3}\ \text{litres}$$

$$\therefore \quad W_2 = -\frac{1}{2} \times \frac{20}{3} \times 10^{-3} \times 1 \times 10^5 = -0.33\ \text{kJ}$$

$\therefore$ Total work done by the gas

$$\Delta W = W_1 + W_2 = 3\ \text{kJ} - 0.33\ \text{kJ} = 2.67\ \text{kJ}$$

ILLUSTRATION 4.4

Find the amount of work done to increase the temperature of one mole of an ideal gas by 30°C, if it is expanding under condition $V \propto T^{2/3}$.

Sol. Ideal gas equation is $PV = nRT$.

For one mole of ideal gas, $PV = RT \Rightarrow P = \dfrac{RT}{V}$

Given that $V \propto T^{2/3}$ or, $V = kT^{2/3}$ ...(i)

Differentiating Eq. (i), $dV = k\dfrac{2}{3}T^{-1/3}dT$ ...(ii)

Work done, $dW = P\,dV = \left(\dfrac{RT}{V}\right)\cdot k\dfrac{2}{3}T^{-1/3}dT$

$$\Rightarrow \quad dW = \frac{RT}{kT^{2/3}}\cdot k\cdot\frac{2}{3}T^{-1/3}dT = \frac{2}{3}RdT \qquad \text{...(iii)}$$

Integrating Eq. (iii), we get

$$W = \int dW = \frac{2R}{3}\int_{T_1}^{T_2} dT = \frac{2}{3}R(T_2 - T_1) = \frac{2}{3}R\,\Delta T$$

Total work done, $W = \dfrac{2}{3}\times R\times 30 = 20R$

Work Done in Isothermal Process

We express work done by an ideal gas by

$$W = \int_{V_i}^{V_f} P\,dV \qquad \text{...(i)}$$

The ideal gas equation is: $PV = nRT$

$$\Rightarrow \quad P = \frac{nRT}{V} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), $W = \displaystyle\int_{V_i}^{V_f}\left(\frac{nRT}{V}\right)dV$

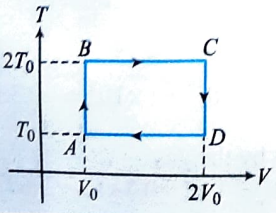
$$\Rightarrow \quad W = nRT\int_{V_i}^{V_f}\frac{dV}{V} = nRT\left[\log_e V\right]_{V_i}^{V_f}$$

$$\Rightarrow \quad W = nRT\log_e\left(\frac{V_f}{V_i}\right) = 2.303\,nRT\log_{10}\left(\frac{V_f}{V_i}\right)$$

$$\Rightarrow \quad W = nRT\log\left(\frac{P_i}{P_f}\right) = 2.303\,nRT\log_{10}\left(\frac{P_i}{P_f}\right)$$

ILLUSTRATION 4.5

T–V curve of cyclic process is shown below. Number of moles of the gas are n. Find the total work done during the cycle.



Sol. Since path AB and CD are isochoric, therefore work done is zero during paths AB and CD. Process BC and DA are isothermal, therefore

$$W_{BC} = nR2T_0\ln\frac{V_C}{V_B} = 2nRT_0\ln 2$$

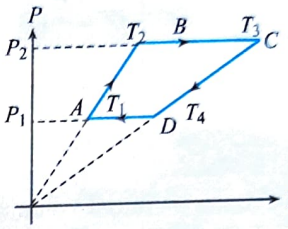
$$W_{DA} = nRT_0\ln\frac{V_A}{V_D} = -nRT_0\ln 2$$

Total work done

$$W_{BC} + W_{DA} = 2nRT_0\ln 2 - nRT_0\ln 2 = nRT_0\ln 2$$

ILLUSTRATION 4.6

P–T curve of a cyclic process is shown. Find out the work done by the gas in the given process if number of moles of the gas are n.



Sol. Since path AB and CD are isochoric therefore work done during AB and CD is zero. Path BC and DA are isobaric.

For isobaric process, $W = P\Delta V = nR\Delta T$ (using ideal gas equation)

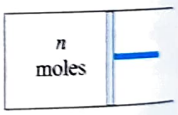
Hence $W_{BC} = nR\Delta T = nR(T_3 - T_2)$

$$W_{DA} = nR(T_1 - T_4)$$

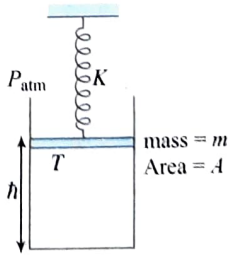
Total work done $= W_{BC} + W_{DA} = nR(T_1 + T_3 - T_4 - T_2)$

CONCEPT APPLICATION EXERCISE 4.1

1. The cylinder shown in the figure has conducting walls and temperature of the surrounding is T, the piston is initially in equilibrium, the cylinder contains n moles of a gas. Now the piston is displaced slowly by an external agent to make the volume double of the initial. Find work done by external agent in terms of n, R, T



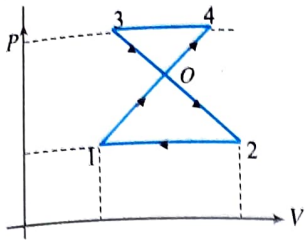
2. A non conducting piston of mass m and area of cross section A is placed on a non conducting cylinder as shown in figure. Temperature, spring constant, height of the piston are given by T, K and h respectively. Initially spring is relaxed and piston is at rest. Find



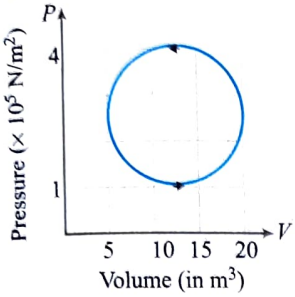
   (a) Number of moles
   (b) Work done by gas to displace the piston by distance d when the gas is heated slowly.

3. Determine the work done by an ideal gas doing $1 \to 4 \to 3 \to 2 \to 1$.

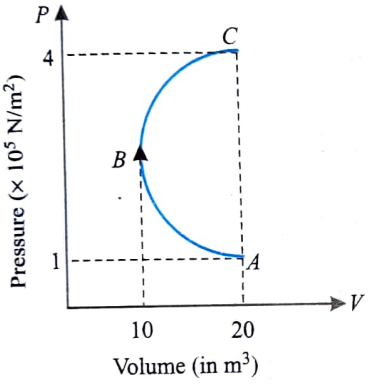
Given $P_1 = 10^5$ Pa, $P_0 = 3 \times 10^5$ Pa, $P_3 = 4 \times 10^5$ Pa and $V_2 - V_1 = 10$ L.



4. Find the work done by gas going through a cyclic process shown in the figure?



5. In the P-V diagram shown in the figure, ABC is a semicircle. Find the work done in the process ABC.

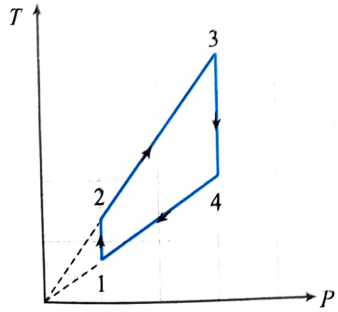


6. A sample of an ideal gas has pressure p_0, volume V_0 and temperature T_0. It is isothermally expanded to twice its original volume. It is then compressed at constant pressure to have the original volume V_0. Finally, the gas is heated at constant volume to get the original temperature.

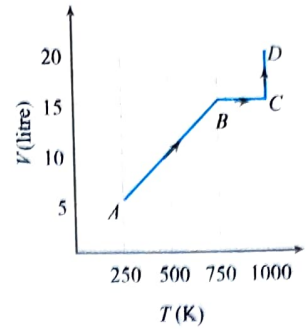
(a) Show the process in a V–T diagram.

(b) Calculate the total work done in the process.

7. Two moles of an ideal monatomic gas undergo a cyclic process as shown in the figure. The temperatures in different states are given as $6T_1 = 3T_2 = 2T_4 = T_3 = 1800$ K. Determine the work done by the gas during the cycle.



8. Two moles of helium gas are taken along the path $ABCD$ (as shown in figure). Find the work done by the gas.



9. Two moles diatomic ideal gas are taken through the process PT = const. Its temperature is increased from T_0 K to $2T_0$ K. Find work done by the system?

ANSWERS

1. $nRT(1-\ln 2)$ **2.** (a) $\dfrac{\left(P_{atm}+\dfrac{mg}{A}\right)Ah}{RT}$ (b) $mgd + P_{atm}dA + \dfrac{1}{2}Kd^2$

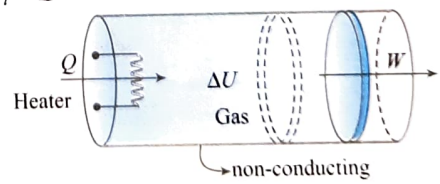
3. 750 J **4.** $-\dfrac{225\pi}{2}$ kJ **5.** 375π kJ **6.** (b) $\dfrac{P_0V_0}{2}[2\ln 2 - 1]$

7. $-1200R$ **8.** $1000R\left(1+2\ln\dfrac{4}{3}\right)$ **9.** $8\,RT_0$

FIRST LAW OF THERMODYNAMICS

Let a system change from a state i to a state f along a particular path. Let Q be the heat absorbed and W be the work done. Measuring both Q and W, let us calculate $(Q - W)$. If we do the same thing a number of times for many different paths which are available between the states i and f, it will be found that $(Q - W)$ is the same for all the paths between i and f. Since Q is the added energy that has been given to the system by the transfer of heat and W is energy that has been extracted from the system in the form of work, $(Q - W)$ represents the change in the internal energy of the system. It is thus clear that the change in the internal energy of a system is independent of the path and is equal to the difference in the internal energies (U_f and U_i) of the system in the final and initial states, i.e.,

$$U_f - U_i = Q - W \qquad \ldots(i)$$



$Q = \Delta U + W$ or $\Delta U = U_f - U_i = Q - W$

When we introduced the law of conservation of energy, we stated that the change in the energy of a system is equal to the sum of all transfers of energy across the system's boundary. The **first law of thermodynamics** is a special case of the law of conservation of energy that describes processes in which only the internal energy changes and the only energy transfers are by heat and work:

According to it, heat given to a system (Q) is equal to the sum of increase in its internal energy (ΔU) and the work done (W) by the system against the surroundings.

$$Q = \Delta U + W$$

It makes no distinction between work and heat as according to it the internal energy (and hence temperature) of a system may be increased either by adding heat to it or doing work on it or both.

Q and W are the path functions but ΔU is the point function.

In the above equation all three quantities Q, ΔU and W must be expressed either in joule or in calorie.

Important Point:

First law of thermodynamics does not indicate the direction of heat transfer. It does not tell anything about the conditions, under which heat can be transformed into work and also it does not indicate as to why the whole heat energy cannot be converted into mechanical work continuously.

Useful sign convention in thermodynamics

Quantity	Sign	Condition
Q	+	When heat is supplied to a system
	–	When heat is drawn from the system
W	+	When work done by the gas (expansion)
	–	When work done on the gas (compression)
ΔU	+	With temperature rise, internal energy increases
	–	With temperature fall, internal energy decreases

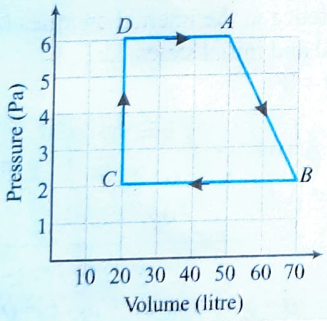
ILLUSTRATION 4.7

In the figure, a thermodynamic system is taken from state A to B, C, D and then back to A, as shown. The system has an internal energy of 400 mJ in state C.

Determine:

(a) The internal energy of the system in the state B if the heat given to the system during the process from B to C be 50 mJ.

(b) Find the work done by the system and heat supplied to it, when it is taken from state C to state B via the path $CDAB$.



Sol.

(a) Work done during the process from B to C

$= -$ area below the line BC

$= -(70-20)\times 10^{-3}\ m^3 \times 2\ Nm^{-2} = -100\times 10^{-3}$ J

$= -100$ mJ

Now for the change of state from B to C, applying first law thermodynamics, $Q = \Delta U + W$

$(+50\ mJ) = (400\ mJ - U_B) - 100\ mJ$

$U_B = 250$ mJ

(b) Work done during the process $CDAB$

$W = W_{CD} + W_{DA} + W_{AB}$

$= 0 + (50-20)\times 10^{-3}\ m^3 \times 6\ Nm^{-2}$

$+\frac{1}{2}\times(70-50)\times 10^{-3}\ m^3 \times (6+2)\ Nm^{-2}$

$= 180\ mJ + 80\ mJ = 260\ mJ$

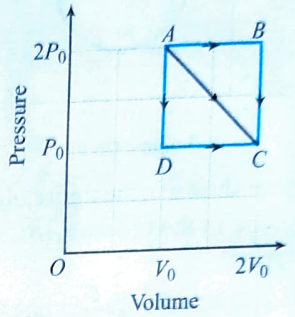
Applying first law of thermodynamics to the process $CDAB$,

$Q = \Delta U + W = (U_B - U_C) + W$

$= (250 - 400)\ mJ + 260\ mJ = 110\ mJ$

ILLUSTRATION 4.8

Figure shows the graph of pressure versus volume of an ideal gas, taken from state A to C, via three different paths such as ABC, AC and ADC. The internal energies of the gas corresponding to the states A and C are (P_0V_0) and $(5P_0V_0)$ respectively. Determine



(a) the change in internal energy of the gas during the process AC.

(b) the work done during the process DC and AB.

(c) the total heat energy supplied to the system, during each of the three processes ABC, AC and ADC.

Sol.

(a) Change in internal energy of the gas during the process

$AC = \Delta U = U_C - U_A = 5P_0V_0 - P_0V_0 = 4P_0V_0$

(b) Work done during the process:

$DC =$ Area under the curve $CD = P_0(2V_0 - V_0) = P_0V_0$

$AB = 2P_0(2V_0 - V_0) = 2P_0V_0$

(c) Applying, the first law of thermodynamics, to the process ABC, we have, $Q = \Delta U + W = 4P_0V_0 + 2P_0(2V_0 - V_0) = 6P_0V_0$

AC, we have, $Q = \Delta U + W$

$= 4P_0V_0 + \frac{1}{2}[2P_0 + P_0]\times(2V_0 - V_0) = 5.5P_0V_0$

ADC, we have, $Q = \Delta U + W = 4P_0V_0 + P_0(2V_0 - V_0) = 5P_0V_0$

ILLUSTRATION 4.9

A vertical hollow cylinder having a cross-sectional area of 5×10^{-3} m^2 fitted with a movable piston contains an 5 kg of an ideal gas. Now the gas is heated from 300 K to 350 K and the piston rises by 0.1 m. The piston is now clamped in this position and the gas is cooled back to 300 K. Find the difference between the heat energy added during heating and the heat energy lost during cooling. (1 atmospheric pressure = 10^5 Nm^{-2} and g = 10 ms^{-2})

Sol. The initial pressure of the gas in the cylinder P = atmospheric pressure + pressure due to weight Mg of piston

$= P_0 + \frac{Mg}{A}$

Here the mass of piston $M = 5$ kg, cross-sectional area of piston $A = 5 \times 10^{-3}$ m^2, $g = 10$ ms^{-2} and atmospheric pressure $P_0 = 10^5$ Nm^{-2}

Hence, $P = 10^5 + \dfrac{5\times10}{5\times10^{-6}} = 1.1\times10^5$ Nm^{-2}

When the piston rises by $x = 0.1$ m, the increase in the volume of the gas is

$$\Delta V = Ax = (5\times10^{-3})\times0.1 = 5\times10^{-4}\text{ m}^3$$

Thus, work done by the gas is $W = \int_{V_1}^{V_2} PdV$

Hence, $W = P(V_2 - V_1) = 1.1\times10^5 \times 5\times10^{-4} = 55$ J

If ΔU is the increase in the internal energy during heating, then from the first law of thermodynamics, the heat energy supplied to the gas is given as

$$Q = \Delta U + W = (\Delta U + 55) \text{ joule}$$

Since the piston is clamped the volume of the gas remains constant during cooling. Hence work done during cooling is zero.

As the gas is cooled back to the initial temperature, the change in the internal energy during cooling is

$$\Delta U' = -\Delta U$$

Thus, heat energy lost by the gas during cooling is

$$Q' = \Delta U' + W' = -\Delta U + 0 = -\Delta U$$

The difference between heat energy added during heating and heat energy lost during cooling is

$$\Delta Q = Q - Q' = (\Delta U + 55) - \Delta U = 55 \text{ joule}$$

HEAT IN THERMODYNAMICS

HEAT (Q)

It is the energy that is transferred between a system and its environment because of the temperature difference between them.

Heat is a path dependent quantity e.g., Heat required to change the temperature of a given gas at a constant pressure is different from that required to change the temperature of same gas through same amount at constant volume.

For gases when heat is absorbed and temperature changes. If we take n moles of a gas and increase the temperature by ΔT, then heat absorbed by gas is given by

$$Q = nC\Delta T$$

Here, C is the specific heat capacity of the gas. The specific heat can be defined by two following ways: (i) Gram specific heat and (ii) Molar specific heat.

Gram specific heat: Gram specific heat of a substance may be defined as the amount of heat required to raise the temperature of unit mass of the substance by unit degree. This definition holds good in case of gases as well.

$$\text{Gram specific heat, } C = \frac{Q}{m\Delta T}$$

Molar specific heat: Molar specific heat of a substance may be defined as the amount of heat required to raise the temperature of one gram mole of the substance by a unit degree,

$$\text{Molar specific heat, } C = \frac{Q}{n\Delta T}$$

SPECIFIC HEAT OF GASES

When heat is supplied to a gas, its volume and pressure change along with a change in its temperature. Therefore, the amount of heat required to raise the temperature of unit mass of a gas through 1 K is not definite or fixed. This means that a gas does not possess a unique or single specific heat. The specific heat of a gas depends upon the conditions of volume and pressure under which it is heated. Since a gas can be heated under a large number of such conditions, its specific heat has innumerable values as its clear from the following discussion.

In case of gases, heat energy supplied to a gas is spent not only in raising the temperature of the gas but also in expansion of gas against atmospheric pressure.

Hence specific heat of a gas, which is the amount of heat energy required to raise the temperature of one gram of gas through a unit degree shall not have a single or unique value.

(i) If the gas is compressed suddenly and no heat is supplied from outside i.e., $Q = 0$, but the temperature of the gas raises on the account of compression.

$$\therefore \quad C = \frac{Q}{n\Delta T} = 0 \qquad \text{(i.e., } C = 0\text{)}$$

(ii) If the gas is heated and allowed to expand at such a rate that rise in temperature due to heat supplied is exactly equal to fall in temperature due to expansion of the gas, i.e., $\Delta T = 0$

$$\therefore \quad C = \frac{Q}{n\Delta T} = \frac{Q}{0} = \infty \qquad \text{(i.e., } C = \infty\text{)}$$

(iii) If rate of expansion of the gas were slow, the fall in temperature of the gas due to expansion would be smaller than the rise in temperature of the gas due to heat supplied. Therefore, there will be some net rise in temperature of the gas, i.e., ΔT will be positive.

$$\therefore \quad C = \frac{Q}{n\Delta T} = \text{a positive quantity} \qquad \text{(i.e., } C = \text{positive)}$$

(iv) If the gas were to expand very fast, fall of temperature of gas due to expansion would be greater than rise in temperature due to heat supplied. Therefore, there will be some net fall in temperature of the gas i.e., ΔT will be negative.

$$C = \frac{Q}{n\Delta T} = \text{a negative quantity} \qquad \text{(i.e., } C = \text{negative)}$$

Hence, the specific heat of gas can have any positive value ranging from zero to infinity. Further, it can even be negative. The exact value depends upon the mode of heating the gas. Out of many values of specific heat of a gas, two are of special significance.

Specific Heat of a Gas at Constant Volume (C_v)

Consider, n moles of an ideal gas, confined in a cylinder, fitted with a fixed piston. If Q be the heat supplied to the gas and as expected, the increase in temperature be ΔT, then, experimentally,

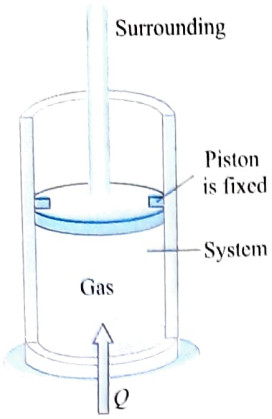
$$Q \propto n;\ \Delta T = \text{constant} \qquad \ldots(i)$$

also $Q \propto \Delta T;\ n = \text{constant}$ ...(ii)

Combining Eqs. (i) and (ii)

$$Q \propto n\Delta T$$

or $$Q = nC_v\Delta T$$



where C_v is a constant, depending upon the nature of gas.

Hence, $C_v = \frac{Q}{n\Delta T}$

If $n = 1$ and $\Delta T = 1$, then $C_v = Q$ (numerically)

Thus, C_v is the amount of heat required (numerically) to raise the temperature of unit mole of a gas by unit degree at constant volume, and is known as the molar heat capacity of the gas at constant volume.

Thus, for any gas, $\Delta U = nC_v\Delta T$ ...(i)

We can write internal energy (U) of a gas at any temperature T as:

$$U = nC_v\Delta T$$

In particular, for an isochoric process,

$$Q = \Delta U = nC_v\Delta T \quad [\because W = 0]$$

Specific Heat of a Gas at Constant Volume in Terms of Degree of Freedom

We know that internal energy of n mole of an ideal gas, having f degrees of freedom can be given by

$$U = n\frac{f}{2}RT$$

If ΔU be the change in its internal energy, corresponding to a change in temperature ΔT,

$$\Delta U = n\frac{fR}{2}\Delta T \quad ...(ii)$$

Comparing Eqs. (i) and (ii) $C_v = \frac{fR}{2}$ ...(iii)

(a) **For a monatomic gas:** According to kinetic energy of gases, an ideal (or monatomic) gas has three degrees of freedom, hence, $C_V = \frac{3}{2}R$.

(b) **For a diatomic gas:** The diatomic gas has five degrees of freedom, hence $C_V = \frac{5}{2}R$.

(c) **For a polyatomic (non-linear) gas:** The degrees of freedom possessed by a polyatomic gas, at ordinary temperature, is six hence $C_v = 3R$.

Specific Heat of a Gas at Constant Pressure (C_p)

Consider n moles of a gas contained in a cylinder fitted with a frictionless movable piston bearing some load. Let heat be supplied to the gas slowly so that the pressure of the gas never exceeds the external pressure more than by an infinitesimal amount. Assuming, no loss of heat to the surrounding, if Q be the total heat supplied to the gas and ΔT the total rise in temperature, then, it is found that

$$Q \propto n;\ \Delta T \text{ constant} \quad ...(i)$$

also $Q \propto \Delta T;\ n = \text{constant}$ ...(ii)

Combining, we get $\Delta Q \propto n\Delta T$

$$Q = C_p n\Delta T \quad ...(iii)$$

where C_p is a constant, depending upon the nature of the gas and is known as the molar heat capacity of the gas at constant pressure.

From Eq. (iii), $C_P = \frac{Q}{n\Delta T}$

If $n = 1$ and $\Delta T = 1$, then $C_p = Q$ (numerically).

Thus, C_p is the amount of heat required (numerically) to raise the temperature of unit mole of a gas by unit degree at constant pressure, and is known as the molar heat capacity of the gas at constant pressure.

From, first law of thermodynamics, $Q = \Delta U + W$

Or $nC_p\Delta T = nC_v\Delta T + P\Delta V$

From equation of state for an ideal gas $PV = nRT$

$$P\Delta V = nR\Delta T$$

$$nC_p\Delta T = nC_v\Delta T + nR\Delta T \quad \text{[using Eq. (iii)]}$$

or $C_P = C_V + R$ ...(iv)

Eq. (iv), i.e., $C_P - C_V = R$ is known as *Mayer's relation.*

Specific Heat of a Gas at Constant Pressure in Terms of Degree of Freedom

We know, $C_v = \frac{fR}{2}$

But $C_P = C_V + R \Rightarrow C_P = \frac{fR}{2} + R = R\left(\frac{f}{2} + 1\right)$

(a) **For a monatomic gas:** According to kinetic energy of gases, an ideal (or monatomic) gas has three degrees of freedom.

Hence, $C_P = R\left(\frac{f}{2} + 1\right) = R\left(\frac{3}{2} + 1\right) = \frac{5R}{2}$

(b) **For a diatomic gas:** The diatomic gas has five degrees of freedom.

Hence, $C_P = R\left(\frac{f}{2} + 1\right) = R\left(\frac{5}{2} + 1\right) = \frac{7R}{2}$

(c) **For a polyatomic (non-linear) gas:** The degrees of freedom possessed by a polyatomic gas, at ordinary temperature, is six hence $C_P = R\left(\frac{f}{2} + 1\right) = R\left(\frac{6}{2} + 1\right) = 4R$

Adiabatic Constant

The ratio of specific heat capacity at constant pressure and constant volume is called adiabatic constant.

$$\gamma = \frac{C_P}{C_V}$$

$$\gamma = \frac{C_P}{C_V} = \frac{R\left(\frac{f}{2} + 1\right)}{\frac{fR}{2}} = \left(1 + \frac{2}{f}\right)$$

(i) **For a monatomic gas:**

$$\gamma = \left(1+\frac{2}{f}\right) = \left(1+\frac{2}{3}\right) = \frac{5}{3}$$

(ii) **For a diatomic gas:**

$$\gamma = \left(1+\frac{2}{f}\right) = \left(1+\frac{2}{5}\right) = \frac{7}{5}$$

(iii) **For a nonlinear polyatomic gas:**

$$\gamma = \left(1+\frac{2}{f}\right) = \left(1+\frac{2}{6}\right) = \frac{4}{3}$$

Internal energy of an ideal gas in terms of adiabatic constant:

We have the relations: $C_P - C_V = R$ ...(i)

and $\gamma = \frac{C_P}{C_V} \Rightarrow C_P = \gamma C_V$ ...(ii)

Solving Eqs. (i) and (ii)

$\gamma C_V - C_V = R$

$\Rightarrow C_V(\gamma - 1) = R$

$\Rightarrow C_V = \frac{R}{(\gamma-1)}$ and $C_P = \frac{\gamma R}{(\gamma-1)}$

We express internal energy of an ideal gas as:

$$U = nC_VT = n\left(\frac{R}{(\gamma-1)}\right)T$$

Hence we can write, $U = \frac{nRT}{(\gamma-1)} = \frac{PV}{(\gamma-1)}$

Also we can express change in internal energy of an ideal gas as:

$$\Delta U = \frac{nR\Delta T}{(\gamma-1)} = \frac{(P_2V_2 - P_1V_1)}{(\gamma-1)}$$

C_p is greater than C_v: When a gas is to be heated at constant volume, the pressure on the gas has to be increased. As the gas is not allowed to expand (volume being kept constant), therefore, in case of C_v, heat is required only for raising the temperature of one mole of gas through 1 K.

When a gas is heated at constant pressure, it expands. In expanding, the gas has to do some work against the external pressure and it gets cooled. More heat has therefore to be supplied for raising its temperature through 1 K. Thus, in case of C_p, heat is required for two purposes:

(i) for increasing the temperature of one mole of gas through 1 K.

(ii) for performing work whole increasing the volume of the gas against external pressure.

Clearly, heat required at constant pressure is more than that at constant volume, i.e., $C_p > C_v$. The difference of the two is the amount of heat spent in doing work for increasing the volume of the gas when heated at constant pressure.

ILLUSTRATION 4.10

The figure shows two identical cylinders A and B which contain equal amounts of an ideal gas with adiabatic exponent γ. The piston of cylinder A is free and that of cylinder B is fixed.

(a) If the same amount of heat is absorbed by each cylinder, then in which cylinder the temperature rise is more?

(b) If the temperature rise in cylinder A is T_0, then determine the temperature rise in cylinder B.

Sol.

(a) The heat absorbed by the cylinder A is utilized in two ways: increasing the internal energy (or temperature) and performing work during expansion.

While the heat absorbed by cylinder B is used up only in increasing the internal energy (or temperature).

Therefore, the temperature rise in cylinder B is more than that in cylinder A.

(b) Heat absorbed by the system at constant pressure,

$Q = nC_P\Delta T$

The change in internal energy, $\Delta U = nC_V\Delta T$

Therefore, the fraction, $\frac{\Delta U}{Q} = \frac{nC_V\Delta T}{nC_P\Delta T} = \frac{C_V}{C_P} = \frac{1}{\gamma}$

Since $\Delta T \propto \Delta U$

$\Rightarrow \frac{\Delta T_B}{\Delta T_A} = \frac{\Delta U_B}{\Delta U_A} = \frac{Q}{Q/\gamma} = \gamma$

$\Rightarrow \Delta T_B = \gamma\Delta T_A$

Since $\Delta T_A = T_0$. Therefore, $\Delta T_B = \gamma T_0$

ILLUSTRATION 4.11

An ideal gas with adiabatic exponent γ is heated at constant pressure. It absorbs Q amount of heat.

Determine the fractions of heat absorbed in raising the internal energy and performing the work.

Sol. At constant pressure heat absorbed by the system is

$Q = nC_P\Delta T$

And internal energy change is $\Delta U = nC_V\Delta T$

Therefore, the fraction, $\frac{\Delta U}{Q} = \frac{nC_V\Delta T}{nC_P\Delta T} = \frac{C_V}{C_P} = \frac{1}{\gamma}$

From the first law of thermodynamics, we know

$W = Q - \Delta U$

Therefore, the fraction, $\frac{W}{Q} = \frac{Q-\Delta U}{Q} = 1 - \frac{\Delta U}{Q} = 1 - \frac{1}{\gamma}$

ILLUSTRATION 4.12

A gas has molar heat capacity $C = 4.5R$ in the process PT = constant. Find the number of degrees of freedom of molecules in the gas.

Sol. Let us take one mole of an ideal gas, the equation of state becomes, $PV = RT$ ...(i)

We are given, PT = constant = K (say)

or $P = \frac{K}{T}$ ...(ii)

From Eqs. (i) and (ii), $\left(\frac{K}{T}\right)V = RT$

or $V = \frac{R}{K}T^2$ ...(iii)

Differentiating Eq. (iii), we get $dW = P\,dV = \left(\frac{K}{T}\right)\frac{2RT}{K}dT$

or $dW = 2R\,dT$ ...(iv)

Now using first law of thermodynamics, $dQ = dU + dW$

$\Rightarrow \quad CdT = C_1 dT + 2RdT$

$\Rightarrow \quad C_V = C - 2R = 4.5R - 2R = \frac{5}{2}R$

But we define $C_V = \frac{f}{2}R$

Hence, degree of freedom, $n = 5$.

ILLUSTRATION 4.13

The volume of one mole of an ideal gas with the adiabatic exponent γ is varied according to the law $V = \frac{a}{T}$, where a is a gas constant. Find the amount of heat obtained by the gas in this process if the gas temperature is increased by ΔT.

Sol. For an ideal gas $PV = nRT$

$$P = \frac{nRT}{V} = \frac{nRT^2}{a} \quad \text{...(i)}$$

$$dV = -\frac{a}{T^2}dT \quad \text{...(ii)}$$

Work done, $dW = P\,dV = \frac{nRT^2}{a}\left(-\frac{a}{T^2}dT\right) = -nR\,dT$ ...(iii)

Integrating Eq. (iii), $\int dW = \int_{T}^{T+\Delta T} -nR\,dT$

$\Rightarrow \quad W = -nR\Delta T$

We define change in internal energy as: $\Delta U = n\left(\frac{R}{\gamma - 1}\right)\Delta T$

From first law of thermodynamics,

$$Q = \Delta U + W = \frac{nR\,\Delta T}{\gamma - 1} - nR\,\Delta T$$

$$\Rightarrow \quad Q = nR\,\Delta T\left[\frac{1}{\gamma - 1} - 1\right] = nR\,\Delta T\left[\frac{2-\gamma}{\gamma - 1}\right]$$

ILLUSTRATION 4.14

One mole of an ideal monatomic gas undergoes the process $P = \alpha T^{1/2}$, where α is a constant.

(a) Find the work done by the gas if its temperature increases by 50 K.

(b) Also, find the molar specific heat of the gas.

Sol.

(a) $P = \alpha T^{1/2}$ where α is constant

$n = 1$ mole, monatomic

$$W = \int P\,dV = \int (\alpha T^{1/2})\,dV$$

As $PV = nRT \Rightarrow T = \frac{PV}{nR}$

$$T = \frac{\alpha T^{1/2} V}{nR} \quad \text{or} \quad T^{1/2} = \frac{\alpha V}{nR}$$

Differentiating both sides,

$$\frac{1}{2\sqrt{T}}dT = \frac{\alpha . dV}{nR}$$

$$\Rightarrow \quad dV = \frac{nR}{\alpha 2\sqrt{T}}dT$$

From Eqs. (i) and (ii),

$$W = \int \alpha T^{1/2} \cdot \frac{nR}{2\alpha T^{1/2}}dT$$

$$= \frac{nR}{2}\int_{T_1}^{T_2} dT = \frac{R}{2}(T_2 - T_1) \quad (\because n = 1)$$

$$= \frac{R}{2}\Delta T = \frac{R}{2} \times 50 = 25R$$

(b) From first law of thermodynamics, $Q = \Delta U + W$

Considering 1 mole of an ideal gas ($n = 1$),

$$1 \cdot C(\Delta T) = 1 \cdot \frac{R}{(\gamma - 1)}(\Delta T) + 1 \cdot \frac{R}{2}(\Delta T)$$

$$C = \frac{R}{(\gamma - 1)} + \frac{R}{2} = \left(\frac{\gamma + 1}{\gamma - 1}\right)\frac{R}{2}$$

ILLUSTRATION 4.15

A ideal gas is kept in closed vessel and an amount Q of heat is added to the gas using a thermodynamic process. In the process gas expand and does a work $Q/2$ on its surrounding. Find the molar specific heat of gas in this process.

Sol. We have first law of thermodynamics: $Q = \Delta U + W$

It is given that in the process gas expand and does a work $Q/2$ on its surrounding hence, $W = \frac{Q}{2}$

Thus $Q = \Delta U + \frac{Q}{2}$ hence $\Delta U = \frac{Q}{2}$

If C is the molar specific heat of the gas in this process then the amount of heat supplied can be given as

$$Q = nC\Delta T$$

The change in internal energy of gas is given by, $\Delta U = nC_V\Delta T$

or $\Delta U = n\left(\frac{3R}{2}\right)\Delta T$

Thus from equation (i) and (ii) and (iii) we have

$$nC_V\Delta T = \frac{1}{2}(nC\Delta T) \quad \text{or} \quad C = 2C_V = 3R$$

ILLUSTRATION 4.16

An ideal gas is kept in closed vessel of volume 0.0083 m^3, at a temperature of 300 K and a pressure of 1.6×10^6 N/m^2. An amount of 2.49×10^4 joule of heat energy is supplied to the gas. Calculate the final temperature and pressure of the gas. (Given: The specific heat at constant pressure $C_p = 5R/2$ and gas constant $R = 8.3$ J/mol-K)

Sol. The number of moles of gas in the container (using gas law): $n = \frac{P_1V_1}{RT_1}$

Here we are given initial pressure, volume and temperature of gas: $P_1 = 1.6 \times 10^6$ N/m^2, $V_1 = 0.0083$ m^3 and $T_1 = 300$ K

$$\Rightarrow \quad n = \frac{P_1V_1}{RT_1} = \frac{1.6\times10^6 \times 0.0083}{8.3\times300} = \frac{16}{3}\text{mol}$$

As the molar specific heat of gas at constant pressure is $C_P = \frac{5R}{2}$, hence the gas is monoatomic, hence its molar specific heat at constant volume should be $C_V = \frac{3R}{2}$

As gas is heated in a closed vessel i.e. at constant volume. If its temperature is raised from T_1 to T_2 then, we have heat supplied to the gas is

$$Q = nC_V(T_2 - T_1)$$

or $$2.94\times10^4 = \frac{16}{3}\times\frac{3}{2}R(T_2 - 300)$$

or $$T_2 = 300 + \frac{3\times2\times2.49\times10^4}{16\times3\times8.3} = 300 + 375 = 675 \text{ K}$$

For constant volume process, we have $\frac{P_2}{P_1} = \frac{T_2}{T_1}$

Thus $$P_2 = \frac{T_2}{T_1}\times P_1 = \frac{675}{300}\times1.6\times10^6 = 3.6\times10^6 \text{ N/m}^2$$

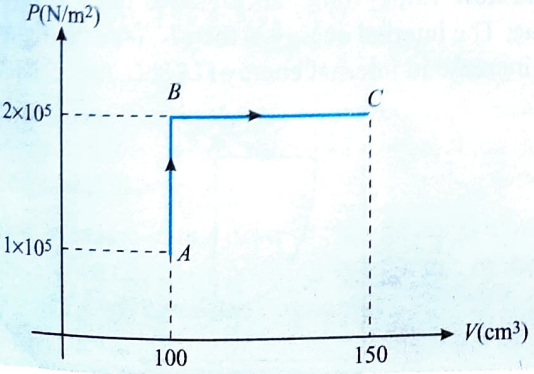
ILLUSTRATION 4.17

A monoatomic ideal gas is taken in a cylinder. It undergoes through the process ABC as shown in figure. If the temperature at the point A is 300 K Find

(i) the temperature at points B and C.

(ii) the work done and heat supplied to the gas in paths AB and BC.

(Use gas constant $R = \frac{25}{3}$ J/mol-K)



Sol.

(i) The number of moles of gas, as $n = \frac{P_AV_A}{RT_A}$ ...(i)

We are given: Initial state pressure, volume and temperature of gas as $P_A = 1\times10^5$ Pa , $V_A = 100\times10^{-6}$ m^3 and $T_A = 300$ K

Substituting these values in (i) we get

$$n = \frac{P_AV_A}{RT_A} = \frac{10^5\times10^{-4}}{25/3\times300} = 0.004\,\text{mol}$$

Thus, at point B, gas temperature: $\frac{P_A}{T_A} = \frac{P_B}{T_B}$

or $$T_B = \frac{P_B}{P_A}\times T_A = \frac{2\times10^5}{10^5}\times300 = 600\,\text{K}$$

Similar at point C, gas temperature: $\frac{T_C}{V_C} = \frac{T_B}{V_B}$

or $$T_C = \frac{V_C}{V_B}\times T_B = \frac{150\times10^{-6}}{100\times10^{-6}}\times600 = 900\,\text{K}$$

(ii) Now in process AB, gas volume is constant thus no work is done by the gas and heat supplied to the gas can be given as

$$Q = nC_V(T_B - T_A) = n\left[\frac{3}{2}R\right](T_B - T_A)$$

[As for a monoatomic gas $C_V = \frac{3R}{2}$]

or $$Q = 0.004\times\frac{3}{2}\times\frac{25}{3}\times[600-300] = 15\,\text{J}$$

In the process BC, gas pressure is constant thus work done by the gas can be given as

$$W = P_B(V_C - V_B) = 2\times10^5[150-100]\times10^{-6} = 10\,\text{J}$$

The heat supplied to the gas in this constant pressure process BC can be given as

$$Q = nC_P(T_C - T_B) = n\left[\frac{5}{2}R\right](T_C - T_B)$$

[As for monoatomic gas the value of $C_P = 5/2$ R]

$$= 0.004\times\frac{5}{2}\times\frac{25}{3}\times(900-600) = 25\,\text{J}$$

GASEOUS MIXTURE

If two non-reactive gases are enclosed in a vessel of volume V. In the mixture n_1 moles of one gas are mixed with n_2 moles of another gas. If N_A is Avogadro's number then

Number of molecules of first gas $N_1 = n_1 N_A$ and number of molecules of second gas $N_2 = n_2N_A$

Total mole fraction $n = (n_1 + n_2)$.

If M_1 is the molecular weight of first gas and M_2 that of second gas.

Then molecular weight of mixture $M = \frac{n_1M_1 + n_2M_2}{n_1 + n_2}$

As there is no loss of internal energy, the internal energy should be same before mixing and after mixing. Let specific heat of the mixture at constant volume be $C_{V_{mix}}$ and adiabatic constant of mixture be γ.

$$(n_1 + n_2)C_{V_{mix}}T = n_1C_{V_1}T + n_2C_{V_2}T \quad \text{...(i)}$$

$$C_{V_{mix}} = \frac{n_1C_{V_1} + n_2C_{V_2}}{n_1 + n_2} \quad \text{...(ii)}$$

In Eq. (i), we can substitute $C_{V_{mix}} = \frac{R}{\gamma - 1}$,

$$C_{V_1} = \frac{R}{\gamma_1 - 1} \text{ and } C_{V_2} = \frac{R}{\gamma_2 - 1}$$

So, $$(n_1 + n_2)\left(\frac{R}{\gamma - 1}\right)T = n_1\left(\frac{R}{\gamma_1 - 1}\right)T + n_2\left(\frac{R}{\gamma_2 - 1}\right)T$$

$$\Rightarrow \quad \frac{(n_1 + n_2)}{\gamma - 1} = \left(\frac{n_1}{\gamma_1 - 1}\right) + \left(\frac{n_2}{\gamma_2 - 1}\right) \qquad \ldots(iii)$$

From Eq. (iii), we can calculate γ of mixture.

Similarly, we can calculate specific heat of the mixture at constant pressure.

$$C_{P_{mix}} = \frac{n_1 C_{P_1} + n_2 C_{P_2}}{n_1 + n_2}$$

ILLUSTRATION 4.18

A gaseous mixture enclosed in a vessel consists of one mole of a gas A with ($\gamma = 5/3$) and another B with ($\gamma = 7/5$) at a temperature T. The gases A and B do not react with each other and assumed to be ideal. Find the number of moles of the gas B, if γ for the gaseous mixture is (19/13).

Sol. We define specific heat capacity at constant volume as:

$$C_V = \frac{R}{(\gamma - 1)}$$

For gas A, $(C_V)_A = \frac{R}{(5/3) - 1} = \frac{3}{2}R$

For gas B, $(C_V)_B = \frac{R}{(7/5) - 1} = \frac{5}{2}R$

For gaseous mixture, $(C_V)_{mix} = \frac{R}{(19/13) - 1} = \frac{13}{6}R$

We define specific heat capacity of mixture as:

$$(C_V)_{mix} = \frac{n_1(C_V)_1 + n_2(C_V)_2}{n_1 + n_2}$$

$$\Rightarrow \quad \frac{13}{6}R = \frac{1 \times \frac{3}{2}R + n \times \frac{5}{2}R}{1 + n} = \frac{(3 + 5n)}{2(1 + n)}$$

or $13 + 13n = 9 + 15n$, i.e., $n = 2$ g mole

ILLUSTRATION 4.19

How much heat energy should be added to the gaseous mixture consisting of one gram of hydrogen and one gram of helium to raise its temperature from 0°C to 100°C

(a) at constant volume

(b) at constant pressure? [$R = 2$ cal/mol K]

Sol. As hydrogen is diatomic and has molecular weight 2, and degree of freedom 5.

Molar heat capacity of hydrogen at constant volume,

$$(C_V)_H = \frac{f}{2}R$$

and number of moles of hydrogen, $n_H = \frac{m_H}{M_H} = \frac{1}{2}$

While He is monatomic and has molecular weight 4. Molar heat capacity of helium at constant volume,

$$(C_V)_{He} = \frac{f}{2}R = \frac{3}{2}R$$

and number of moles of helium, $n_{He} = \frac{m_{He}}{M_{He}}\frac{1}{4}$

Molar heat capacity of gaseous mixture at constant volume,

$$(C_V)_{mix} = \frac{n(C_V)_1 + n_2(C_V)_2}{n_1 + n_2} = \frac{\frac{1}{2} \times \frac{5}{2}R + \frac{1}{4} \times \frac{3}{2}R}{\frac{1}{2} + \frac{1}{4}}$$

$$\Rightarrow \quad (C_V)_{mix} = \frac{13}{8} \times \frac{4}{3}R = \frac{13}{6}R$$

(a) So, heat energy that should be added to the gaseous mixture at constant volume,

$$(\Delta Q)_V = (n_1 + n_2)C_V\Delta T$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) \times \frac{13}{6} \times 2 \times (100 - 0) = 325 \text{ cal}$$

(b) Now, molar heat capacity of gaseous mixture at constant pressure

$$(C_P)_{mix} = (C_V)_{mix} + R = \frac{13}{6}R + R = \frac{19}{6}R$$

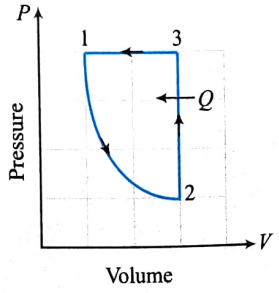
So, heat energy that should be added to the gaseous mixture at constant pressure,

$$(\Delta Q)_P = nC_P\Delta T$$

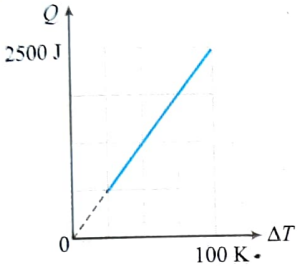
$$= \left(\frac{1}{2} + \frac{1}{4}\right) \times \frac{19}{6} \times 2 \times (100 - 0) = 475 \text{ cal}$$

CONCEPT APPLICATION EXERCISE 4.2

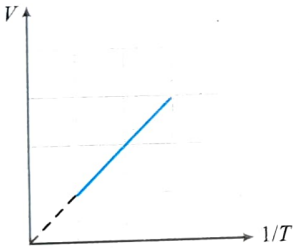
1. A cylinder contains 3 moles of oxygen at a temperature of 27°C. The cylinder is provided with a frictionless piston which maintains a constant pressure of 1 atm on the gas. The gas is heated until its temperature rises to 127°C.
   (a) How much work is done by the gas in the process?
   (b) What is the change in the internal energy of the gas?
   (c) How much heat was supplied to the gas?
   For oxygen $C_p = 7.03$ cal mol^{-1} °C^{-1}.
2. As a result of the lsobaric heating by $\Delta T = 72$ K, one mole of a certain ideal gas receives heat $Q = 1.6$ kJ. Find the work performed by the gas, the increment of its internal energy, and the value of its adiabatic exponent γ.
3. A sample of an ideal gas initially having internal energy U_1 is allowed to expand adiabatically performing work W. Heat Q is then supplied to it, keeping the volume constant at its new value, until the pressure rises to its original value. The internal energy is then U_3 (see the figure). Find the increase in internal energy $(U_3 - U_1)$?



4. An amount Q of heat is added to a monatomic ideal gas in a process in which the gas perform a work $Q/2$ on its surrounding. Find the molar heat capacity for the process.
5. An ideal gas is taken through a process in which the pressure and the volume are changed according to the equation $p = kV$. Show that the molar heat capacity of the gas for the process is given by $C = C_V + R/2$.
6. Two ideal gases have the same value of $C_P/C_V = \gamma$. What will be the value of this ratio for a mixture of the two gases in the ratio 1: 2?
7. A mixture contains 1 mole of helium ($C_P = 2.5\ R$, $C_V = 1.5\ R$) and 1 mole of hydrogen ($C_P = 3.5\ R$, $C_V = 2.5\ R$). Calculate the value of C_P, C_V and γ for mixture.
8. One mole of a gas mixture is heated under constant pressure, and heat required Q is plotted against temperature difference acquired. Find the value of γ for mixture.



9. One mole of an ideal monatomic gas undergoes a process as shown in the figure. Find the molar specific heat of the gas in the process.



10. In a process, the molar heat capacity of a diatomic gas is $\frac{10}{3}R$. When heat Q is supplied to the gas, find the work done by the gas.

ANSWERS

1. (a) 2523 J (b) 1506 cal (c) 2109 cal
2. $W = 597.6$ J, $\Delta U = 1002.4$ J, $\gamma = 1.6$ **3.** $Q - W$ **4.** $3R$
6. γ **7.** $3R, 2R, 1.5$ **8.** $\frac{3}{2}$ **9.** $\frac{R}{2}$ **10.** $\frac{Q}{4}$

DIFFERENT PROCESSES IN FIRST LAW OF THERMODYNAMICS

ISOBARIC PROCESS

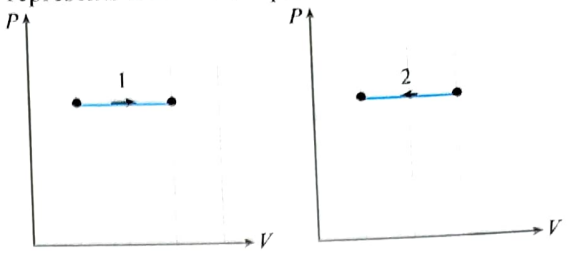
When a thermodynamic system undergoes a physical change in such a way that its pressure remains constant, then the change is known as isobaric process.

Equation of state: In this process V and T change but P remains constant. Hence Charles's law is obeyed in this process.

Hence, if pressure remains constant

$$V \propto T \Rightarrow \frac{V_1}{T_1} = \frac{V_2}{T_2}$$

Indicator diagram: Graph 1 represents isobaric expansion, graph 2 represents isobaric compression.



(a) Expansion (b) Compression

Effect of heating/cooling in isobaric process

	Isobaric heating	Isobaric cooling
Temperature and internal energy	Increases; so, ΔU is positive.	Decreases; so, ΔU is negative.
Volume and work	Increases; so, W is positive.	Decreases; so, W is negative.
Heat	$Q = nC_P\Delta T$ Here, ΔT is positive, hence, heat flows into the system. So, Q is positive.	Here ΔT, is negative, hence, heat flows out of the system. So, Q is negative.

Some important parameters in isobaric process

Specific heat	$C_P = \left(\frac{f}{2}+1\right)R$
Bulk modulus of elasticity	$K = \frac{\Delta P}{\left(\frac{-\Delta V}{V}\right)}$ Here, $\Delta P = 0$, hence, $K = 0$
Work done	$W = \int_{V_i}^{V_f} P\,dV = P\int_{V_i}^{V_f} dV$ $= P(V_f - V_i) = nR[T_f - T_i] = nR\Delta T$
First law of thermodynamics	$Q = \Delta U + W$ Here, $Q = nC_P\,\Delta T = n\left(\frac{\gamma R}{\gamma - 1}\right)\Delta T$, $\Delta U = nC_V\,\Delta T = n\frac{R}{(\gamma - 1)}\Delta T$

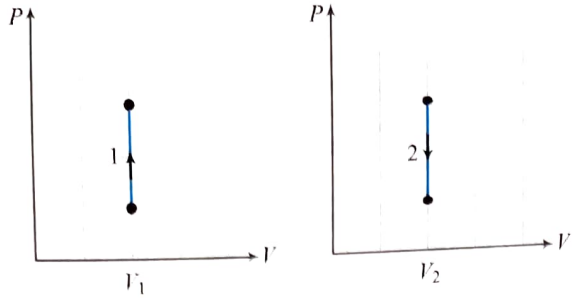
ISOCHORIC OR ISOMETRIC PROCESS

When a thermodynamic process undergoes a physical chang such a way that its volume remains constant, then the chang known as isochoric process.

Equation of state: In this process P and T change but V rem constant. Hence, Gay-Lussac's law is obeyed in this process

$$P \propto T \Rightarrow \frac{P_1}{T_1} = \frac{P_2}{T_2} = \text{constant.}$$

Indicator diagram: Graph 1 and 2 represent isochoric increase in pressure at volume V_1 and isometric decrease in pressure at volume V_2, respectively.



Effect of heating/cooling on various parameters

	Isochoric heating	Isochoric cooling
Temperature and internal energy	Increases; so, ΔU is positive.	Decreases; so, ΔU is negative.
Pressure and work	Pressure increases but volume remains constant, so W is zero.	Pressure decreases but volume remains constant so, W is zero.
Heat	$Q = nC_v\Delta T$ Here, ΔT is positive, hence, heat flows into the system. So, Q is positive.	Here, ΔT is negative, hence, heat flows out from the system. So, Q is negative.

Some important parameters in isochoric process

Specific heat	$C_V = \frac{f}{2}R$
Bulk modulus of elasticity	$K = \frac{\Delta P}{\left(\frac{-\Delta V}{V}\right)}$, Here, $\Delta V = 0$ hence $K = \infty$
Work done	As V = constant, $W = \int_{V_i}^{V_f} P\,dV = 0$
First law of thermodynamics	$Q = \Delta U + W$ Here, $Q = nC_V\Delta T = n\left(\frac{R}{\gamma-1}\right)\Delta T$, $\Delta U = nC_V\Delta T = n\frac{R}{(\gamma-1)}\Delta T = \frac{P_fV_f - P_iV_i}{\gamma-1}$

ISOTHERMAL PROCESS

When a thermodynamic system undergoes a physical change in such a way that its temperature remains constant, then the change is known as isothermal changes.

Essential Conditions for Isothermal Process

(i) The walls of the container must be perfectly conducting to allow free exchange of heat between the gas and its surrounding.

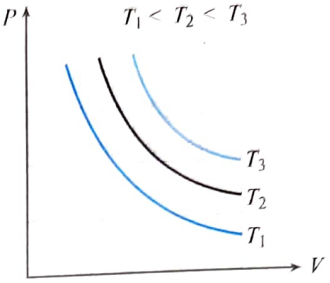
(ii) The process of compression or expansion should be so slow so as to provide time for the exchange of heat.

Since these two conditions are not fully realised in practice, therefore, no process is perfectly isothermal.

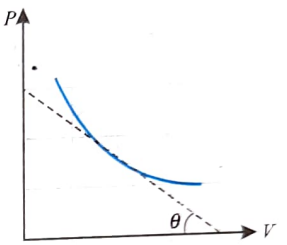
Equation of state: In this process, P and V change but T remains constant i.e., change in temperature $\Delta T = 0$.

Boyle's law is obeyed i.e., PV = constant $\Rightarrow P_1V_1 = P_2V_2$

Indicator diagram: According to PV = constant, graph between P and V is a part of rectangular hyperbola. The graphs at different temperatures are parallel to each other and are called isotherms.



Slope of Isothermal Curve



By differentiating PV = constant, we get $PdV + VdP = 0$

$\Rightarrow \quad PdV = -VdP$

$\Rightarrow \quad \text{Slope} = \tan\theta = \frac{dP}{dV} = -\frac{P}{V}$

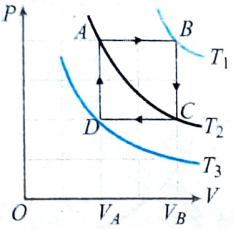
Some important parameters in isothermal process

Work done	$W = \int_{V_i}^{V_f} PdV = nRT\log_e\left(\frac{V_f}{V_i}\right)$ Area between the isotherm and volume axis represents the work done in isothermal process. If volume increases, W = +ve (Area under curve) and if volume decreases, W = –ve (Area under curve).
Specific heat	Here, $\Delta T = 0$ $C = \frac{Q}{m\Delta T} = \frac{Q}{m\times 0} = \infty$ Hence specific heat of gas during isothermal change is infinite.
Isothermal elasticity (E_θ)	For isothermal process PV = constant. $\Rightarrow PdV = -VdP$ $\Rightarrow P = \frac{dP}{-dV/V} = \frac{\text{Stress}}{\text{Strain}} = E_\theta$ $\Rightarrow E_\theta = P$ i.e., isothermal elasticity is equal to pressure.

First law of thermodynamics	$Q = \Delta U + W = 0 + W$ [As $\Delta T = 0$] $\Rightarrow Q = \Delta W$ i.e., heat supplied in an isothermal change is used to do work against external surrounding. Or if the work is done on the system then equal amount of heat energy will be liberated by the system.

ILLUSTRATION 4.20

Figure shows three isotherms at temperature $T_1 = 4000$ K, $T_2 = 2000$ K, $T_3 = 1000$ K. When one mole of an ideal monatomic gas is taken through the paths AB, BC, CD and DA, find the change in internal energy ΔU, the work done by the gas W and the heat Q absorbed by the gas in each path. Also, find these quantities for complete cycle $ABCDA$.



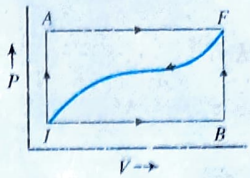
Sol. When one mole of an ideal gas undergoes a change in temperature ΔT, the change in its internal energy equals $C_V \Delta T$, where C_V is the molar specific heat of the gas at constant volume. For monatomic gas $C_V = (3/2)R$.

Thus, the change of internal energy is given by

Process	ΔU	W	Q
$A \to B$	$= nC_V(T_B - T_A)$ $= 1 \times \frac{3}{2} \times (4000 - 2000)$ $= 3000R$	$= P(V_B - V_A)$ $= R(T_B - T_A)$ $= R(4000 - 2000)$ $= 2000\ R$	$= \Delta U + W$ $= 3000\ R + 2000\ R$ $= 5000\ R$
$B \to C$	$= nC_V(T_C - T_B)$ $= 1 \times \frac{3}{2} \times (2000 - 4000)$ $= -3000R$	Zero	$= \Delta U + W$ $= -3000\ R + 0$ $= -3000\ R$
$C \to D$	$= nC_V(T_D - T_C)$ $= 1 \times \frac{3}{2} \times (1000 - 2000)$ $= -1500R$	$= P(V_D - V_C)$ $= R(T_D - T_C)$ $= R(1000 - 2000)$ $= -1000\ R$	$= \Delta U + W$ $= -1500R - 1000\ R$ $= -2500\ R$
$D \to A$	$= nC_V(T_A - T_D)$ $= 1 \times \frac{3}{2} \times (2000 - 1000)$ $= 1500R$	Zero	$= \Delta U + W$ $= 1500\ R + 0$ $= 1500\ R$
Total	$= 3000\ R + (-3000\ R) + (-1500\ R) + 1500\ R = 0$	$= 2000\ R + 0 + (-1000\ R) + 0$ $= 1000\ R$	$= 5000\ R + (-3000\ R) + (-2500\ R) + 1500\ R = 1000\ R$

ILLUSTRATION 4.21

When a thermodynamic system is taken from an initial state I to a final state F along the path IAF, as shown in the given figure, the heat energy absorbed by the system is $Q = 55$ J and the work done by the system is $W = 25$ J. If the same system is taken along the path IBF, the value of $Q = 35$ J.



(a) Find the work done along the path IBF.

(b) If $W = -15$ J for the curved path FI, how much heat energy is lost by the system along this path?

(c) If $U_I = 10$ J, what is U_F?

(d) If $U_B = 20$ J, what is Q for the process BF and IB?

Sol. The first law of thermodynamics states that

$$\Delta Q = \Delta U + \Delta W \quad \text{or} \quad Q = (U_F - U_I) + W$$

Here U_I and U_F are the internal energies in the initial and the final state, respectively. Given that for path IAF, $Q = 55$ J and $W = 25$ J. Therefore,

$$\Delta U = U_F - U_I = Q - W = 55 - 25 = 30 \text{ J}$$

The internal energy is independent of the path; it depends only on the initial and final states of the system. Thus, the internal energy between I and F states is 30 J irrespective of the path followed by the system.

(a) For path IBF, $Q = 35$ J and $\Delta U = 30$ J. Therefore,
$W = Q - \Delta U = 35 - 30 = 5$ J

(b) $W = -15$ J, but $\Delta U = -30$ J. Therefore,
$Q = W + \Delta U = -15 - 30 = -45$ J.

(c) Given $U_I = 10$ J. Therefore,
$U_F = \Delta U + U_I = 30 + 10 = 40$ J.

(d) The process BF is isochoric, i.e., the volume is constant. Hence, $W = 0$. Therefore, $Q = (\Delta U)_{BF} = U_F - U_B = 40 - 20 = 20$ J.
The process IB is isobaric (constant pressure). Therefore,
$Q = (Q)_{IBF} - (Q)_{BF} = 35 - 20 = 15$ J.

ADIABATIC PROCESS

When a thermodynamic system undergoes a change in such a way that no exchange of heat takes place between system and surroundings, the process is known as adiabatic process.

In this process P, V and T change but $\Delta Q = 0$.

Essential Conditions for Adiabatic Process

(i) There should not be any exchange of heat between the system and its surroundings. All walls of the container and the piston must be perfectly insulated.

(ii) The system should be compressed or allowed to expand suddenly so that there is no time for the exchange of heat between the system and its surroundings.

Since, these two conditions are not fully realised in practice, so no process is perfectly adiabatic.

Relation Between Pressure (P) and Volume (V) During an Adiabatic Process

Let n moles of a gas (ideal) be made to expand from an initial state (P_1V_1) to a final state (P_2V_2) adiabatically.

From $dU + dW = 0$ at any state, $dU = nC_V dT$ and $dW = PdV$

$\therefore \quad nC_V dT + PdV = 0$

Let us eliminate ndT, by making use of the equation of state from

$$PV = nRT$$

Differentiating, and remembering that P, V and T all of them change,

$$PdV + VdP = nRdT$$

or $$ndT = \frac{1}{R}(PdV + VdP)$$

The above equation reduces to $$\frac{C_V(PdV + VdP)}{R} + PdV = 0$$

$$(C_V + R)\,PdV + C_V VdP = 0$$

or $$C_P\,PdV + C_V VdP = 0 \qquad \text{[From Mayer's relation]}$$

Dividing throughout by $C_V VP$, $$\frac{C_P}{C_V}\frac{dV}{V} + \frac{dP}{P} = 0$$

Substituting $\frac{C_P}{C_V} = \gamma$ a constant, known as the ratio of specific heats of a gas, we get $$\gamma\frac{dV}{V} + \frac{dP}{P} = 0$$

or $$\gamma\int_{V_1}^{V_2}\frac{dV}{V} = -\int_{P_1}^{P_2}\frac{dP}{P}$$

or $$\gamma\ln\left(\frac{V_2}{V_1}\right) = -\ln\left(\frac{P_2}{P_1}\right) = \ln\left(\frac{P_1}{P_2}\right) \quad \text{or} \quad \left(\frac{V_2}{V_1}\right)^{\gamma} = \frac{P_1}{P_2}$$

or $$P_1V_1^{\gamma} = P_2V_2^{\gamma}$$

i.e., $$PV^{\gamma} = k \text{ constant}$$

- **For temperature and volume**

  We have equation of adiabatic process $PV^{\gamma} = k$

  From ideal gas equation, $P = \frac{nRT}{V}$

  Hence, $$\left(\frac{nRT}{V}\right)V^{\gamma} = k$$

  $$TV^{\gamma-1} = \text{constant}$$

  $$\Rightarrow \quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \quad \text{or} \quad T \propto V^{1-\gamma}$$

- **For temperature and pressure**

  From ideal gas equation, $V = \frac{nRT}{P}$

  Hence, $$P\left(\frac{nRT}{P}\right)^{\gamma} = \text{constant}$$

  $$\Rightarrow \quad P\left(\frac{T}{P}\right)^{\gamma} = \text{constant}$$

  $$\Rightarrow \quad \frac{T^{\gamma}}{P^{\gamma-1}} = \text{constant}$$

  $$\Rightarrow \quad T_1^{\gamma}P_1^{1-\gamma} = T_2^{\gamma}P_2^{1-\gamma}$$

  $$\Rightarrow \quad T \propto P^{\frac{\gamma-1}{\gamma}} \quad \text{or} \quad P \propto T^{\frac{\gamma}{\gamma-1}}$$

Some Examples of Adiabatic Process

- Sudden compression or expansion of a gas in a container with perfectly non-conducting walls.
- Sudden bursting of the tube of bicycle tyre.
- Propagation of sound waves in air and other gases.

Work Done in Adiabatic Process

If an ideal gas expands adiabatically from initial volume V_i to final volume V_f, then work done,

$$W = \int_{V_i}^{V_f} P\,dV = \int_{V_i}^{V_f}\frac{K}{V^{\gamma}}dV = K\int_{V_i}^{V_f}V^{-\gamma}dV = K\left[\frac{V^{-\gamma+1}}{-\gamma+1}\right]_{V_i}^{V_f}$$

$$= \frac{1}{\gamma-1}\left[\frac{K}{V_i^{\gamma-1}} - \frac{K}{V_f^{\gamma-1}}\right] = \frac{1}{\gamma-1}\left[\frac{P_iV_i^{\gamma}}{V_i^{\gamma-1}} - \frac{P_fV_f^{\gamma}}{V_f^{\gamma-1}}\right]$$

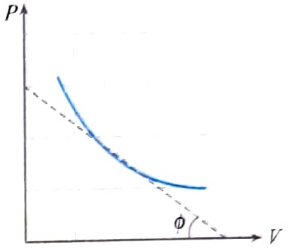
$$= \frac{[P_iV_i - P_fV_f]}{(\gamma-1)} = \frac{nR(T_i - T_f)}{(\gamma-1)}$$

Indicator Diagram

Curve obtained on P-V graph is called adiabatic curve.

Slope of adiabatic curve: PV^{γ} = constant

By differentiating, we get $dP\,V^{\gamma} + P\gamma V^{\gamma-1}\,dV = 0$



$$\Rightarrow \quad \frac{dP}{dV} = -\gamma\frac{PV^{\gamma-1}}{V^{\gamma}} = -\gamma\left(\frac{P}{V}\right)$$

$\therefore$ Slope of adiabatic curve, $\tan\phi = -\gamma\left(\frac{P}{V}\right)$

But we also know that slope of isothermal curve, $\tan\theta = \frac{-P}{V}$

Hence, $(\text{Slope})_{\text{Adi}} = \gamma \times (\text{Slope})_{\text{Iso}}$

or $$\frac{(\text{Slope})_{\text{Adi}}}{(\text{Slope})_{\text{Iso}}} > 1$$

Some important parameters in adiabatic process

Specific heat	Here, $Q = 0$ $C = \frac{Q}{m\Delta T} = 0$ Hence, specific heat of a gas during adiabatic change is zero.
Adiabatic elasticity (E_φ)	For adiabatic process PV^{γ} = constant Differentiating both sides $d\,PV^{\gamma} + P\gamma V^{\gamma-1}\,dV = 0$ $\Rightarrow \quad \gamma P = \frac{dP}{-dV/V} = \frac{\text{Stress}}{\text{Strain}} = E_\varphi$ Thus, $E_\varphi = \gamma P$ i.e., adiabatic elasticity is γ times that of pressure. Also, isothermal elasticity, $E_\theta = P \Rightarrow \frac{E_\varphi}{E_\theta} = \gamma = \frac{C_P}{C_V}$ i.e., the ratio of two elasticities of gases is equal to the ratio of two specific heats.

First law of thermodynamics	$Q = \Delta U + W$ For adiabatic process, $Q = 0$ $\Rightarrow \quad \Delta U = -W$ If W = positive then ΔU = negative. So, temperature decreases i.e., adiabatic expansion causes cooling. If W = negative then ΔU = positive. So, temperature increases i.e., adiabatic compression causes heating.

Comparison between isothermal and adiabatic indicator diagrams

Adiabatic curves are more steeper than isothermal curves

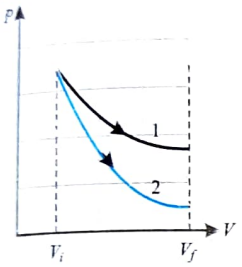
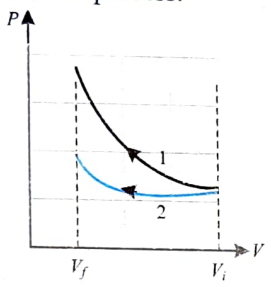
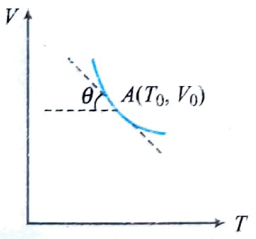
Equal expansion	Equal compression
Graph 1 represents isothermal process and graph 2 represents adiabatic process.	Graph 1 represents adiabatic process and graph 2 represents isothermal process.
	
$W_{isothermal} > W_{adiabatic}$ $P_{isothermal} > P_{adiabatic}$ $T_{isothermal} > T_{adiabatic}$ $(Slope)_{Isothermal} < (Slope)_{Adiabatic}$	$W_{Adiabatic} > W_{Isothermal}$ $P_{Adiabatic} > P_{Isothermal}$ $T_{Adiabatic} > T_{Isothermal}$ $(Slope)_{Isothermal} < (Slope)_{Adiabatic}$

ILLUSTRATION 4.22

A gas is undergoing an adiabatic process. At a certain stage A, the values of volume and temperature are V_0 and T_0, respectively. From the details given in the graph, find the value of C_P and C_V.



Sol. We have, for an adiabatic process, the relation between temperature and volume as: $TV^{\gamma-1}$ = constant

Differentiating with respect to T,

$$V^{\gamma-1} \cdot 1 + T \cdot (\gamma-1)V^{\gamma-2}\frac{dV}{dT} = 0$$

$$\Rightarrow \quad \frac{dV}{dT} = -\frac{V}{T(\gamma-1)}$$

But from the graph, $\left|\frac{dV}{dT}\right|_{T_0,V_0} = \tan(\pi-\theta) = -\tan\theta$

$$\Rightarrow \frac{V_0}{T_0(\gamma-1)} = \tan\theta \Rightarrow \gamma - 1 = \frac{V_0}{T_0\tan\theta} \Rightarrow \gamma = \frac{V_0}{T_0\tan\theta} + 1$$

Now, $C_P = \frac{\gamma R}{\gamma-1} = \frac{(V_0 + T_0\tan\theta)R}{T_0\tan\theta . V_0}T_0\tan\theta$

$$\Rightarrow \quad C_P = \left(1 + \frac{T_0}{V_0}\tan\theta\right)R \text{ and } C_V = \frac{R}{\gamma-1} = \frac{RT_0\tan\theta}{V_0}$$

ILLUSTRATION 4.23

In a cylinder, 2.0 moles of an ideal monatomic gas initially at 1.0×10^6 Pa and 300 K expands until its volume doubles. Compute the work done if the expansion is **(a)** isothermal, **(b)** adiabatic and **(c)** isobaric.

Sol. Let $P_1 = 1\times10^6\,\text{Pa} = 10^6\,\text{N/m}^2$; $T_1 = 300$ K; $n = 2$ moles.

Final volume = 2 (initial volume) $\Rightarrow V_2 = 2V_1$

(a) $W_{\text{isothermal}} = 2.303\,nRT\log(V_2/V_1)$

$$\Rightarrow \quad W = 2.303\times2\times8.3\times300\log 2$$

$$W = 3452\text{ J}$$

(b) $W_{(\text{adiabatic})} = \frac{nR(T_1 - T_2)}{\gamma-1}$

For adiabatic process: $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

$$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = 300\times\left(\frac{1}{2}\right)^{(5/3)-1}$$

$$\Rightarrow \quad T_2 = 189\,\text{K}$$

$$W = \frac{nR(T_1 - T_2)}{\gamma-1} = \frac{2\times8.3\times(300-189)}{(5/3)-1}$$

$$\Rightarrow \quad W = 2764\text{ J}$$

(c) $W_{(\text{isobaric})} = P(V_2 - V_1) \Rightarrow P_1V_1 = nRT_1$

$$V_1 = \frac{nRT_1}{P_1} = \frac{2\times8.3\times300}{10^6}$$

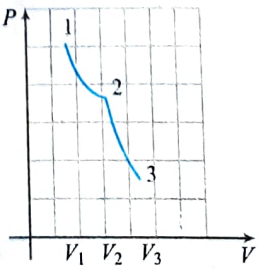
$$\Rightarrow \quad V_1 = 4.98\times10^{-3}\,\text{m}^3$$

$$W = P(V_2 - V_1) = P(2V_1 - V_1) = PV_1$$

$$\Rightarrow \quad W = 10^6\times4.98\times10^{-3}\text{ J} \quad \Rightarrow W = 4980\text{ J}$$

ILLUSTRATION 4.24

One mole of a perfect gas, initially at a pressure and temperature of 10^5 N/m^2 and 300 K, respectively, expands isothermally until its volume is doubled and then adiabatically until its volume is again doubled. Find the final pressure and temperature of the gas. Find the total work done during the isothermal and adiabatic processes. Given $\gamma = 1.4$. Also draw the P–V diagram for the process.



Sol. Let P_1, V_1, T_1 be the initial pressure, volume and the temperature of the gas, respectively. For isothermal expansion (1 to 2), we have

$$V_2 = 2V_1; \quad P_1V_1 = P_2V_2$$
$$P_2 = P_1(V_1/V_2) = P_1/2 = 0.5 \times 10^5 \text{ N/m}^2$$
$$T_2 = T_1 = 300 \text{ K}$$

For adiabatic expansion (2 to 3), $P_2V_2^{\gamma} = P_3V_3^{\gamma}$

$$\Rightarrow \quad V_3 = 2V_2$$

$$\Rightarrow \quad P_3 = P_2\left(\frac{V_2}{V_3}\right)^{\gamma} = 0.5\times10^5 \times (0.5)^{1.4}$$

$$\Rightarrow \quad P_3 = 1.9\times10^4 \text{ N/m}^2$$

$$T_2V_2^{\gamma-1} = T_3V_3^{\gamma-1}$$

$$\Rightarrow \quad T_3 = T_2\left(\frac{V_2}{V_3}\right)^{\gamma-1} = 300\left(\frac{1}{2}\right)^{0.4} = 227.35 \text{ K}$$

Hence, final pressure and temperatures are 1.9×10^4 N/m^2 and 227.35 K, respectively.

Work done is $W = W_{1\to2} + W_{2\to3}$.

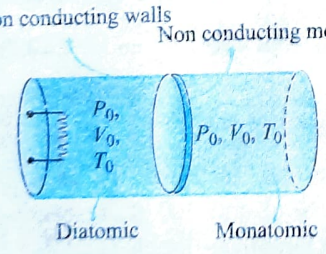
$$W = 2.303\, nRT_1 \log\left(\frac{V_2}{V_1}\right) + \frac{nR(T_2 - T_3)}{(\gamma - 1)}$$

$$W = R(2.303\times300\times\log 2) + \frac{R(300 - 227.35)}{0.4}$$

$$\Rightarrow \quad W = 3233.56 \text{ J}$$

ILLUSTRATION 4.25

A cylindrical container having nonconducting walls is partitioned in two equal parts such that the volume of the each parts is equal to V_0. A movable nonconducting piston is kept between the two parts. Gas on left is slowly heated so that the gas on right is compressed upto volume $V_0/8$. Find pressure and temperature on both sides if initial pressure and temperature, were P_0 and T_0 respectively. Also find heat given by the heater to the gas. (number of moles in each part is n)



Sol. Since the process on right is adiabatic therefore

$$PV^{\gamma} = \text{constant} \quad \left(\gamma = \frac{5}{3}\right)$$

$$\Rightarrow \quad P_0V_0^{\gamma} = P_{\text{final}}\left(\frac{V_0}{8}\right)^{\gamma} \quad \Rightarrow \quad P_{\text{final}} = 32P_0$$

$$T_0V_0^{\gamma-1} = T_{\text{final}}\left(\frac{V_0}{8}\right)^{\gamma-1} \quad \Rightarrow \quad T_{\text{final}} = 4T_0$$

Let volume of the left part is V_1

$$\Rightarrow \quad 2V_0 = V_1 + \frac{V_0}{8} \quad \Rightarrow \quad V_1 = \frac{15V_0}{8}$$

Since number of moles on the left parts remains constant therefore for the left part $\frac{PV}{T}$ = constant.

Final pressure on both sides will be same

$$\Rightarrow \quad \frac{P_0V_0}{T_0} = \frac{P_{\text{final}}V_1}{T_{\text{final}}} \quad \Rightarrow \quad T_{\text{final}} = 60T_0$$

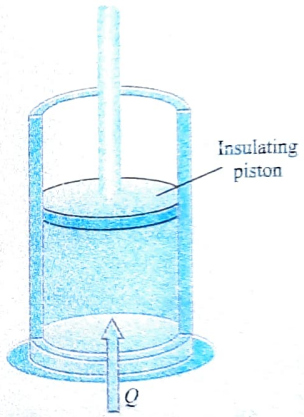
$$\Delta Q = \Delta U + \Delta W$$

$$\Delta Q = n\frac{5R}{2}(60T_0 - T_0) + n\frac{3R}{2}(4T_0 - T_0)$$

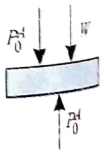
$$\Delta Q = \frac{5nR}{2}\times59T_0 + \frac{3nR}{2}\times3T_0$$

ILLUSTRATION 4.26

An adiabatic vertical cylinder fitted with and adiabatic frictionless piston contains n moles of an ideal diatomic gas. The initial volume and temperature of the gas are V_0 and T_0 respectively, with the piston fixed tightly. The atmospheric pressure is P_0. If the piston be released, find the equilibrium temperature and volume of the gas, if the heat capacity of the cylinder and piston is negligibly small. Weight of the piston and cross-sectional area of the cylinder are W and A respectively.



Sol. The system is isolated from the surrounding, the work done by the gas will be at the cost of its own internal energy. Let the final temperature and volume be T and V, respectively. Let us suppose that the final pressure P of the gas will be just enough to hold the piston in equilibrium against its weight and the atmospheric pressure.



i.e., $PA = P_0A + W$

$$\Rightarrow \quad P = P_0 + \frac{W}{A}$$

Change in internal energy of the gas,

$$\Delta U = nC_V\Delta T = \frac{nR\Delta T}{(\gamma - 1)} = \frac{nR[T - T_0]}{\left(\frac{7}{5} - 1\right)} = \frac{5nR(T - T_0)}{2}$$

Work done by the gas against a constant pressure, $W = P\Delta V$

$$\Rightarrow \quad W = \left(P_0 + \frac{W}{A}\right)(V - V_0)$$

Using equation of state $PV = nRT$, we have

$$\left(P_0 + \frac{W}{A}\right)V = nRT$$

Since $W+\Delta U=0$ $[\because Q=0]$

$$\Rightarrow \left(P_0+\frac{W}{A}\right)(V-V_0)+\frac{5nR(T-T_0)}{2}=0 \quad \ldots(iii)$$

Solving Eqs. (ii) and (iii), for V and T,we get

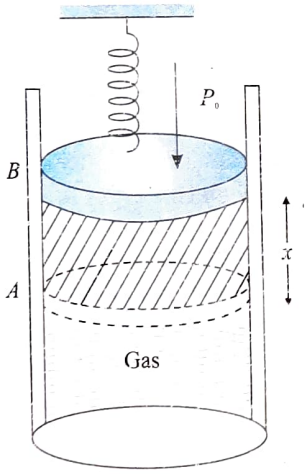
$$T=\frac{5T_0}{7}+\left(P_0+\frac{W}{A}\right)\frac{2V_0}{7nR} \text{ and } V=\frac{2V_0}{7}+\frac{5nRT_0}{7\left(P_0+\frac{W}{A}\right)}$$

ILLUSTRATION 4.27

Three moles of an ideal monoatomic gas are confined within a cylinder by a massless spring loaded with a frictionless piston of negligible mass and of cross – sectional area 4×10^{-3} m^2. The spring is initially in its relaxed state. Now the gas is heated by a heater for some time. During this time the gas expands and does 50J of work in moving the piston through a distance of 0.1m. The temperature increased by 50 K. Calculate the spring constant and the heat supplied by the heater.

(Given: Atmospheric pressure $P_0=10^5$ N/m^2 and gas constant $R=\frac{25}{3}$ J/mol-K)

Sol. Initially the piston is in equilibrium position and spring is relaxed. When the gas is heated, it expands and pushes the piston up by a distance, say, x. Now the spring get compressed. If k is the force constant of the spring and A the area of cross – section of the piston (which is equal to the cross-sectional area of the cylinder), the force in the spring should be $F=kx$



If P_0 is the atmospheric pressure, at equilibrium of piston the pressure of the gas in the cylinder is

$$P=P_0+P_s=P_0+\frac{kx}{A}$$

The increase in the volume of the gas by infinitesimal movement dx of the piston is

$$dV=Adx$$

Thus work done is given as

$$W=\int PdV=\int_0^x\left(P_0+\frac{kx}{A}\right)Adx=P_0A\int_0^x dx+k\int_0^x xdx$$

or $$W=P_0Ax+\frac{1}{2}kx^2 \quad \ldots(i)$$

Substituting the given values $A=4\times10^{-3}$ m^2, $x=0.1$ m, $W=50$ J in equation (i) and solving for k, we get, $k=2000$ Nm^{-1}

To find heat energy Q supplied by the heater, we use the first law of thermodynamics.

$$Q=\Delta U+W$$

Now $$\Delta U=\frac{f}{2}R\Delta T=nC_V\Delta T$$

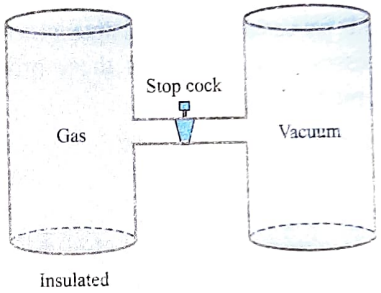
$$\Delta U=\frac{3}{2}nR\Delta T=\frac{3}{2}\times3\times\frac{25}{3}\times50=1875\text{ J}$$

[For a monoatomic gas we have $C_V=\frac{3}{2}R$]

Thus we have, heat supplied given as $Q=1875+50=1925$ J

FREE EXPANSION

Free expansion is adiabatic process in which no work is performed on or by the system. Consider two vessels placed in a system which is enclosed with thermal insulation (asbestos-covered). One vessel contains a gas and the other is evacuated. When suddenly the stopcock is opened, the gas rushes into the evacuated vessel and expands freely.



$W=0$ (Because walls are rigid)

$Q=0$ (Because walls are insulated)

$\Delta U=U_f-U_i=0$ (Because ΔQ and ΔW are zero.

Thus, the final and initial energies are equal in free expansion.

POLYTROPIC PROCESS

A polytropic process is one in which the equation of the pressure versus volume curve is of the form PV^N = constant; where N is a constant throughout the process and may adopt any numerical value.

The thermodynamic processes discussed earlier, may, in some instances, be regarded as particular cases of the polytropic process, where each process has a corresponding value for the exponent N = constant.

It is easy to see that,

for the isochoric process, $N\to\pm\infty$

for the isobaric process, $N=0$

for the isothermal process, $N=1$;

and for the adiabatic process, $N=\gamma=C_P/C_V$ (the adiabatic index).

The values of N for the last three above mentioned processes are apparent; for the isochoric process, if we extract the Nth root on both sides of the equation PV^N = constant, we get $P^{1/N}\times V=$ constant.

For $V=$ constant, $P^{1/N}=1$ or $\frac{1}{N}=0\Rightarrow N\to\pm\infty$

Thus, for a polytropic process, in the general case, $PV^N=$ constant

Work Done in Polytropic Process

We have calculated it in adiabatic process.

Here also, work done, $W = \frac{nR}{N-1}[T_i - T_f]$

or $W = \frac{-nR}{N-1}[T_f - T_i]$ or $W = -\frac{nR\Delta T}{N-1}$

For one mole of a gas, $n = 1$,

$\therefore \quad W = \frac{-R\Delta T}{N-1}$

Molar specific heat: If C is the molar specific heat, then heat required to increase the temperature of one mole of a gas by ΔT,

$Q = C\Delta T$

From first law of thermodynamics, $Q = \Delta U + W$

or $C\Delta T = C_V \Delta T - \frac{R\Delta T}{N-1}$

$\therefore \quad C = C_V - \frac{R}{N-1} = \frac{R}{\gamma - 1} - \frac{R}{N-1}$

Standard Processes as a Special Case of Polytropic Process

All the standard processes are the special cases of general polytropic processes. Let us discuss these processes again in terms of polytropic process

(i) **Isochoric Process:** In an isochoric process, the process equation for the process is given by V= constant

The general equation for polytropic process PV^N = constant, approaches this result when $N \to \infty$ and in this case The molar specific heat of the process is given by

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-N} \text{ as } N \to \infty \; C = \frac{R}{\gamma - 1} = C_V$$

Which is true for the isochoric process

(ii) **Isobaric process:** In isobaric process the process equation is given as P= constant

The general equation of polytropic process PV^N = constant, above equation is obtained by substituting polytropic constant $N = 0$

Hence the molar specific heat of this process can be given by

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-N} \quad \text{or} \quad C = \frac{R}{\gamma - 1} + R = C_V + R = C_P$$

Which is true for the isobaric process

(iii) **Isothermal process:** In isothermal process equation is given as: PV= constant

If we compare above equation with general equation of polytropic process, we can see that for an isothermal process the polytropic constant $N = 1$. Hence, we can find molar specific heat for isothermal process as

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-N} \quad \text{or } C \to \infty$$

The temperature of gas in isothermal process never changes and molar heat capacity is the amount of heat required to change the temperature for one mole of gas by one degree. Hence if we continuously supply heat to a gas infinitely then also its temperature will not change then it is undergoing an isothermal process.

(iv) **Adiabatic process:** The process equation of an adiabatic process is given as PV^γ = Constant

Hence, for an adiabatic process, the polytropic constant is equal to the adiabatic exponent of the gas, $N = \gamma$. Now the molar heat capacity for a gas in adiabatic process is

$$C = \frac{R}{\gamma - 1} + \frac{R}{1-\gamma} = 0$$

Now we can say that in an adiabatic expansion or compression temperature of gas changes without any supply of heat. Thus, in this process no heat is required to raise the temperature of gas.

ILLUSTRATION 4.28

An ideal gas ($C_p/C_v = \gamma$) is taken through a process in which the pressure and the volume vary as $P = aV^b$. Find the value of b for which the specific heat capacity in the process is zero.

Sol. The equation of the process is given as $P = aV^b$

or $PV^{-b} = a$, this is a type of polytropic process

Comparing with PV^N = Constant, we have $N = -b$.

We know that, $C = C_V - \frac{R}{N-1}$

Here, $C = 0, C_V = \frac{R}{\gamma - 1}$

$\therefore \quad 0 = \frac{R}{\gamma - 1} - \frac{R}{-b-1}$

or $b = -\gamma$

ILLUSTRATION 4.29

The heat supplied to one mole of an ideal monatomic gas in increasing temperature from T_0 to $2T_0$ is $2RT_0$. Find the equation of process to which the gas follows.

Sol. The heat supplied to the gas, $Q = nC(\Delta T)$

Here, n is the number of moles and C is the molar heat capacity of the gas. In this case, $n = 1$.

For polytropic process molar heat capacity is given by,

$$C = \frac{R}{(\gamma - 1)} + \frac{R}{(1-N)} = \frac{R}{(5/3)-1} + \frac{R}{(1-N)} = \frac{3}{2}R + \frac{R}{(1-N)}$$

$$Q = n\left[\frac{3R}{2} + \frac{R}{(1-N)}\right](2T_0 - T_0)$$

$$= (1)RT_0\left[\frac{5-3N}{2(1-N)}\right] = 2RT_0 \text{ (given)}$$

or $5 - 3N = 4(1 - N)$

or $N = -1$

Hence equation of process, PV^{-1} = constant

ILLUSTRATION 4.30

One mole of an ideal gas with adiabatic exponent γ whose pressure changes with volume as $P = \alpha V$, where α is a constant, is expanded so that its volume increase η times. Find the change in internal energy and heat capacity of the gas. Initial volume of gas is V_0.

Sol. Given process: $P = \alpha V$ ...(i)

Rearranging above equation we can write

$PV^{-1} = \alpha$ (a constant) ...(ii)

Equation (ii) shows a polytropic process with the value of polytropic constant $N = -1$. Thus, the molar heat capacity of gas in this process can be given as

$$C = \frac{R}{\gamma - 1} + \frac{R}{1 - N} = \frac{R}{\gamma - 1} + \frac{R}{2} = \frac{R}{2}\left[\frac{\gamma + 1}{\gamma - 1}\right]$$

Let during the process, temperature of gas changes from T_1 to T_2 then change in internal energy of gas;

$$\Delta U = C_V \Delta T = \frac{R}{\gamma - 1}(T_2 - T_1) \quad \text{...(iii)}$$

As $n = 1$ mole and $C_V = \frac{R}{\gamma - 1}$

As initial and final volumes of the gas are V_0 and ηV_0. Thus the respective pressures in initial and final states are: $P = \alpha V_0$ and $P = \alpha \eta V_0$

From gas law we can find initial temperatures as

$$T_1 = \frac{P_1 V_1}{R} = \frac{\alpha V_0 . V_0}{R} \text{ and } T_2 = \frac{P_2 V_2}{R} = \frac{\alpha \eta V_0 . \eta V_0}{R}$$

Now from equation (iii) change in internal energy of gas is given as

$$\Delta U = \frac{R}{\gamma - 1}\left[\frac{\alpha \eta^2 V_0^2}{R} - \frac{\alpha V_0^2}{R}\right] = \frac{\alpha V_0^2}{\gamma - 1}[\eta^2 - 1]$$

CYCLIC AND NON-CYCLIC PROCESSES

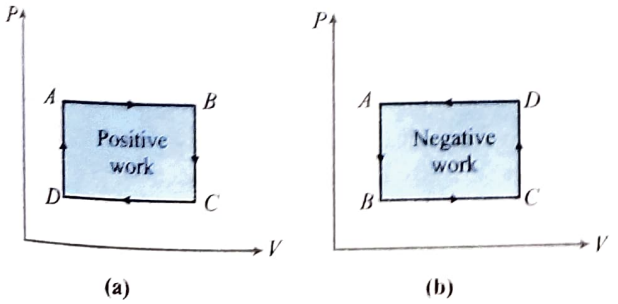
A cyclic process consists of a series of changes which return the system back to its initial state.

In non-cyclic process, the series of changes involved do not return the system back to its initial state.

(i) In case of cyclic process, $U_f = U_i$ so $\Delta U = U_f - U_i = 0$ i.e., change in internal energy for cyclic process is zero and also $\Delta U \propto \Delta T$ so $\Delta T = 0$ i.e., temperature of system remains constant.

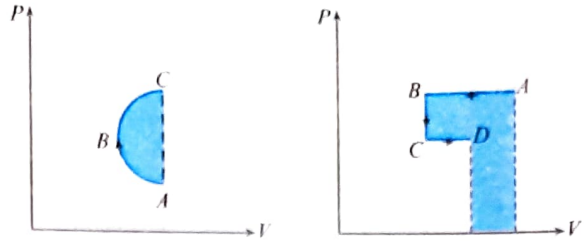
(ii) From first law of thermodynamics, $Q = \Delta U + W \Rightarrow Q = \Delta W$ i.e., heat supplied is equal to the work done by the system.

(iii) For cyclic process, P-V graph is a closed curve and area enclosed by the closed path represents the work done. If the cycle is clockwise work done is positive and if the cycle is anticlockwise work done is negative.



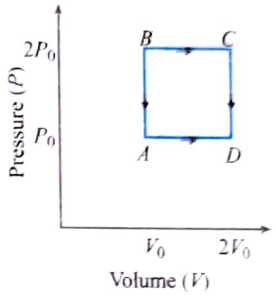
(a) (b)

(iv) Work done in non cyclic process depends upon the path chosen or the series of changes involved and can be calculated by the area covered between the curve and volume axis on PV diagram.



Efficiency of Cyclic Process

When a system is subjected to a cyclic process, heat is supplied during some part of the process, while heat is abstracted during other part.



Evidently, the net heat supplied will be the work done by the system ($\because Q = \Delta W$). However, the gross heat supplied will be more than that of the net heat.

Efficiency (η) of a cycle is defined as the ratio of the work performed (net heat given) to the gross heat supplied to the system per cycle,

Thus, $\eta = \dfrac{\text{work done per cycle}}{\text{gross heat supplied per cycle}}$

ILLUSTRATION 4.31

Four moles of an ideal gas (monatomic) at pressure P_0 and temperature T_0 is isothermally expanded to twice its initial volume. The gas is then compressed at constant pressure to its original volume. Finally, the gas is compressed at constant volume to its original pressure (P_0).

(a) Sketch the P–V and P–T diagrams for the process.

(b) Calculate the net work done by the gas and heat supplied to the gas during the complete cycle.

Sol.

(a) Let the initial and the two successive intermediate states be symbolised by A, B and C, respectively.

For state A: $P_A = P_0$; $T_A = T_0$ and $V_A = \dfrac{4RT_0}{P_0}$ ($\because PV = nRT$)

For state B: $V_B = \dfrac{8RT_0}{P_0}$; $T_B = T_0$

(the process $A \rightarrow B$ being isothermal)

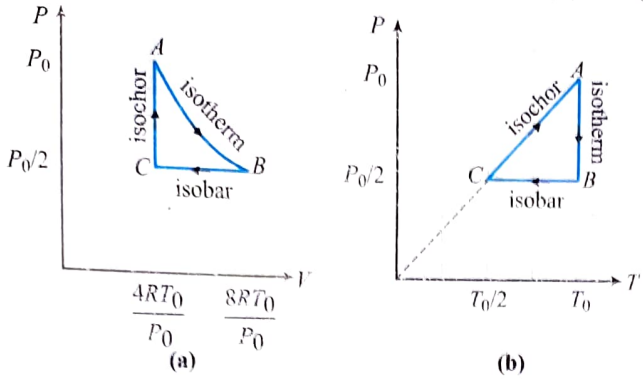
and $P_B = \dfrac{P_0}{2}$

For state C: $V_C = \dfrac{4RT_0}{P_0}$; $P_C = \dfrac{P_0}{2}$

(the process $B \rightarrow C$ being isobaric)

$\therefore$ $V \propto T$ and $T_C = \dfrac{T_0}{2}$

The P–V and P–T curves are shown in Figs. (a) and (b).



(b) For the process $A \to B$ (isothermal):

$$\text{Work done, } W_{AB} = nRT \ln\left(\frac{V_B}{V_A}\right) = 4RT_0 \ln 2$$

For the process $B \to C$ (isobaric):

$$W_{BC} = P\Delta T = nR\Delta T = 4R\left(\frac{T_0}{2} - T_0\right) = -2RT_0$$

The work done in part $C \to A$ is zero as the volume remains constant. Hence, $W_{CA} = 0$.

Thus net work done during the complete cycle,

$$W = W_{AB} + W_{BC} + W_{CA}$$

$$= 4RT_0 \ln 2 - 2RT_0 + 0 = 2RT_0 (2 \ln 2 - 1)$$

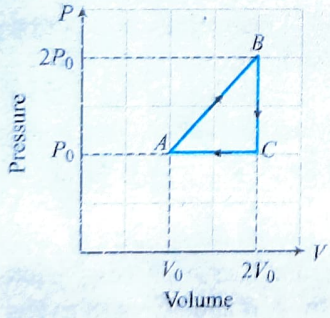
As the process is cyclic, the change in internal energy is zero. The heat given to the system is then equal to the work done by it.

i.e., heat given to the gas during the complete cycle,

$$Q = 2RT_0 (2 \ln 2 - 1)$$

ILLUSTRATION 4.32

Find the efficiency of the thermodynamic cycle shown in figure, for an ideal diatomic gas.



Sol. Let the number of moles of the gas be n and the temperature at point A be T_0.

Now, work done during the cycle = Area enclosed

$$\Rightarrow \quad W = \frac{1}{2} \times (2V_0 - V_0)(2P_0 - P_0) = \frac{1}{2} P_0 V_0$$

Now, consider the heat (Q_{AB}) given during the process $A \to B$.

We have, $Q_{AB} = W_{AB} + \Delta U_{AB}$ ...(i)

W_{AB} = area under the straight line AB

$$\Rightarrow \quad W_{AB} = \frac{1}{2}(P_0 + 2P_0)(2V_0 - V_0) = \frac{3P_0V_0}{2}$$

Applying equation of state for the gas in the state A and B,

$$\frac{P_0V_0}{RT_0} = \frac{(2P_0)(2V_0)}{RT_B} \Rightarrow T_B = 4T_0$$

Change in internal energy during process $A \to B$,

$$\Delta U_{AB} = nC_V\Delta T = n\left(\frac{5R}{2}\right)(4T_0 - T_0)$$

$$\Rightarrow \quad \Delta U_{AB} = \frac{15nRT_0}{2} = \frac{15P_0V_0}{2}$$

Substituting the values of W_{AB} and ΔU_{AB} in eq.(i), we get

$$Q_{AB} = \frac{3}{2}P_0V_0 + \frac{15}{2}P_0V_0 = 9P_0V_0$$

The process $B \to C$ is an isochoric process, in this process

$$P \propto T \text{ or } \frac{P_B}{T_B} = \frac{P_C}{T_C} \Rightarrow \frac{2P_0}{P_0} = \frac{4T_0}{T_C} \Rightarrow T_C = 2T_0$$

The heat given during the process $B \to C$,

$$Q_{BC} = nC_V\Delta T = n\left(\frac{5R}{2}\right)(T_C - T_B)$$

$$\Rightarrow \quad Q_{BC} = n\left(\frac{5R}{2}\right)(2T_0 - 4T_0) = -5nRT_0$$

Q_{BC} is negative, which means that the process $B \to C$ involves the abstraction of heat from the gas.

The process $C \to A$ is an isobaric process.

The heat given during the process $C \to A$,

$$Q_{AC} = nC_P\Delta T = n\left(\frac{7R}{2}\right)(T_A - T_C)$$

$$\Rightarrow \quad Q_{AC} = n\left(\frac{7R}{2}\right)(T_0 - 2T_0) = -\frac{7}{2}nRT_0$$

Q_{AC} is negative which means that the process $A \to C$ also involves the abstraction of heat from the gas.

Obviously, the processes $B \to C$ and $C \to A$ involve the abstraction of heat from the gas.

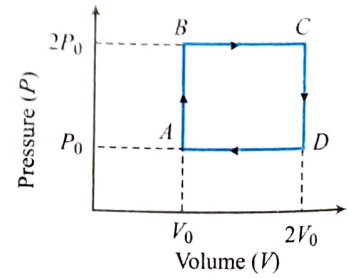
$\therefore$ Efficiency of the cycle,

$$\eta\% = \frac{\text{Work done per cycle}}{\text{Total heat supplied per cycle}} \times 100$$

$$\Rightarrow \quad \eta = \frac{\frac{1}{2}P_0V_0}{9P_0V_0} \times 100 = \frac{50}{9}\%$$

ILLUSTRATION 4.33

The given figure shows the indicator diagram corresponding to n moles of an ideal gas taken along the cyclic process $ABCDA$. Find the efficiency of the cycle.



Sol. Evidently, work done by the gas during the cyclic process is
$W = (2P_0 - P_0)(2V_0 - V_0)$ unit $= P_0V_0$ unit

For the process A to B which is isochoric, the heat supplied will be given by

$$\Delta Q_1 = \Delta U + \Delta W = nC_V\Delta T + 0$$
$$= n\left(\frac{3R}{2}\right)\left[\frac{V_0(2P_0 - P_0)}{nR}\right] \qquad (\because nR\Delta T = V\Delta P)$$
$$= \frac{3}{2}P_0V_0 \text{ unit}$$

And for the process B to C which is isobaric, the heat absorbed will be given by

$$\Delta Q_2 = nC_P\Delta T$$
$$= n\left(\frac{5R}{2}\right)\left[\frac{2P_0(2V_0 - V_0)}{nR}\right] \qquad (\because nR\Delta T = P(\Delta V))$$
$$= 5P_0V_0 \text{ unit}$$

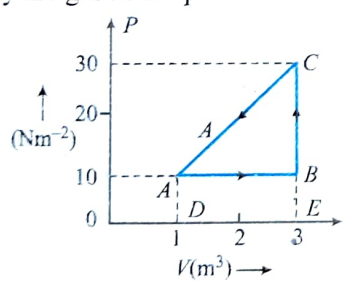
For the processes C to D and D to A, the heat given will be obviously negative, which implies that heat is abstracted from the system.

Therefore, efficiency (η) of the cycle

$$= \frac{\text{Work done per cycle}}{\text{Total heat given per cycle}}\%$$
$$= \frac{P_0V_0}{\left(\frac{3P_0V_0}{2}\right) + 5P_0V_0} = \frac{2}{13} \times 100 = \frac{200}{13}\% = 15.38\%$$

ILLUSTRATION 4.34

An ideal gas is taken round a cyclic thermodynamic process $ABCA$ as shown in the given figure. If the internal energy of the gas at point A is assumed zero while at B it is 50 J. The heat absorbed by the gas in the process BC is 90 J.



(a) What is the internal energy of the gas at point C?
(b) How much heat energy is absorbed by the gas in the process AB?
(c) Find the heat energy rejected or absorbed by the gas in the process CA.
(d) What is the net work done by the gas in the complete cycle $ABCA$?

Sol. Given that $U_A = 0$, $U_B = 50$ J and $Q_{BC} = 90$ J.
Also $P_A = P_B = 10\ \text{Nm}^{-2}$, $P_C = 300\ \text{Nm}^{-2}$, $V_A = 1\ \text{m}^3$
and $V_B = V_C = 3\ \text{m}^3$

(a) In process BC as volume of gas remains constant, work done by gas in this process is zero, thus $W_{BC} = 0$
Heat absorbed by the gas is $Q_{BC} = 90$ J. From the first law of thermodynamics.

$$(\Delta U)_{BC} = U_C - U_B = Q_{BC} - W_{BC} = 90\text{ J} - 0 = 90\text{ J}$$
$$U_C = (\Delta U)_{BC} + U_B = 90\text{ J} + 50\text{ J} = 140\text{ J}$$

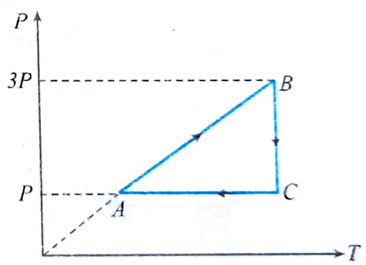
(b) In process AB, we have $(\Delta U)_{AB} = U_B - U_A = 50 - 0 = 50$ J
Work done is given as
W_{AB} = area under AB in P–V diagram
= area of rectangle $ABED$
$= AB \times AD = (3\ \text{m}^3 - 1\ \text{m}^3) \times 10\ \text{Nm}^{-2} = 20$ J
Thus, heat absorbed by the system is
$Q_{AB} = (\Delta U)_{AB} + W_{AB} = 50 + 20 = 70$ J

(c) For process CA
$(\Delta U)_{CA} = U_A - U_C = 0 - 140 = -140$ J
Work done is given as
W_{CA} = area $ACED$
= area of triangle ACB + area of rectangle $ABED$
$= 1/2 \times AB \times BC + AB \times AD$
$= 1/2 \times (3 - 1)\ \text{m}^3 \times (30 - 10)\text{Nm}^{-2} + 20$ J
$= 20 + 20 = 40$ J
In this process, the volume decreases, the work is done on the gas. Hence, the work done is negative,
Thus, $W_{CA} = -40$ J.
Thus heat rejected by the gas is
$Q_{CA} = (\Delta U)_{CA} + W_{CA} = -140 - 40 = -180$ J

(d) Net work done in the complete cyclic process $ABCA$ is
W = area of triangle $ABC = \frac{1}{2} \times 2 \times 20 = 20$ J
As the cycle is anticlockwise, net work is done on the gas.

ILLUSTRATION 4.35

A fixed mass of oxygen gas performs a cyclic process $ABCA$ as shown. Find the efficiency of the process.



Sol. Process $A \to B$ is isochoric.

$$W_{AB} = 0$$
$$Q_{AB} = nC_V(3T_A - T_A) = \left(\frac{5}{2}\right)nR2T_A = 5nRT_A$$
$$W_{BC} = nR3T_A \ln\left(\frac{P_B}{P_C}\right) = 3nRT_A \ln 3 = 3nR\,T_A \ln 3 = Q_{BC}$$
$$W_{CA} = P(V_A - V_C) = nR(T_A - 3T_A) = -2nRT_A$$

Efficiency of a cyclic process is given by

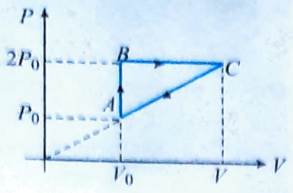
$$\eta = \frac{\text{Work done in cyclic process}}{\text{Heat supplied to the system}} = \frac{W}{Q_{in}}$$

Efficiency, $\eta = \frac{3\ln 3 - 2}{5 + 3\ln 3}\% = \frac{3\ln 3 - 2}{5 + 3\ln 3}\%$

ILLUSTRATION 4.36

In moles of a diatomic gas has undergone a cyclic process ABC as shown in figure. Temperature at a is T_0.

(a) Find volume at C.
(b) Find maximum temperature.
(c) Find total heat given to gas.
(d) Is heat rejected by the gas. If yes, how much heat is rejected?
(e) Find out the efficiency.



Sol.

(a) Since triangle OAV_0 and OCV are similar therefore

$$\frac{2P_0}{V} = \frac{P_0}{V_0} \quad \Rightarrow \quad V = 2V_0$$

(b) Since process AB is isochoric hence

$$\frac{P_A}{T_A} = \frac{P_B}{T_B} \quad \Rightarrow \quad T_B = 2T_0$$

Since process BC is isobaric, $\frac{T_B}{V_B} = \frac{T_C}{V_C} \Rightarrow T_C = 2T_B = 4T_0$

(c) Since process is cyclic therefore

$$\Delta Q = \Delta W = \text{area under the cycle} = \frac{1}{2}P_0V_0 .$$

(d) Since Δu and ΔW both are negative in process CA

$\therefore$ ΔQ is negative in process CA and heat is rejected in process CA

$$\Delta Q_{CA} = \Delta W_{CA} + \Delta U_{CA} = -\frac{1}{2}[P_0 + 2P_0]V_0 - \frac{5}{2}nR(T_c - T_a)$$

$$= -\frac{1}{2}[P_0 + 2P_0]V_0 - \frac{5}{2}nR\left(\frac{4P_0V_0}{nR} - \frac{P_0V_0}{nR}\right)$$

$$= -9P_0V_0 = \text{Heat injected.}$$

(e) η = efficiency of the cycle

$$= \frac{\text{work done by the gas}}{\text{heat injected}} = \eta = \frac{P_0V_0/2}{Q_{\text{injected}}} \times 100$$

$$\Delta Q_{\text{inj}} = \Delta W_{AB} + \Delta U_{BC}$$

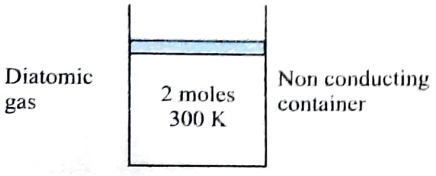
$$= \left[\frac{5}{2}nR(2T_0 - T_0)\right] + \left[\frac{5}{2}nR(2T_0) + 2P_0(2V_0 - V_0)\right]$$

$$= \frac{19}{2}P_0V_0$$

CONCEPT APPLICATION EXERCISE 4.3

1. One mole of argon expands polytropically, the polytropic constant being 1.5, that is, the process proceeds according to the law $pV^{1.5}$ = constant. In the process, its temperature changes by $\Delta T = -26$ K. Find
(a) the amount of heat obtained by the gas,
(b) the work performed by the gas.

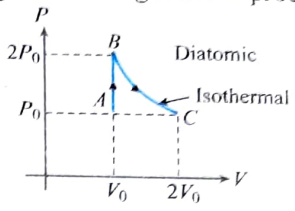
2. Two moles of a diatomic gas at 300 K are kept in a nonconducting container enclosed by a piston. Gas is now compressed to increase the temperature from 300 K to 400 K. Find work done by the gas



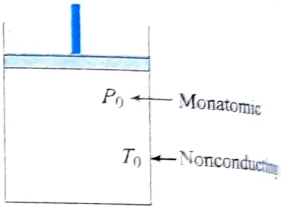
3. The internal energy of a monatomic ideal gas is $1.5\,nRT$. One mole of helium is kept in a cylinder of cross-section 8.5 cm^2. The cylinder is closed by a light frictionless piston. The gas is heated slowly in a process during which a total of 42 J heat is given to the gas. If the temperature rises through 2°C, find the distance moved by the piston. Atmospheric pressure = 100 kPa.

4. A sample of ideal gas ($f = 5$) is heated at constant pressure. If an amount 140 J of heat is supplied to the gas, find
(a) the change in internal energy of the gas and
(b) the work done by the gas.

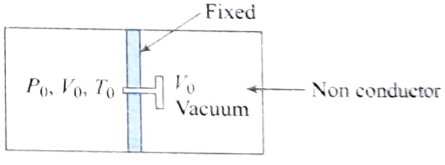
5. P–V curve of a diatomic gas is shown in the figure. Find the total heat given to the gas in the process $A \to B \to C$



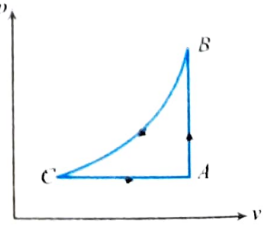
6. A monatomic gas is enclosed in a nonconducting cylinder having a piston which can move freely. Suddenly gas is compressed to 1/8 of its initial volume. Find the final pressure and temperature if initial pressure and temperature are P_0 and T_0 respectively.



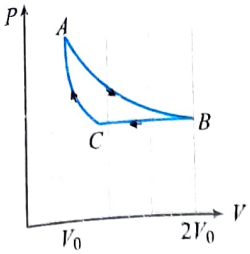
7. A nonconducting cylinder having volume $2V_0$ is partitioned by a fixed non conducting wall in two equal part. Partition is attached with a valve. Right side of the partition is a vacuum and left part is filled with a gas having pressure and temperature P_0 and T_0 respectively. If valve is opened find the final pressure and temperature of the two parts.



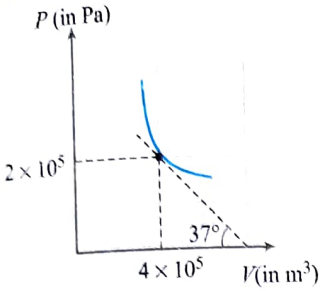
8. A sample of an ideal gas is taken through the cyclic process $A \to B \to C \to A$ as shown in the figure. It absorbs 50 J of heat during the part $A \to B$, no heat during $B \to C$ and rejects 70 J of heat during $C \to A$. If −40 J of work is done on the gas during the part $B \to C$.
(a) Find the internal energy of the gas at B and C if it is 1500 J at A.
(b) Calculate the work done by the gas during the part $C \to A$.



9. In a cycle $ABCA$ consisting of isothermal expansion AB, isobaric compression BC and adiabatic compression CA, find the efficiency of cycle
(Given: $T_A = T_B = 400$ K, $\gamma = 1.5$)

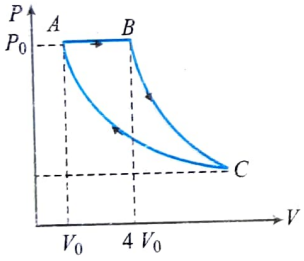


10. P–V graph for an ideal gas undergoing polytropic process PV^m = constant is shown here. Find the value of m.



11. A container of volume 1 m^3 is divided into two equal compartment by a partition. One of these compartments contains an ideal gas at 300 K. The other compartment is vacuum. The whole system is thermally isolated from its surroundings. The partition is removed and the gas expands to occupy the whole volume of the container. Find the new temperature.

12. A fixed mass of a gas is taken through a process $A \to B \to C \to A$ (as shown in the figure). Here $A \to B$ is isobaric, $B \to C$ is adiabatic and $C \to A$ is isothermal. Find efficiency of the process (take $\gamma = 1.5$).



13. An ideal gas is taken through a cyclic thermodynamic process through four steps. The amount of heat involved in these steps are Q_1 = 5960 J, Q_2 = –5585 J, Q_3 = –2980 J and Q_4 = 3645 J, respectively. The corresponding works involved are: $W_1 = 2200$ J, $W_2 = -825$ J, $W_3 = -1100$ J and W_4.
(a) Find the value of W_4.
(b) What is the efficiency of the cycle?

ANSWERS

1. (a) 432 J (b) 108 J 2. –4157 J 3. 20 cm

4. (a) 100 J (b) 40 J 5. $\frac{5}{2}P_0V_0 + 2P_0V_0 \ln 2$

6. $P_{final} = 32P_0$, $T = 4T_0$ 7. $T_{final} = T_0$, $P_{final} = P_0/2$

8. (a) 1550 J, 1590 J (b) 20 J 9. $1 - \frac{3\left(1 - \frac{1}{2^{1/3}}\right)}{\ln 2}$

10. $\frac{3}{2}$ 11. 300 K 12. $n = \frac{3 - 2\ln 2}{3}$

13. (a) 765 J (b) 10.8%

REVERSIBLE AND IRREVERSIBLE PROCESSES

REVERSIBLE PROCESS

A reversible process is one which can be reversed in such a way that all changes occurring in the direct process are exactly repeated in the opposite order and inverse sense and no change is left in any of the bodies taking part in the process or in the surroundings. For example, if heat is absorbed in the direct process, the same amount of heat should be given out in the reverse process, if work is done on the working substance in the direct process then the same amount of work should be done by the working substance in the reverse process. The conditions for reversibility are:

(i) There must be complete absence of dissipative forces such as friction, viscosity, electric resistance etc.
(ii) The direct and reverse processes must take place infinitely slowly.
(iii) The temperature of the system must not differ appreciably from its surroundings.

Some examples of reversible process are:

(i) All isothermal and adiabatic changes are reversible if they are performed very slowly.
(ii) When a certain amount of heat is absorbed by ice, it melts. If the same amount of heat is removed from it, the water formed in the direct process will be converted into ice.
(iii) An extremely slow extension or contraction of a spring without setting up oscillations.
(iv) When a perfectly elastic ball falls from some height on a perfectly elastic horizontal plane, the ball rises to the initial height.
(v) If the resistance of a thermocouple is negligible there will be no heat produced due to Joule's heating effect. In such a case heating or cooling is reversible. At a junction where a cooling effect is produced due to Peltier effect when current flows in one direction and equal heating effect is produced when the current is reversed.
(vi) Very slow evaporation or condensation.

It should be remembered that the conditions mentioned for a reversible process can never be realised in practice. Hence, a reversible process is only an ideal concept. In actual process, there is always loss of heat due to friction, conduction, radiation etc.

IRREVERSIBLE PROCESS

Any process which is not reversible exactly is an irreversible process. All natural processes such as conduction, radiation, radioactive decay etc. are irreversible. All practical processes such as free expansion, Joule-Thomson expansion, electrical heating of a wire are also irreversible. Some examples of irreversible processes are given below:

(i) When a steel ball is allowed to fall on an inelastic lead sheet, its kinetic energy changes into heat energy by friction. The heat energy raises the temperature of lead sheet. No reverse transformation of heat energy occurs.
(ii) The sudden and fast stretching of a spring may produce vibrations in it. Now a part of the energy is dissipated. This is the case of irreversible process.
(iii) Sudden expansion or contraction and rapid evaporation or condensation are examples of irreversible processes.

(iv) Produced by the passage of an electric current through a resistance is irreversible.

(v) Heat transfer between bodies at different temperatures is also irreversible.

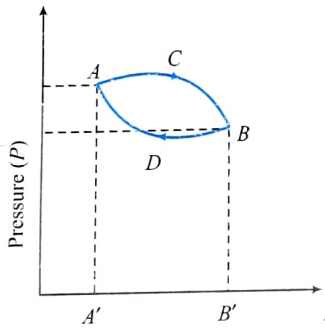
(vi) Joule-Thomson effect is irreversible because on reversing the flow of gas a similar cooling or heating effect is not observed.

HEAT ENGINE

The first law of thermodynamics speaks of the equivalence and mutual conversion of work into heat and vice-versa. Without going into the details of practical feasibility, a *heat engine*, in brief, can be defined as a device which converts heat into mechanical work.

In any heat engine, i.e., device which transforms heat into work, there should be an agent, known as the working body (substance) which can affect the transformation.

It is, by the process of expansion, that the working body performs positive work. The continued action of the engine resulting in an uninterrupted supply of work requires the constant repetition of the expansion process by the working body. Such repetition can be obtained in two ways:



(i) a certain portion of the working body is allowed to execute work by expanding and is removed from the engine, once the work is obtained and a further portion is introduced in the same quantity and in the same initial state, executing the process of expansion over and over again.

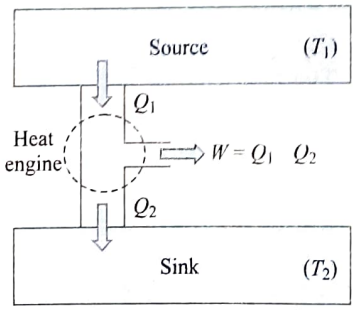
(ii) The working body, having delivered the requisite amount of work by the expansion process, is made to return to the initial state, to restart the cycle again and again.

The essential parts of a heat engine are:

Source: It is a reservoir of heat at high temperature and infinite thermal capacity. Any amount of heat can be extracted from it.

Working substance: Steam, petrol etc.

Sink: It is a reservoir of heat at low temperature and infinite thermal capacity. Any amount of heat can be given to the sink.



The working substance absorbs heat Q_1 from the source, does an amount of work W, returns the remaining amount of heat to the sink and comes back to its original state and there occurs no change in its internal energy.

By repeating the same cycle over and over again, work is continuously obtained.

The performance of heat engine is expressed by means of "efficiency" η which is defined as the ratio of useful work obtained from the engine to the heat supplied to it.

$$\eta = \frac{\text{Work done}}{\text{Heat input}} = \frac{W}{Q_1}$$

For cyclic process, $\Delta U = 0$.

Hence, from first law of thermodynamics $\Delta Q = \Delta W$

So, $W = Q_1 - Q_2$

$$\Rightarrow \quad \eta = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

A perfect heat engine is one which converts all heat into work i.e., $W = Q_1$ so that $Q_2 = 0$ and hence $\eta = 1$.

But practically, efficiency of an engine is always less than 1.

ILLUSTRATION 4.37

A heat engine rejects 75% of the heat supplied to the surrounding (acting as a sink) per cycle. Estimate its efficiency.

Sol. Given, heat rejected per cycle, $Q_2 = 0.75\ Q_1$, where Q_1 is the heat received per cycle.

$\therefore \quad W = Q_1 - Q_2 = Q_1 = 0.75\ Q_1 = 0.25\ Q$

$\therefore$ Efficiency, $\eta = \dfrac{W}{Q_1} = 0.25$ or 25%

ILLUSTRATION 4.38

A motor cycle engine delivers a power of 10 kW by consuming petrol at the rate of 2.4 kg/hour. Estimate the efficiency of the engine and the rate of heat rejection by the exhaust, if the calorific value of petrol is 35.5 MJ/kg.

Sol. Heat received (or produced by the burning of petrol) in one hour will be = 2.4 kg/hour) 35 MJ/kg) = 95.2 × 106 J hour

$\therefore$ The rate at which heat is received $= \dfrac{85.2 \times 10^6\ \text{J}}{(3600\ \text{s})}$

$= 2.37 \times 104\ \text{J/s} = 23.7\ \text{kW}$

Now, efficiency of the engine $\eta = \dfrac{\text{rate at which work is obtained}}{\text{rate at which heat is supplied}}$

$= \dfrac{10\ \text{kW}}{23.7\ \text{kW}} = 42.2\%$

Evidently, the rate of heat rejection = rate at which heat is produced – rate at which work is obtained = 23.7 kW – 10 kW = 13.7 kW

ILLUSTRATION 4.39

A heat engine receives 50 kcal of heat from the source per cycle, and operates with an efficiency of 20%. Determine the work obtained from the engine and the heat rejected to the sink, per cycle.

Sol. Given $Q_1 = 50$ kcal and $\eta = 2\%$

Using $\eta = \frac{W}{Q_1}$, we have $W = \eta \times Q_1$

i.e., work obtained per cycle $W = 20\% \times 50$ kcal $= 10$ kcal $= 42$ kJ

Since $Q_1 = Q_2 + W$

so, $Q_2 = Q - W$

i.e., heat rejected to the sink per cycle Q_2

$= 50$ kcal $- 10$ kcal $= 40$ kcal

SECOND LAW OF THERMODYNAMICS

As already mentioned in the beginning, that the first law of thermodynamics, though establishes an equivalence relation between work and heat, it is entangled amidst several natural limitations.

First law of thermodynamics merely explains the equivalence of work and heat. It does not explain why heat flows from bodies at higher temperatures to those at lower temperatures. It cannot tell us why the converse is not possible. It cannot explain why the efficiency of a heat engine is always less than unity. It is also unable to explain why cool water on stirring gets hotter whereas there is no such effect on stirring warm water in a beaker. Second law of thermodynamics provides answers to these questions. Statements of this law are as follows:

Clausius Statement

It is impossible for a self-acting machine to transfer heat from a colder body to a hotter one without the aid of an external agency.

From Clausius statement, it is clear that heat cannot flow from a body at low temperature to one at higher temperature unless work is done by an external agency. This statement is in fair agreement with our experiences in different branches of physics. For example, electrical current cannot flow from a conductor at lower electrostatic potential to that at higher potential unless an external work is done. Similarly, a body at a lower gravitational potential level cannot move up to higher level without work done by an external agency.

Kelvin's Statement

It is impossible for a body or system to perform continuous work by cooling it to a temperature lower than the temperature of the coldest one of its surroundings. A Carnot engine cannot work if the source and sink are at the same temperature because work done by the engine will result into cooling the source and heating the surroundings more and more.

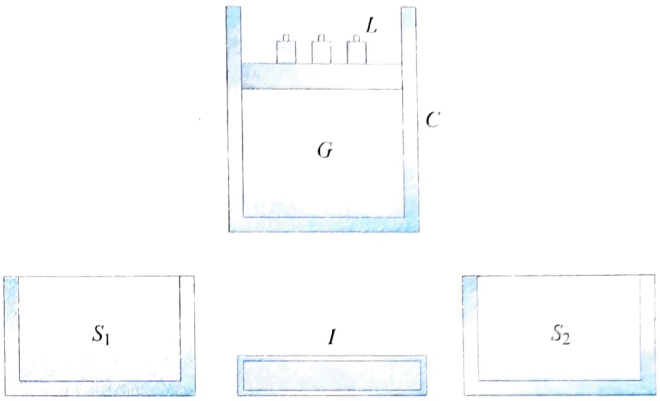
Kelvin-planck's Statement

It is impossible to design an engine that extracts heat and fully utilises into work without producing any other effect.

From this statement it is clear that any amount of heat can never be converted completely into work. It is essential for an engine to return some amount of heat to the sink. An engine essentially requires a source as well as sink. The efficiency of an engine is always less than unity because heat cannot be fully converted into work.

CARNOT ENGINE

Carnot designed a theoretical engine which is free from all the defects of a practical engine. Although this engine is practically impossible due to a number of reasons, nevertheless, its study forms the basis of all real heat engines. Disregarding the practical difficulties of a heat engine, Carnot focussed mainly on the cycle of operations needed to be carried out by a heat engine, in order to yield an uninterrupted and maximum supply of work. The working substance of a Carnot's engine is an ideal gas and each of the processes of a cycle consisted of reversible processes.



It consists of the following parts:

Source (S_1): It is heat reservoir of large thermal capacity maintained at a constant temperature t_1. It delivers a certain amount of heat Q_1 per cycle.

Insulating cylinder (C): It is fitted with a frictionless and insulating piston bearing some load; the cylinder containing a fixed mass of an ideal gas (G). The base of the cylinder is a perfect conductor.

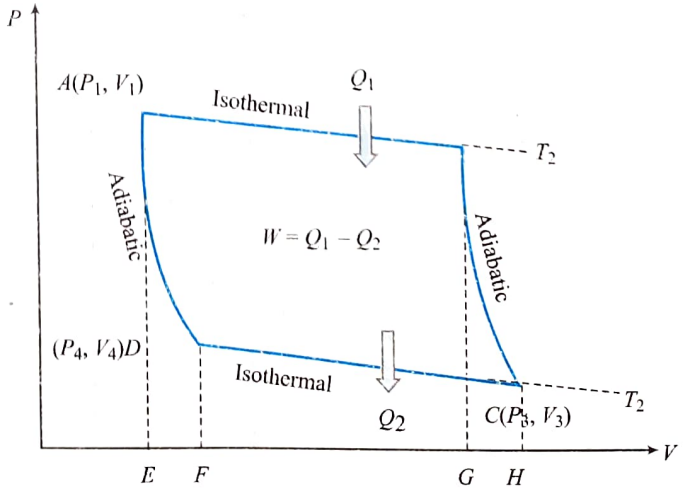
Sink (S_2): It is a also a heat reservoir, which receives or stores the remaining heat Q_2 rejected by the working substance per cycle. The sink is maintained at a lower temperature T_2 than the source.

Insulating pad (I): It is large enough to cover the entire cross section of the cylinder.

CARNOT CYCLE

As the engine works, the working substance of the engine undergoes a cycle known as Carnot cycle. The Carnot cycle consists of the following four strokes:

(i) **First stroke (isothermal expansion) (curve AB):** The cylinder containing ideal gas as working substance allowed to expand slowly at this constant temperature T_1.



Work done = Heat absorbed by the system

$$\Rightarrow \quad W_1 = Q_1 = \int_{V_1}^{V_2} P\,dV$$

$$= RT_1 \log_e\left(\frac{V_2}{V_1}\right) = \text{Area } ABGE$$

(ii) **Second stroke (Adiabatic expansion) (curve *BC*):** The cylinder is then placed on the non-conducting stand and the gas is allowed to expand adiabatically till the temperature falls from T_1 to T_2.

$$\therefore \quad W_2 = \int_{V_2}^{V_3} P\,dV = \frac{R}{(\gamma-1)}[T_1 - T_2] = \text{Area } BCHG$$

(iii) **Third stroke (Isothermal compression) (curve *CD*):** The cylinder is placed on the sink and the gas is compressed at constant temperature T_2.

Work done = Heat released by the system

$$\Rightarrow \quad W_3 = Q_2 = -\int_{V_3}^{V_4} P\,dV = -RT_2 \log_e \frac{V_4}{V_3}$$

$$= RT_2 \log_e \frac{V_3}{V_4} = \text{Area } CDFH$$

(iv) **Fourth stroke (adiabatic compression) (curve *DA*):** Finally the cylinder is again placed on non-conducting stand and the compression is continued so that gas returns to its initial stage.

$$\therefore \quad W_4 = -\int_{V_4}^{V_1} P\,dV = -\frac{R}{\gamma-1}(T_2 - T_1)$$

$$= \frac{R}{\gamma-1}(T_1 - T_2) = \text{Area } ADFE$$

Efficiency of Carnot Cycle

The efficiency of engine is defined as the ratio of work done to the heat supplied i.e.,

$$\eta = \frac{\text{Work done}}{\text{Heat input}} = \frac{W}{Q_1}$$

Net work done during the complete cycle

$= W_1 - W_3 = \text{Area } ABCD$ [As $W_2 = W_4$]

$$\therefore \quad \eta = \frac{W}{Q_1} = \frac{W_1 - W_3}{W_1} = \frac{Q_1 - Q_2}{Q_1} = 1 - \frac{W_3}{W_1} = 1 - \frac{Q_2}{Q_1}$$

or $$\eta = 1 - \frac{RT_2 \log_e(V_3/V_4)}{RT_1 \log_e(V_2/V_1)}$$

Since points B and C lie on same adiabatic curve

$$\therefore \quad T_1V_2^{\gamma-1} = T_2V_3^{\gamma-1} \text{ or } \frac{T_1}{T_2} = \left(\frac{V_3}{V_2}\right)^{\gamma-1} \quad \ldots(i)$$

Also point D and A lie on the same adiabatic curve

$$\therefore \quad T_1V_1^{\gamma-1} = T_2V_4^{\gamma-1} \text{ or } \frac{T_1}{T_2} = \left(\frac{V_4}{V_1}\right)^{\gamma-1} \quad \ldots(ii)$$

From Eqs. (i) and (ii), $\frac{V_3}{V_2} = \frac{V_4}{V_1}$ or $\frac{V_3}{V_4} = \frac{V_2}{V_1}$

$$\Rightarrow \quad \log_e\left(\frac{V_3}{V_4}\right) = \log_e\left(\frac{V_2}{V_1}\right)$$

So efficiency of Carnot engine, $\eta = 1 - \frac{T_2}{T_1}$

Thus, we can say that

(i) efficiency of a heat engine depends only on temperatures of source and sink and is independent of all other factors.

(ii) all reversible heat engines working between same temperatures are equally efficient and no heat engine can be more efficient than Carnot engine (as it is ideal).

(iii) as on Kelvin scale, temperature can never be negative (as 0 K is defined as the lowest possible temperature) and T_1 and T_2 are finite, efficiency of a heat engine is always lesser than unity, i.e., whole of heat can never be converted into work which is in accordance with second law.

CARNOT THEOREM

It states that "out of the various heat engines operating between the same two temperatures, a Carnot's engine possesses the highest efficiency and it depends only on the temperatures of the two reservoirs between which it operates".

The efficiency of Carnot's heat engine depends only on the temperature of source (T_1) and temperature of sink (T_2), i.e., $\eta = 1 - \frac{T_2}{T_1}$.

Carnot stated that no heat engine working between two given temperatures of source and sink can be more efficient than a perfectly reversible engine (Carnot engine) working between the same two temperatures. Carnot's reversible engine working between two given temperatures is considered to be the most efficient engine.

Important Points:

- The ratio of the larger to the smaller volume during a process is said to be the expansion or compression ratio, accordingly as the final volume is greater or lesser than the initial volume.
- In Carnot's cycle, the ratios $\frac{V_2}{V_1}$ and $\frac{V_3}{V_4}$ are, respectively, known as isothermal expansion ratio and isothermal compression ratio, denoted by the letter 'r'.
- The ratios $\frac{V_3}{V_2}$ and $\frac{V_4}{V_1}$ are, respectively, known as adiabatic expansion ratio and adiabatic compression ratio, denoted by the letter 'ρ'.
- The efficiency η can be expressed in terms of ρ as follows:

$$\frac{T_1}{T_2} = \rho^{\gamma-1}$$

$$\therefore \quad \eta = 1 - \frac{T_2}{T_1} = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1}$$

ILLUSTRATION 4.40

Calculate the thermal efficiency of a Carnot's heat engine working between ice and steam point.

Sol. Ice point = 0°C = 273 K and steam point = 100°C = 373 K. We know that the higher temperature in a heat engine corresponds to the source (T_1) and the lower one to that of the sink (T_2).

Thermal efficiency, $\eta = 1 - \frac{T_2}{T_1} = 1 - \frac{273}{373} = 0.2681$ or 26.81%

ILLUSTRATION 4.41

A Carnot's engine operates with an efficiency of 40% with its sink at 27°C. By what amount should the temperature of the source be increased with an aim to increase the efficiency by 10%?

Sol. Let T_1 be the initial temperature of the source, then, using $\eta = 1 - \frac{T_2}{T_1}$, we have, $\frac{40}{100} = 1 - \frac{(273 + 27\ \text{K})}{T_1}$

or $T_1 = 500$ K

For the efficiency to be 10% more, i.e., 50%, let T'_1 be the new temperature of the sink, then, $\frac{50}{100} = 1 - \frac{(273 + 27\ \text{K})}{T'_1}$

or $T'_1 = 600$ K

The required increase in the temperature of the source,

$T'_1 = T = 600\ \text{K} - 500\ \text{K} = 100\ \text{K}$

ILLUSTRATION 4.42

A Carnot's heat engine works with an ideal monatomic gas, and an adiabatic expansion ratio 2. Determine its efficiency.

Sol. Given, $\rho = \frac{V_3}{V_2} = 2$ and γ for a monatomic gas = 5/3.

Using, $\eta = 1 - \left(\frac{1}{\rho}\right)^{\gamma - 1}$, we have

Required efficiency, $\eta = 1 - \left(\frac{1}{2}\right)^{\frac{5}{3} - 1} = 1 - 0.63 = 0.37$ or 37%

ILLUSTRATION 4.43

A Carnot's heat engine works between the temperatures 427°C and 27°C. What amount of heat should it consume per second to deliver mechanical work at the rate of 1.0 kW?

Sol. The efficiency of the heat engine is

$$\eta = 1 - \frac{T_2}{T_1} = 1 - \left[\frac{273 + 27\ \text{K}}{273 + 427\ \text{K}}\right] = \frac{4}{7}$$

But $\eta = \frac{W}{Q_1}$

$\therefore \quad Q_1 = \frac{W}{\eta} = \frac{1.0\ \text{kW}}{4/7} = 1.75\ \text{kW} = 0.417$ kcal/s

Thus, the engine would require 417 cal of heat per second, to deliver the requisite amount of work.

A Few Remarks

(i) Since η for a Carnot's heat engine is given by $1 - \frac{T_2}{T_1}$, it is obvious that all Carnot's engines, working between the same temperature limits, possess the same efficiency, irrespective of the amounts of heat absorbed or rejected per cycle. Further, the absence of any physical quantity pertaining to one or more properties of the working substance in the expression for r) reveals the fact that, the efficiency is absolutely independent of the nature of the working substance (which is the fundamental notion conveyed in Carnot's theorem).

(ii) Since $\frac{T_2}{T_1}$ is always positive and less than unity (since $T_2 < T_1$), the efficiency of an ideal heat engine given by $1 - \frac{T_2}{T_1}$ is also positive and less than unity (or 100%).

In the unwanted case of $T_2 = T_1$, the adiabatic expansion and the adiabatic compression (denoted by the curves $B \to C$ and $D \to A$ in the second and fourth operation, of a Carnot's cycle) is not possible and in consequence the curves $A \to B$ and $C \to D$ superimpose, reducing the area of the loop $ABCDA$ to zero. Accordingly the work done per cycle is zero and hence the efficiency is zero. It can also be seen mathematically that when

$$T_2 = T_1 \quad \text{then} \quad \eta = 1 - \frac{T_2}{T_1} = 0$$

(iii) For η to be 1 or (100%), T_2 should be zero, or T_1 should be large (tending to infinity).
Since either condition is impossible, in practice, so, even if, to say, that somehow we obtain a Carnot's engine, still the efficiency cannot be 1 (or 100%).

(iv) For η to be large, $1 - \frac{T_2}{T_1}$ should be large, or $\frac{T_2}{T_1}$ should be small.

i.e., the larger the value of T_1 (temperature of the source) or the smaller the value of T_2 (temperature of the sink), the larger will be the efficiency of the engine. In other words, the efficiency of a Carnot's engine can be increased either by raising the temperature of the source or by lowering the temperature of the sink.

Consider a Carnot's heat engine working between temperatures T_1 (that of source) and T_2 (that of sink).
Let us examine the cases of increasing its efficiency by the two ways.

Initial efficiency of the engine $\eta = 1 - \frac{T_2}{T_1}$

(a) Let the temperature of sink be lowered by an amount ΔT; the new efficiency will be

$$\eta_1 = 1 - \frac{(T_2 - \Delta T)}{T_1} = \eta + \frac{\Delta T}{T_1}$$

$$\eta_1 = -\frac{\eta T_1 + \Delta T}{T_1} \qquad \ldots(i)$$

(b) Now, keeping the temperature of the sink T_2, let the temperature of the source be raised by the same amount, i.e., ΔT; the changed efficiency will be

$$\eta_2 = 1 - \frac{T_2}{T_1 + \Delta T} = \frac{(T_1 + \Delta T) - (1 - \eta)T_1}{T_1 + \Delta T}$$

or $$\eta_2 = \frac{\eta T_1 + \Delta T}{T_1 + \Delta T} \quad \ldots(ii)$$

From Eqs. (i) and (ii), it is evident that $\eta_1 > \eta_2$ i.e., for the same temperature change (fall for sink and rise far source), the increase in efficiency is more corresponding to the case of lowering of temperature of the sink.

ILLUSTRATION 4.44

An ideal Carnot's engine works between temperatures 127°C and –73°C. Find, which of the two processes, increasing the temperature of source by 50°C, or decreasing the temperature of sink by 50°C, would result in a more increase in the efficiency of the engine.

Sol. First, let us find the efficiency of the engine initially (which is not required here). Using

$$\eta = 1 - \frac{T_2}{T_1}$$

we have $$\eta = 1 - \frac{(273 - 73)\text{ K}}{(273 + 127)\text{ K}} = 0.5$$

When the temperature of the source is increased by 50°C

i.e., $T_1 = 273 + 127 + 50\text{ K} = 450\text{ K}$

$$\therefore \quad \eta_1 = 1 - \frac{200\text{ K}}{450\text{ K}} = 55.55\%$$

When the temperature of the sink is decreased by 50°C

i.e, $T_2 = 273 - 73 - 50\text{ K} = 150\text{ K}$

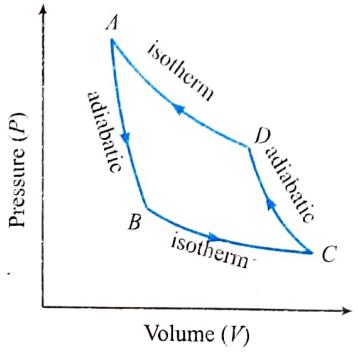
$$\therefore \quad \eta_2 = 1 - \frac{150\text{ K}}{400\text{ K}} = 62.5\%$$

Evidently, $\eta_2 > \eta_1$ i.e., reducing the temperature of the sink by 50°C proves to be more efficient.

REFRIGERATOR OR HEAT PUMP

A refrigerator works along the reverse direction of a heat engine. However, it does not mean that the shaft of a heat engine, rotating in a clockwise sense, is now made to rotate in the anticlockwise sense in a refrigerator. Operating in a reverse direction, simply implies that the sequence of operations involved in a heat engine is reversed.

An ideal refrigerator would, in principle, perform the operations of a Carnot's cycle in the reverse order of a heat engine. Figure shows the *P-V* diagram of the four processes performed by an ideal gas along a Carnot's cycle.



The working substance

(i) expands adiabatically along the curve $A \rightarrow B$;

(ii) expands isothermally along the curve $B \rightarrow C$, taking the requisite amount of heat from the body at a lower temperature to execute the work of expansion;

(iii) gets compressed adiabatically along the curve $C \rightarrow D$, and finally

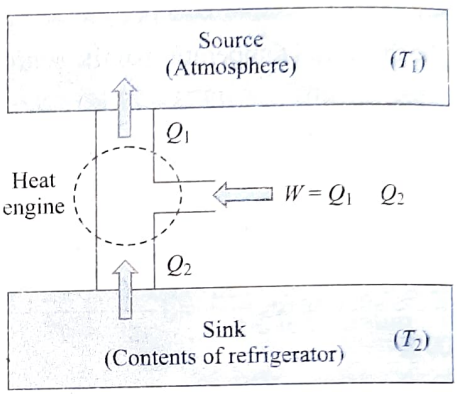
(iv) gets compressed isothermally along the curve $D \rightarrow A$ thereby rejecting the entire heat to the surrounding and returns to its initial state.

It essentially consists of following parts:

Source: At higher temperature T_1.

Working substance: It is called refrigerant liquid ammonia and freon works as a working substance.

Sink: At lower temperature T_2.



The working substance takes heat Q_2 from a sink (contents of refrigerator) at lower temperature, has a net amount of work done W on it by an external agent (usually compressor of refrigerator) and gives out a larger amount of heat Q_1 to a hot body at temperature T_1 (usually atmosphere). Thus, it transfers heat from a cold to a hot body at the expense of mechanical energy supplied to it by an external agent. The cold body is thus cooled more and more.

The performance of a refrigerator is expressed by means of "coefficient of performance" β which is defined as the ratio of the heat extracted from the cold body to the work needed to transfer it to the hot body.

i.e., $$\beta = \frac{\text{Heat extracted}}{\text{Work done}} = \frac{Q_2}{W} = \frac{Q_2}{Q_1 - Q_2}$$

A perfect refrigerator is one which transfers heat from cold to hot body without doing work

i.e., $W = 0$ so that $Q_1 = Q_2$ and hence $\beta = \infty$

Carnot refrigerator: For Carnot refrigerator $\frac{Q_1}{Q_2} = \frac{T_1}{T_2}$

$$\Rightarrow \quad \frac{Q_1 - Q_2}{Q_2} = \frac{T_1 - T_2}{T_2} \quad \text{or} \quad \frac{Q_2}{Q_1 - Q_2} = \frac{T_2}{T_1 - T_2}$$

So coefficient of performance $\beta = \frac{T_2}{T_1 - T_2}$

where T_1 = temperature of surrounding, T_2 = temperature of cold body. It is clear that $\beta = 0$ when $T_2 = 0$, i.e., the coefficient of performance will be zero if the cold body is at the temperature equal to absolute zero.

Relation between coefficient of performance and efficiency of refrigerator:

We know that $\beta = \dfrac{Q_2}{Q_1 - Q_2} = \dfrac{Q_2/Q_1}{1-(Q_2/Q_1)}$ ...(i)

But the efficiency $\eta = 1 - \dfrac{Q_2}{Q_1}$ or $\dfrac{Q_2}{Q_1} = 1 - \eta$ ...(ii)

From Eqs. (i) and (ii) we get, $\beta = \dfrac{1-\eta}{\eta}$

ILLUSTRATION 4.45

A refrigerator is devised by operating a Carnot's heat engine in the reverse direction. If the inside temperature is –10°C and that of outside is 30°C, find its coefficient of performance.

Sol. Given, $T_1 = 273 + 30\,\text{K} = 303\,\text{K}$ and $T_2 = 273 - 10\,\text{K} = 263\,\text{K}$

∴ Coefficient of performance is given by

$$\omega = \frac{T_2}{T_1 - T_2} = \frac{263\,\text{K}}{303\,\text{K} - 263\,\text{K}} = 5.9$$

ILLUSTRATION 4.46

A Carnot's engine, with a perfect gas acting as the working substance, is operated backwards and is employed to freeze water at 0°C. The engine is driven by a 200 W electric motor. Find the amount of water frozen per minute. The outside temperature is 20°C. Specific latent heat of fusion of ice = 80 cal /g and J = 4.2 J /cal.

Sol. Coefficient of performance of the refrigerator, being ideal, is given by

$$\omega = \frac{T_2}{T_1 - T_2} = \frac{273\,\text{K}}{(273+20\,\text{K}) - 273\,\text{K}} = \frac{273}{20}$$

Now, if W be the average work done on the working substance per second, and Q_2 be the average heat extracted from a body at a lower temperature, during that time, then

$$\omega = \frac{Q_2}{W}$$

or $Q_2 = \omega W = \dfrac{273}{20} \times 200\ \text{J/s} = 2730\ \text{J/s} = 650\ \text{cal/s}$

Now, if m be the mass of water frozen per minute, then, heat needed to be extracted will be $Q = mL = m \times 80$ cal/g

But the actual heat extracted per minute

$$= 650\,\frac{\text{cal}}{\text{s}} \times \frac{60\ \text{s}}{\text{min}} = 39 \times 10^3\ \text{cal/min.}$$

$\Rightarrow$ $80\ m\ \text{cal/g} = 39 \times 10^3\ \text{cal/min}$

or $m = 487.5\ \text{g/min}$

CONCEPT APPLICATION EXERCISE 4.4

1. Two Carnot's engines A and B are operated in series. The first one, A, receives heat at 800 K and rejects to a reservoir at temperature T. The second engine, B, receives the heat rejected by the first engine and in turn rejects to a heat reservoir at 300 K. Calculate the temperature T for the following situations:
   (i) The work outputs of the two engines are equal and
   (ii) The efficiencies of two engines are equal.
2. A reversible engine converts one sixth of heat input into work. When the temperature of the sink is reduced by 62°C, its efficiency is doubled. Find the temperatures of the source and sink.
3. A refrigerator freezes water at 0°C into 10 kg ice at 0°C in a time interval of 30 min. Assuming the room temperature to be 20°C, calculate the minimum amount of power needed to make 10 kg of ice.
4. A refrigerator whose coefficient of performance is 5, extracts heat from the cooling compartment at the rate of 250 J/cycle. **(a)** How much work per cycle is required to operate the refrigerator? **(b)** How much heat is discharged to the room?
5. A Carnot engine is designed to operate between 480 K and 300 K. Assuming that the engine actually produces 1.2 kJ of mechanical energy per kcal of heat absorbed, compare the actual efficiency with the theoretical maximum efficiency.
6. The efficiency of a Carnot cycle is 1/6. By lowering the temperature of sink by 65 K, it increases to 1/3. Calculate the initial and final temperatures of the sink.
7. In which case will the efficiency of a Carnot cycle be higher?
   (a) When the temperature of the source is increased by ΔT.
   (b) When the temperature of the sink is lowered by ΔT?
8. A Carnot heat engine has an efficiency of 10%. If the same engine is worked backward to obtain a refrigerator, then find its coefficient of performance.
9. In a cold storage, ice melts at the rate of 2 kg/h when the external temperature is 20°C. Find the minimum power output of the motor used to drive the refrigerator which just prevents the ice from melting. Latent heat of fusion of ice = 80 cal/g.
10. Refrigerator A works between –10°C and +27°C while refrigerator B works between –20°C and +17°C, both removing 2000 J from the freezer. Which of the two is the better refrigerator?

ANSWERS

1. (i) 550 K (ii) 490 K **2.** 372 K, 310 K **3.** 150 W
4. (a) 50 J (b) 300 J
5. Actual efficiency $= \frac{3}{4}$ Maximum efficiency
6. 390 K and 325 K **7.** In case (b) **8.** 9
9. 13.6 W **10.** Refrigerator A

Solved Examples

EXAMPLE 4.1

An ideal gas (2.0 moles) is carried round a cycle as shown. If the process $b \to c$ is isothermal and $C_v = 3$ cal/mol/K. Determine
(a) work done,
(b) change in internal energy,
(c) heat supplied to the system during processes $a \to b$, $b \to c$ and $c \to a$. (Given, 1 atm = 1.01×10^5 N/m^2)

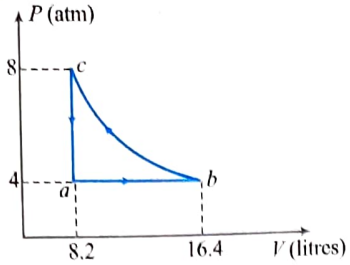
Sol.

(a) Work done: W

Process $a \to b$ is clearly isobaric at $P = 4$ atm.

$$W = P(V_b - V_a) = 4(16.4 - 8.2) = 32.8\ \text{L atm}$$

$\Rightarrow \quad W_{a-b} = 32.8 \times 10^{-3} \times 1.01 \times 10^5$



Process $b \to c$ is isothermal (given)

$$W = 2.303\, nRT \log\left(V_c/V_b\right)$$

$\Rightarrow \quad W = 2.303\left(P_b V_b\right)\log\left(V_c/V_b\right)$

$$W = 2.303(4\times16.4\times101)\log(8.2/16.4)$$

$$W_{b\to c} = -4592.5\,\text{J}$$

Process $c \to a$ is isochoric (volume is constant)

$$W_{c\to a} = 0\,\text{J}$$

(b) Change in internal energy ΔU

Process $a \to b$: $\Delta U = nC_v\Delta T$

$$\Delta U = nC_v\left(\frac{P_bV_b}{nR} - \frac{P_aV_a}{nR}\right) \qquad (\text{since } pV = nRT)$$

$$= \frac{C_v}{R}\left(P_bV_b - P_aV_a\right)$$

$$= \frac{3}{0.0821}(4\times16.4 - 4\times8.2)$$

$\Rightarrow \quad \Delta U = 1200$ cal

Process $b \to c$: isothermal, $\Delta T = 0$

$\Rightarrow \quad \Delta U = 0$

Process $c \to a$: ΔU for complete cyclic process is zero because it depends on initial and final states, i.e., temperature difference only.

$$(\Delta U)_{ab} + (\Delta U)_{bc} + (\Delta U)_{ca} = 0$$

$\Rightarrow \quad 1200 + 0 + (\Delta U)_{ca} = 0$

$\Rightarrow \quad (\Delta U)_{ca} = -1200\,\text{cal}$

(c) Heat supplied: ΔQ

Using the first law of thermodynamics:

$$(\Delta Q)_{ab} = (\Delta U)_{ab} + W_{ab}$$

$$= (1200 + 3313/4.18)\,\text{cal} = 1992.5\,\text{cal}$$

$\Rightarrow \quad (\Delta Q)_{ab} = 1992.5$ cal

$\Rightarrow \quad (\Delta Q)_{bc} = (\Delta U)_{bc} + W_{bc}$

$$= (0 - 4592.5/4.18)\,\text{cal} = 1098.7\,\text{cal}$$

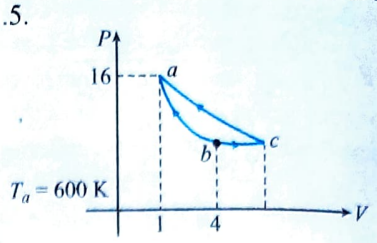
$\Rightarrow \quad (\Delta Q)_{bc} = 1098.7$ cal

$$(\Delta Q)_{ca} = (\Delta U)_{ca} + W_{ca} = -1200 + 0 = -1200\ \text{cal}$$

$\Rightarrow \quad (\Delta Q)_{ac} = 1200$ cal

EXAMPLE 4.2

The given figure shows three processes for an ideal gas. The temperature at 'a' is 600 K, pressure 16 atm, and volume 1 L. The volume at 'b' is 4 L. Out of two processes, ab and ac, one is adiabatic and other is isothermal. The ratio of specific heats of the gas is 1.5.



(a) Which of the processes ab and ac is adiabatic? Why?
(b) Compute the pressure of the gas at b and c.
(c) Compute the temperature at b and c.
(d) Compute the volume at c.

Sol.

(a) The process ab is adiabatic because it is steeper than ac.

(b) $P_a = 16$ atm, $V_b = 4$ L, $V_a = 1$ L

$\Rightarrow \quad T_a = 600$ K

Process ab is adiabatic.

$\Rightarrow \quad P_aV_a^{\gamma} = P_bV_b^{\gamma}$

$$P_b = P_a\left(\frac{V_a}{V_b}\right)^{\gamma} = 16\times\left(\frac{1}{4}\right)^{1.5}$$

$\Rightarrow \quad P_b = 2$ atm

As bc is isobaric.

$\Rightarrow \quad P_b = P_c = 2$ atm

(c) $T_c = T_a = 600$ K because ac is isothermal.

Process ab is adiabatic.

Hence, T_b can be calculated using

$$T_aV_a^{\gamma-1} = T_bV_b^{\gamma-1}$$

$$\Rightarrow \quad \left(\frac{P_aV_a}{T_a}\right) = \left(\frac{P_bV_b}{T_b}\right)$$

Hence, $T_b = T_a\left(\frac{V_a}{V_b}\right)^{\gamma-1} = 600\left(\frac{1}{4}\right)^{0.5} = 300\,\text{K}$

(d) Process ac is isothermal.

$\Rightarrow \quad P_aV_a = P_cV_c$

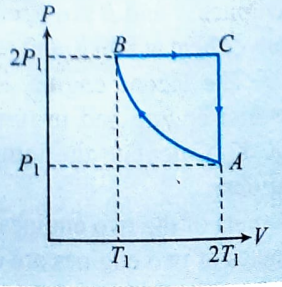
$$V_c = \frac{P_aV_a}{P_c} = \frac{(16\times1)}{2} = 8\,\text{L}$$

EXAMPLE 4.3

Two moles of an ideal monatomic gas is taken through a cycle $ABCA$ as shown in the P–T diagram. During the process AB, pressure and temperature of the gas vary such that PT = constant. If $T_1 = 300$ K, calculate

(a) the work done of the gas in the process AB, and

(b) heat absorbed or released by the gas in each of the process. Give answers in terms of the gas constant R.



Sol. For the process A–B, it is given that PT = constant

Differentiating above equation partially, we have

$$PdT + TdP = 0 \quad ...(i)$$

Equation of state for two moles of a gas

$$PV = 2RT \quad \text{or} \quad P = \frac{2RT}{V} \quad ...(ii)$$

After differentiating Eq. (ii) partially, we get $PdV + VdP = 2R\,dT$

From Eq. (i) partially, we get $PdV + VdP = 2R\,dT$ ...(iii)

From Eqs. (i) and (ii), we have $\left(\frac{2RT}{V}\right)dT + T\,dP = 0$

or $2RTdT + VTdP = 0$

or $VdP = -2\,RdT$ ...(iv)

Now from Eqs. (iii) and (iv), we have $-2RdT + VdP = 2RdT$

or $PdV = 4\,RdT$ ...(v)

(a) The work done in the process AB

$$W_{AB} \int PdV = \int_{600}^{300} 4R\,dT = 4R|T|_{600}^{300} = 4R(300-600)$$

$$= -1200\,R$$

(b) i. As process $B \rightarrow C$ is isobaric, so

$$Q_{BC} = nC_P\Delta T = 2\times\frac{5R}{2}\times(600-300) = 1500\,R$$

ii. Process $C \rightarrow A$ is isothermal, so $\Delta U = 0$

$$Q_{CA} = \Delta U + W_{CA} = W_{CA}$$

$$W_{CA} = nRT\ln\left(P_C/P_A\right)$$

$$= 2R\times600\ln\left(2P_1/P_1\right) = 1200\,R\ln 2$$

$$Q_{CA} = 1200\,R\ln 2$$

Again for process $A \rightarrow B$

$$Q_{AB} = \Delta U + W_{AB} = nC_V\Delta T + W_{AB}$$

$$= 2\times\left(\frac{3R}{2}\right)\times(300-600) - 1200R$$

$$= -900\,R - 1200\,R = -2100R$$

EXAMPLE 4.4

Two moles of helium gas ($\gamma = 5/2$) is initially at temperature 27°C and occupies a volume of 20 L. The gas is first expanded at constant pressure until the volume is doubled. Then, it undergoes an adiabatic change until the temperature returns to its initial value.

(a) Sketch the process on a P–V diagram.

(b) What are the final volume and pressure of the gas?

(c) What is the work done by the gas?

Sol. For a perfect gas $PV = nRT$

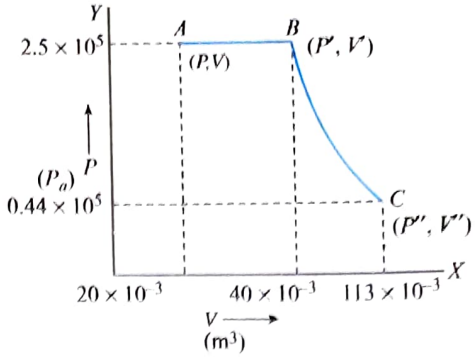
Given that $V = 20\text{ L} = 20\times10^{-3}\text{ m}^3$

$T = 27°\text{C} = 300\text{ K}$ and number of molecules $n = 2$

Thus, initial pressure is given as

$$P = \left(\frac{nRT}{V}\right) = \frac{(2\times8.3\times300)}{(20\times10^{-3})} = 2.5\times10^5\text{ N/m}^2$$

(a) The given figure shows the indicator diagram of the complete process.



(b) At point B,

Pressure $P' = P = 2.5\times10^5\text{ N/m}^2$, and

$V' = 2V = 40\times10^{-3}\text{ m}^3$

As pressure is constant in the process AB, making its volume doubled, its temperature will also be doubled.

Thus, temperature at point B is $T' = 600$ K.

The gas now undergoes adiabatic expansions to cool down at $T'' = T = 300$ K

We know for an adiabatic process $TV^{\gamma-1}$ = constant

$$T'(V')^{\gamma-1} = T''(V'')^{\gamma-1}$$

$$\left(\frac{V''}{V'}\right) = \left(\frac{T'}{T''}\right)^{1/(\gamma-1)} = \left(\frac{600}{300}\right)^{\frac{1}{\frac{5}{3}-1}} = (2)^{3/2} = 2\sqrt{2}$$

Thus, final volume is

$$V'' = (2\sqrt{2})V'$$

$$2\times1.414\times40\times10^{-3} = 113.14\times10^{-3}\text{ m}^3$$

Similarly, final pressure is given by process equation as

$$P'V'^{\gamma} = P''V''^{\gamma}$$

or $$P'' = P'\left(\frac{V'}{V''}\right)^{\gamma}$$

$$= 2.5\times10^5\times\left(\frac{40\times10^{-3}}{113.14\times10^{-3}}\right)^{5/3} = 4.42\times10^4\text{ Pa}$$

(c) Work done under isobaric process AB is

or $W_1 = P\Delta V$

or $W_1 = 2.5\times10^5\times(40-20)\times10^{-3} = 4980$ J

Work done during adiabatic process BC is given as

or $$W_2 = \left(\frac{nR}{\gamma-1}\right)[T_1 - T_2]$$

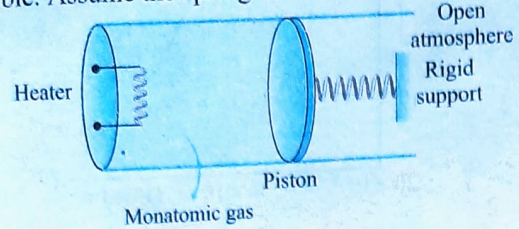
or $$= \frac{(2\times8.3)}{[1-(5/3)]}[300-600] = 7470\text{ J}$$

Total work done $= W_1 + W_2 = 4980 + 7470 = 12450$ J

EXAMPLE 4.5

An ideal monatomic gas is confined in a cylinder by a spring–loaded piston of cross section $8\times10^{-3}\text{ m}^2$. Initially, the gas is at 300 K and occupies a volume of $2.4\times10^{-3}\text{ m}^3$ and the spring is in its relaxed (unstretched, uncompressed) state (see figure). The gas is heated by a small electric heater until the piston moves out slowly by 0.1 m. Calculate the final temperature of the gas and the heat supplied (in joule) by the heater. The force constant of the spring is 8000 N/m, atmospheric pressure $1\times10^5\text{ N/m}^2$. The cylinder and the piston are thermally insulated. The piston is massless and there is no friction between the

piston and the cylinder. Neglect heat loss through the lead wires of the heart. The heat capacity of the heater coil is negligible. Assume the spring to be massless.



Sol. Initially, the pressure of the gas in the cylinder is atmospheric pressure as spring is in relaxed state. Therefore,

P_1 = atmospheric pressure = 1.0×10^5 N/m^2

V_1 = initial volume = 2.4×10^{-3} m^3

T_1 = initial temperature = 300 K

When the heat is supplied by the heater, the piston is compressed by 0.1 m. The reaction force of compression of spring is equal to kx which acts on the piston or on the gas as

$$F = kx = 8000 \times 0.1 \times 800 \text{ N}$$

Pressure exerted on the piston by the spring is

$$\Delta P = \frac{F}{A} = \frac{800}{8\times10^{-3}} = 1\times10^5 \text{ N/m}^2$$

The total pressure P_2 of the gas inside cylinder is

$$P_2 = P_{atm} + \Delta P = 1 \times 10^5 + 1 \times 10^5 = 2 \times 10^5 \text{ N/m}^2$$

Since the piston has moved outwards, there has been an increase of ΔV in the volume of the gas, i.e.,

$$\Delta V = A \times x = (8 \times 10^{-3}) \times (0.1) = 8 \times 10^{-4} \text{ m}^3$$

The final volume of the gas

$$V_2 = V_1 + \Delta V = 2.4 \times 10^{-3} + 8 \times 10^{-4} = 3.2 \times 10^{-3} \text{ m}^3$$

Let T_2 be the final temperature of gas. Then

$$(P_1V_1/T_1) = (P_2V_2/T_2)$$

$$\Rightarrow \quad T_2 = (P_2V_2/P_1V_1)T_1$$

$$= 300\times(2\times10^5\times3.2\times10^{-3})/(10^5\times2.4\times10^{-3}) = 800\,\text{K}$$

Let the heat supplied by the heater be Q. This is used in two parts, i.e., a part is used in doing external work W due to expansion of the gas and the other part is used in increasing h internal energy of the gas. Hence, $Q = W + \Delta U$

$$\text{Now, } W = \int_{V_1}^{V_2} PdV = \int_0^x \left(P_{atm} + \frac{kx}{A}\right)Adx$$

[as pressure is $(P_{atm} + kx/A$ and $dV = Adx)$]

$$\text{or} \quad W = P_{atm}Ax + \left(\frac{kx^2}{2}\right)$$

$$= \left[10^5\times(8\times10^{-3})(0.1) + \frac{8000\times(0.1)^2}{2}\right] = 120\,\text{J}$$

Further, $\Delta U = nC_v\Delta T$

Number of moles of gas can be obtained from initial conditions and gas law as

$$n = \left(\frac{PV}{RT}\right) = \left(\frac{1\times10^5\times2.4\times10}{8.314\times300}\right) = 0.096\,\text{mol}$$

Thus, change in internal energy of gas is given as

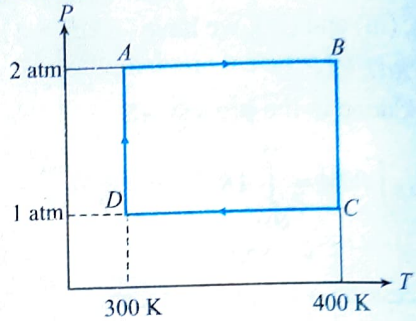
$$\Delta U = n\left(\frac{3}{2}R\right)\Delta T \qquad \left(\text{as for monatomic gas } C_v = \frac{3}{2}R\right)$$

$$\text{or} \quad U = 0.096 \times \left(\frac{3}{2}\right) \times 8.314 \times 500 = 598.6 \text{ J}.$$

Heat supplied by the heater = (120 + 598.6) = 718.6 J.

EXAMPLE 4.6

Two moles of helium gas undergoes a cyclic process as shown in the given figure. Assuming the gas to be ideal, calculate the following quantities in this process.



(a) The net change in the heat energy.
(b) The net work done.
(c) The net change in internal energy.

Sol. As we know in a cyclic process, the change in heat energy or heat supplied to the gas is equal to the net work done by the gas.

Here, AB is isobaric process. Hence, work done during the process from A to B is

$$W_{AB} = P(V_2 - V_1) = nR(T_2 - T_1)$$

$$\text{or} \quad W_{AB} = 2 \times 8.314 \times (400 - 300) = 1662.8 \text{ J}$$

Work done during isothermal process from B to C is

$$W_{BC} = nRT_C \ln (V_2/V_1) = nRT_C \ln (P_1/P_2)$$

$$= 2 \times 8.314 \times 400 \times \ln (2)$$

$$= 2 \times 8.314 \times 400 \times 0.693 = 4610.2 \text{ J}$$

Work done during isobaric process from C to D

$$W_{CD} = nR(T_D - T_C) = 2 \times 8.314 \times (300 - 400) = -1662.8$$

Work done during isothermal process from D to A

$$W_{DA} = nRT_D \ln (P_D/P_A) = nRT_D \ln (2)$$

$$= -2 \times 8.314 \times 300 \times 0.693 = -3457.7 \text{ J}$$

Net work done = $W_{AB} + W_{BC} + W_{CD} + W_{DA}$

$$= 1662.8 + 4610.2 - 1662.5 - 3457.7 = 1152.5$$

Now from the first law of thermodynamics $\Delta Q = \Delta U + \Delta W$

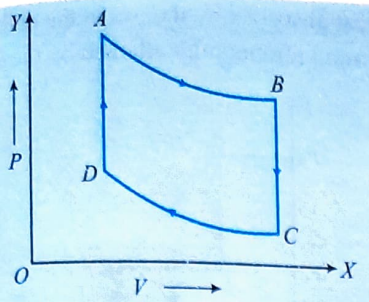
Here $\Delta U = 0$, thus we have $\Delta Q = \Delta W = 1152.5$ J

So the heat given to the system is 1152.5 J

As the gas returns to its original state, there is no change in internal energy.

EXAMPLE 4.7

One mole of monatomic ideal gas is taken through the cycle as shown in the given figure.



$A \rightarrow B$ adiabatic, expansion
$B \rightarrow C$ cooling at constant volume
$C \rightarrow D$ adiabatic compression
$D \rightarrow A$ heating at constant volume

The pressure and temperature at A, B, etc., are denoted by P_A, P_B, ... and, T_A, T_B ... T, respectively. Given that $T_A = 1000$ K, $P_B = (2/3)P_A$ and $P_C = (1/3)P_A$.

Calculate the following quantities:

(a) The work done by the gas in process $A \rightarrow B$.

(b) The work done by the gas in process $B \rightarrow C$.

(c) The temperature T_D. (Given that $(2/3)^{2/5} = 0.85$)

Sol.

(a) The work done in adiabatic process AB is given by

$$W_1 = \frac{R(T_A - T_B)}{\gamma - 1} = \frac{R(T_A - T_B)}{(5/3) - 1}$$

(as for monatomic gas $\gamma = 5/3$)

or $$W_1 = \frac{3}{2}R(T_A - T_B) \quad ...(i)$$

For adiabatic change, we have

$$P_A^{\gamma-1}T_A^{-\gamma} = P_B^{\gamma-1}T_B^{-\gamma}$$

or $$\left(\frac{P_A}{P_B}\right)^{\gamma-1} = \left(\frac{T_A}{T_B}\right)^{\gamma}$$

or $$\left(\frac{T_A}{T_B}\right) = \left(\frac{P_A}{P_B}\right)^{2/5} \quad \left[\text{Using } \gamma = \frac{5}{3}\right]$$

or $$\left(\frac{T_A}{T_B}\right) = \left(\frac{3}{2}\right)^{2/5} \quad (\text{as } P_B = (2/3)P_A)$$

or $$T_B = T_A\left(\frac{2}{3}\right)^{2/5} \quad ...(ii)$$

$$= 1000 \times 0.85 = 850 \text{ K}$$

From Eq. (i), $W_1 = \left(\frac{3}{2}\right) \times 8.31 \times (1000 - 850) = 1869.83$ J

(b) Heat lost by the gas in process BC is given by

$$C_V(T_B - T_C) = \left(\frac{R}{\gamma - 1}\right)(T_B - T_C) = \left(\frac{3}{2}\right)R(T_B - T_C) \quad ...(iii)$$

Process BC is under constant volume; hence

$$\left(\frac{P_B}{P_C}\right) = \left(\frac{T_B}{T_C}\right) \quad \text{or} \quad T_C = \left(\frac{P_C}{P_B}\right) \times T_B \quad ...(iv)$$

or $$T_C = \frac{(P_A/3)}{2P_A/3}T_B = \left(\frac{T_B}{2}\right) = 425 \text{K}$$

From Eq. (ii), we get

Heat lost $= \left(\frac{3}{2}\right) \times 8.31 \times (850 - 425) = 5297.63$ J

(c) For path AB

$$P_A^{\gamma-1}T_A^{-\gamma} = P_B^{\gamma-1}T_B^{-\gamma}$$

or $$\left(\frac{P_A}{P_B}\right)^{\gamma-1} = \left(\frac{T_A}{T_B}\right)^{\gamma} \quad ...(v)$$

For path BC

$$\left(\frac{P_B}{T_B}\right) = \left(\frac{P_C}{T_C}\right) \quad \text{or} \quad \left(\frac{P_B}{P_C}\right) = \left(\frac{T_B}{T_C}\right) \quad ...(vi)$$

For path CD

$$\left(\frac{P_D}{P_C}\right)^{\gamma-1} = \left(\frac{T_D}{T_C}\right)^{\gamma} \quad ...(vii)$$

For path AD

$$\left(\frac{P_A}{P_D}\right) = \left(\frac{T_A}{T_D}\right) \quad ...(viii)$$

Dividing Eq. (v) by Eq. (vii), we get

$$\left(\frac{P_A}{P_B} \times \frac{P_C}{P_D}\right)^{\gamma-1} = \left(\frac{T_A}{T_B} \times \frac{T_C}{T_D}\right)^{\gamma} \quad ...(ix)$$

Dividing Eq. (viii) by Eq. (iv), we get

$$\left(\frac{P_A}{P_B} \times \frac{P_C}{P_D}\right) = \left(\frac{T_A}{T_B} \times \frac{T_C}{T_D}\right)$$

or $$\left(\frac{P_A}{P_B} \times \frac{P_C}{P_D}\right)^{\gamma-1} = \left(\frac{T_A}{T_B} \times \frac{T_C}{T_D}\right)^{\gamma-1} \quad ...(x)$$

$$\left(\frac{T_A}{T_B} \times \frac{T_C}{T_D}\right)^{\gamma} = \left(\frac{T_A}{T_B} \times \frac{T_C}{T_D}\right)^{\gamma-1}$$

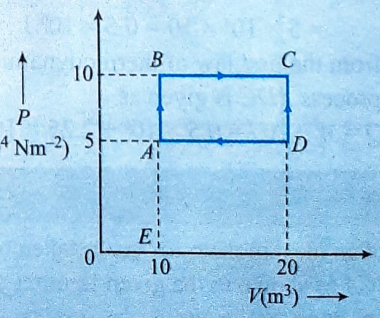
$$T_A T_C = T_D T_B$$

$$1000 \times 425 = T_D \times 850$$

$$T_D = 500 \text{ K}$$

EXAMPLE 4.8

A sample of 2 kg of monatomic helium (assumed ideal) is taken through the process ABC and another sample of 2 kg of the same gas is taken through the process ADC as shown in the given figure. Given molecular mass of helium = 4.



(a) What is the temperature of helium in each of the states A, B, C and D?

(b) Is there any way of telling afterwards which sample of helium went through the process *ABC* and which went through the process *ADC*? Write yes or no.

(c) How much is the heat involved in each of the processes *ABC* and *ADC*?

Sol. Given that mass of helium used in the process is $m = 2$ kg, number of moles can be given as

$$n = \left(\frac{m}{M}\right) = \frac{2}{(2\times10^{-3})} = 500$$

At different states, from the given figure, the pressure and volume of gas are also givens as

$P_A = P_D = 5 \times 10^4$ N/m^2

$P_B = P_C = 10^5$ N/m^2

$V_A = V_B = 10$ m^3

$V_C = V_D = 20$ m^3

(a) From gas law, we have

$$T_A = \frac{(P_AV_A)}{(nR)} = \frac{(5\times10^4\times10)}{(500\times8.314)} = 120.3\text{ K}$$

$$T_B = \frac{(P_BV_B)}{(nR)} = \frac{(10^5\times20)}{(500\times8.314)} = 481.11\text{K}$$

$$T_D = \frac{(P_DV_D)}{(nR)} = \frac{(5\times10^4\times20)}{(500\times8.314)} = 120.3\text{ K}$$

(b) Since the gas is taken from same initial state to same final state *C*, no matters whatever be the path, the answer is no.

(c) In process *ABC*, the change in internal energy is

$$\Delta U_{ABC} = U_C - U_A = (f/2)nR(T_C - T_A)$$
$$= (3/2) \times 500 \times 8.314\ (481.11 - 120.3)$$
$$= 2.25 \times 10^6\text{ J}$$

Net work done in process *ABC* is

$$W_{ABC} = W_{AB} + W_{BC} = 0 + \text{area below curve } BC$$
$$= 0 + 10^5 \times 10 = 10^6\text{ J}$$

Thus, from the first law of thermodynamics, heat supplied in process *ABC* is

$$Q = W + \Delta U$$

or $\quad Q = 10^6 + 2.25 \times 10^6 = 3.25 \times 10^6$ J

Similarly, in process *ADC* as being a state function, change in internal energy remains same as initial and final states are same. Thus,

$$\Delta U_{ADC} = 2.25 \times 10^6\text{ J}$$

Thus, work done by the gas in process *ADC* is

$$W_{ADC} = W_{AD} + W_{DC} = \text{area below curve } AD + 0$$
$$= 5 \times 10^4 \times 10 = 0.5 \times 10^6\text{ J}$$

Thus, from the first law of thermodynamics, heat supplied in the process *ADC* is given as

$$Q = W + \Delta U = 0.5 \times 10^6 + 2.25 \times 10^6 = 2.75 \times 10^6\text{ J}$$

EXAMPLE 4.9

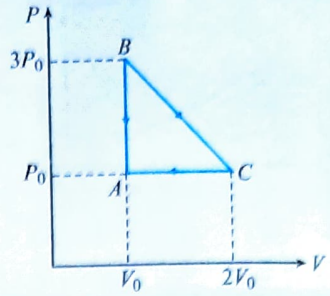
One mole of an ideal monatomic gas is taken round the cyclic process *ABCA* as shown in the given figure. Calculate

(a) the work done by the gas,

(b) the heat rejected by the gas in the path *CA* and the heat absorbed by the gas in the path *AB*,

(c) the net heat absorbed by the gas in the path *BC*,

(d) the maximum temperature attained by the gas during the cycle.



Sol.

(a) The work done by the gas is equal to the area under the closed curve. Thus, work done in cycle is

$$W = \frac{1}{2}(2V_0 - V_0)(3P_0 - P_0) = \frac{1}{2}V_0 \times 2P_0 = P_0V_0$$

(b) Heat rejected in path *CA* is given as

$$Q_{CA} = nC_P\Delta T = nC_P(T_C - T_A)$$
$$= 1 \times (5/2)R\left[\frac{P_0 2V_0}{R} - \frac{P_0V_0}{R}\right] = \frac{5}{2}P_0V_0$$

(as $n = 1$ mole)

Heat absorbed in path *AC* is $Q_{AC} = nC_P(T_B - T_A)$

$$= 1\times\frac{3}{2}R\times\left[\frac{3P_0V_0}{R} - \frac{P_0V_0}{R}\right] = 3P_0V_0$$

(c) For cycle *ABC*, we have

heat supplied = work done by the gas

or $\quad -\left(\frac{5}{2}\right)P_0V_0 + 3P_0V_0 + Q_{BC} = P_0V_0$

Heat supplied in path *BC* is given by

$$Q_{BC} = P_0V_0 + \left(\frac{5}{2}\right)P_0V_0 - 3P_0V_0 = P_0V_0/2$$

(d) We know that PV/T = constant. So, when PV is maximum T is also maximum. PV is maximum for part *BC*. Hence temperature will be maximum between *B* and *C*.

Let equation of *BC* be $P = kV + k'$ satisfying both the points *B* and *C*

For point *B*, $3P_0 = kV_0 + k'$

For point *C*, $P_0 = k(2V_0) + k'$

Solving these equations, we get

$$k = -2(P_0/V_0) \text{ and } k' = 5P_0$$

So the equation for time *BC* is

$$P = -2\left(\frac{P_0}{V_0}\right)\times V + 5P_0$$

or $\quad \left(\frac{RT}{V}\right) = -\left(\frac{2P_0V}{V_0}\right) + 5P_0$

or $\quad T = \frac{P_0}{R}\left[5V - 2\left(\frac{V^2}{V_0}\right)\right] = 0$

For maximum, $dT/dV = 0$

So $\frac{dT}{dV}=\left(\frac{P_0}{R}\right)\left[5-\left(\frac{4V}{V_0}\right)\right]=0$

Hence, $5-\left(\frac{4V}{V_0}\right)=0$

or $5V_0-4V=0$

or $V=\left(\frac{5}{4}\right)V_0$ ...(ii)

Substituting the value of V from Eq. (ii) in Eq. (i), we get

$$T_{max}=\frac{P_0}{R}\left[5\times\left(\frac{5}{4}V_0\right)-2\left(\frac{5V_0}{4}\right)^2\frac{1}{V_0}\right]$$

$$=\frac{P_0}{R}\left[\frac{25V_0}{4}-\frac{25V_0}{8}\right]=\frac{P_0}{R}\times\frac{25V_0}{8}=\frac{25P_0V_0}{8R}$$

EXAMPLE 4.10

A massless piston divides a closed, thermally insulated cylinder into two equal parts. One part contains $M = 28$ g of nitrogen. At this temperature, one-third of molecules are dissociated into atoms and the other part is evacuated. The piston is released and the gas fills the whole volume of the cylinder at temperature T_0. Then, the piston is slowly displaced back to its initial position. Calculate the increase in internal energy of the gas. Neglect further dissociation of molecules during, the motion of the piston.

Sol. Molar mass of N_2 gas is $M = 28$ g

Number of moles of N_2, $n_0=(m/M)=1$

One-third molecules are dissociated into atoms.

No. of moles of diatomic $N_2 = 2/3$

No. of moles of monatomic nitrogen $= 2\times(1/3)$

For monatomic gas, $C_{V_1}=(3/2)R$

For diatomic gas, $C_{V_2}=(5/2)R$

Average molar specific heat at constant volume,

$$C_V=\frac{n_1C_{V_1}+n_2C_{V_2}}{n_1+n_2}=\frac{\left(\frac{2}{3}\right)\left(\frac{3}{2}\right)R+\left(\frac{2}{3}\right)\left(\frac{5}{2}\right)R}{\frac{2}{3}+\frac{2}{3}}=2R$$

Now, $C_P-C_V=R \Rightarrow C_P=3R$ and $\gamma=\frac{C_P}{C_V}=1.5$

Piston is displaced back to its initial position, and during the adiabatic compression, volume of the gas mixture is halved.

For adiabatic compression, $T_1V_1^{\gamma-1}=T_2V_2^{\gamma-1}$

where $T_1=T_0$; $V_1=V_0$; $T_2=?$; $V_2=V_0/2$

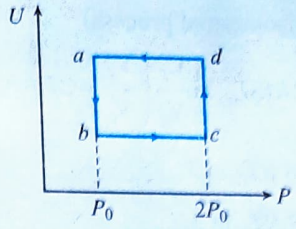
$$T_0V_0^{\gamma-1}=T_2\left(\frac{V_0}{2}\right)^{\gamma-1} \Rightarrow T_2=T_0\sqrt{2}$$

$$\Delta U=(n_1+n_2)C_V(T_2-T_0)$$

$$=\frac{4}{3}\times0.2R(T_0\sqrt{2}-T_0)=\frac{8}{3}RT_0(\sqrt{2}-1)$$

EXAMPLE 4.11

The given figure shows the variation of internal energy (U) with the pressure (P) of 2.0 mole gas in cyclic process *abcda*. The temperatures of gas at c and d are 300 K and 500 K, respectively. Calculate the heat absorbed by the gas during the process.



Sol. Change in internal energy for cyclic process (ΔU) = 0

For process $a \rightarrow b$, (P – constant)

$$W_{a\rightarrow b}=P\Delta V=nR\Delta T=-400R$$

For process $b \rightarrow c$ (T – constant)

$$W_{b\rightarrow c}=-2R(300)\ln 2$$

For process $c \rightarrow d$ (P – constant)

$$W_{c\rightarrow d}=+400\,R$$

For process $d \rightarrow a$ (T – constant)

$$W_{d\rightarrow a}=+2R(500)\ln 2$$

Net work, $(\Delta W)=W_{a\rightarrow b}+W_{b\rightarrow c}+W_{c\rightarrow d}+W_{d\rightarrow a}$

$$\Delta W=400\,R\ln 2$$

$\Delta Q=\Delta U+\Delta W$, first law of thermodynamics.

$$\Delta Q=400\,R\ln 2$$

EXAMPLE 4.12

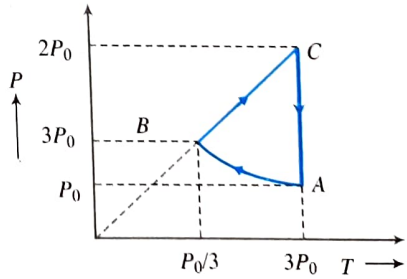
Two moles a monatomic gas in state 'A' having critical pressure P_0 and temperature $3T_0$ is taken to a state B having pressure $3P_0$ and temperature $T_0/3$ by the process of equation P^2T = constant. Then state B is taken to state C keeping the volume constant and it comes back to initial state 'A' keeping temperature constant.

(a) Plot a P and T graph. (P on the y-axis and T on the x-axis).

(b) Find the net work done and heat supplied to the gas during the complete cycle.

Sol.

(a)



(b) Work done in process AB

$$\Delta W_{AB}=\frac{2R\left[\frac{T_0}{3}-3T_0\right]}{\left(\frac{1}{3}-1\right)}$$

$$W = -8\,RT_0$$

$$\Delta Q_{AB} = \Delta U + \Delta W = -nC_v\Delta T - 8RT_0$$

$$\Delta Q_{AB} = -16\,RT_0$$

For process $B \to C$, $W_{BC} = 0$

$$\Delta U = 8RT_0, \quad \Delta Q_{BC} = 8RT_0$$

and $C \to A$ (isothermal process)

$$\Delta W_{CA} = 2R(3T_0)\ln\frac{V_f}{V_i}\left(\frac{V_f}{V_i} = 27\right)$$

$$\Delta W_{CA} = 18\,RT_0 \ln 3$$

$$\Delta Q_{CA} = 18\,RT_0 \ln 3$$

Net work done,

$$\Delta W = \Delta W_{AB} + \Delta W_{BC} + \Delta W_{CA} = -8RT_0 + 0 + 18\,RT_0 \ln 3$$

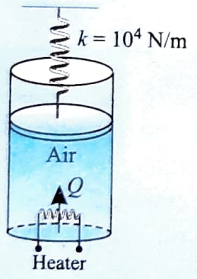
$$\Delta W = RT_0 + (18 \ln 3 - 8)$$

$$\Delta Q_{net} = \Delta Q_{BC} + \Delta Q_{CA} = -8\,RT_0 + 18\,RT_0 \ln 3$$

$$\Delta Q_{net} = RT_0(18 \ln 3 + 8)$$

EXAMPLE 4.13

Air is contained in a piston–cylinder arrangement as shown in the given figure with a cross-sectional area of 4 cm² and an initial volume of 20 cc. The air is initially at a pressure of 1 atm and temperature of 20°C. The piston is connected to a spring whose spring constant is $k = 10^4$ N/m, and the spring is initially undeformed. How much heat must be added to the air inside cylinder to increase the pressure to 3 atm (for air, $C_v = 718$ J/kg °C, molecular mass of air 28.97)?



Sol. When pressure changes from 1 atm to 3 atm, the change in pressure

$$P = 2 \text{ atm} = 2 \times 1 \times 10^5 \text{ N/m}^2$$

The force exerted on the piston $F = PA = 2 \times 10^5 \times 4 \times 10^{-4} = 80$ N

The compression of the spring $x = \frac{F}{k} = \frac{80}{10^4} = 0.08$ m

The change in volume of air due to displacement of piston by x

$$\Delta V = Ax = 4 \times 10^{-4} \times 0.008 = 3.2 \times 10^{-6} \text{ m}^3$$

$\therefore$ Final volume, $V_2 = V_1 + \Delta V = 20 \times 10^{-6} + 3.2 \times 10^{-6}$

By equation of state $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$

$$T_2 = \frac{P_2V_2T_1}{P_1V_1} = \left(\frac{3}{1}\right) \times \frac{(23.2 \times 10^{-6})}{(20 \times 10^{-6})} \times (273 + 20) = 1020 \text{ K}$$

The change in internal energy of air $\Delta U = mC_v\Delta T$

$$= (2.38 \times 10^{-5}) \times 718 \times (1020 - 293) = 12.42 \text{ J}$$

Work done in compressing the spring by x

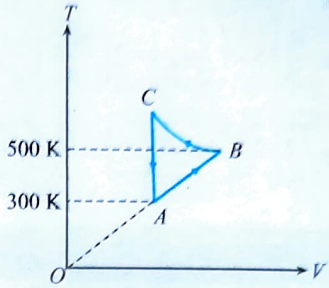
$$W = \frac{1}{2}kx^2 = \frac{10^4}{2} \times (0.008)^2 = 0.32 \text{ J}$$

From the first law of thermodynamics

$$Q = \Delta U + W = 12.42 + 0.32 = 12.74 \text{ J}$$

EXAMPLE 4.14

Consider the cyclic process $ABCA$, shown in the given figure, performed, on a sample of 2.0 moles of an ideal gas. A total of 1200 J of heat is withdrawn from the sample in the process. Find the work done by the gas during the part BC.



Sol. In a cyclic process $\Delta U = 0$

From the first law of thermodynamics, for the cyclic process

$$Q = \Delta U + W$$

$\therefore$ $W = Q - \Delta U = -1200 - 0 = -1200$ J

From C to A, $\Delta V = 0$

$\therefore$ $W_{CA} = 0$

For the whole cycle $W_{AB} + W_{BC} + W_{CA} = W = -1200$ J

As $W_{CA} = 0$

$\therefore$ $W_{AB} + W_{BC} = -1200$ J ...(i)

Work done from A to B: In the process $V \propto T$, so pressure remains constant.

We know that $PV = nRT$

or $P\Delta V = nR\Delta T$

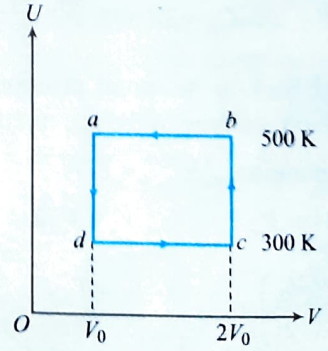
$\therefore$ $W_{AB} = P\Delta V = nR\Delta T = 2 \times 8.31 \times (500 - 300) = 3324$ J

Substituting this value in Eq. (i), we get $3324 + W_{BC} = -1200$

$\therefore$ $W_{BC} = -4524$ J

EXAMPLE 4.15

The given figure shows the variation in the internal energy U with the volume V of 2.0 moles of an ideal gas in cyclic process $abcda$. The temperatures of the gas at b and c are 500 K and 300 K, respectively. Calculate the heat absorbed by the gas during the process.



Sol. In the process a to b and c to d as $\Delta U = 0$, therefore $\Delta T = 0$

or T = constant

$$W = \int_{V_i}^{V_f} PdV$$

We have $PV = nRT \Rightarrow P = \dfrac{nRT}{V}$

$$\therefore \quad W = \int_{V_i}^{V_f} (nRT)\frac{RV}{V} = nRT\,|\ln V|_{V_i}^{V_f} = nRT \ln \frac{V_f}{V_i}$$

$$W_{ab} = nRT_b \ln \frac{2V_0}{V_0} = 2R \times 500 \ln 2 = 1000\,R \ln 2$$

and $$W_{cd} = nRT_c \ln \frac{V_0}{2V_0} = 2R \times 300 \ln \frac{1}{2} = -600\,R \ln 2$$

There is no volume change from b to c and from d to a, so

$$W_{bc} = W_{da} = 0$$

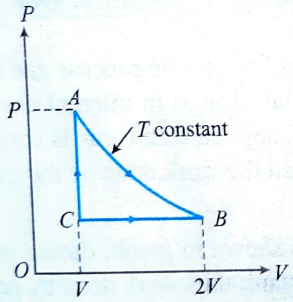
The work done in complete cycle

$$W = W_{ab} + W_{bc} + W_{cd} + W_{da}$$

$$= 1000\,R \ln 2 + 0 - 600\,R \ln 2 + 0 = 400\,R \ln 2$$

EXAMPLE 4.16

One mole of a gas is carried through the cycle shown in the given figure. The gas expands at constant temperature T from volume V to 2V. It is then compressed to the initial volume at constant pressure and is finally brought back to its original state by heating at constant volume. Calculate the work done by the gas in complete cycle.



Sol. Work done in isothermal process A to B

$$W_{AB} = nRT \ln \frac{V_f}{V_i} = RT \ln \frac{2V}{V} \qquad [\because \ n = 1]$$

If pressure at B is P_B, then $P_A V_A = P_B V_B$

or $PV = P_B \times 2V \Rightarrow P_B = P/2$

Work done in isobaric process from B to C at constant pressure $P/2$

$$W_{BC} = P_B(V_f - V_i) = \frac{P}{2}(V - 2V) = -\frac{PV}{2} = \frac{-RT}{2}$$

Work done in isochoric process C to A, $W_{CA} = 0$

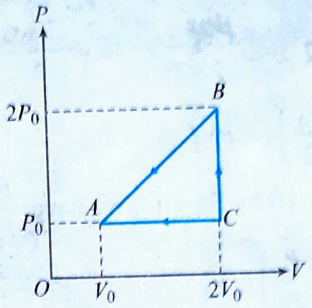
Therefore work done in the whole cycle,

$$W = W_{AB} + W_{BC} + W_{CA} = RT \ln 2 - \frac{RT}{2} + 0$$

$$\Rightarrow \quad W = RT\left(\ln 2 - \frac{1}{2}\right)$$

EXAMPLE 4.17

One mole of a monatomic gas is taken from a point A to another point B along the path ACB. The initial temperature at A is T_0. Calculate the heat absorbed by the gas in the process $A \to C \to B$.



Sol. If T_B be the temperature at B, then by gas law

$$\frac{P_A V_A}{T_A} = \frac{P_B V_B}{T_B}$$

$$\therefore \quad T_B = \frac{P_B V_B}{P_A V_A} T_A = \frac{(2P_0)(2V_0)}{P_0 V_0} T_0$$

The change in internal energy from A to B

$$\Delta U = nC_v \Delta T = 1 \times \frac{3R}{2} \times (4T_0 - T_0) = \frac{9RT_0}{2}$$

Work done in the process A to C

$$W_{AC} = P\Delta V = P_0(2V_0 - V_0) = P_0 V_0 = RT_0 \text{ and } W_{CB} = 0$$

$\therefore$ Total work done from $A \to C \to B$

$$W_{AC} + W_{CB} = RT_0 + 0 = RT_0$$

From the first law of thermodynamics, $Q = \Delta U + W$

$$= \frac{9RT_0}{2} + RT_0 = \frac{11RT_0}{2}$$

Thus heat absorbed by the gas from $A \to C \to B$ is $\dfrac{11RT_0}{2}$

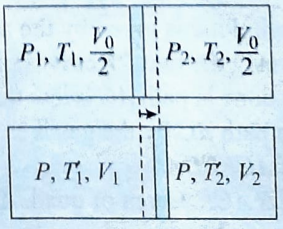
EXAMPLE 4.18

The given figure shows an adiabatic cylindrical tube of volume V_0, divided in two parts by a frictionless adiabatic separator. Initially, the separator is kept in the middle, an ideal gas at pressure P_1 and temperature T_1 is injected into the left part and another ideal gas at pressure P_2 and temperature T_2 is injected into the right part. $C_P/C_V = \gamma$ is the same for both the gases. The separator is slid slowly and is released at a position where it can stay in equilibrium. Find

(a) the volumes of the two parts,

(b) the heat given to the gas in the left part and

(c) the final common pressure of the gases.



Sol. The separator will stop in the position when pressures of both the parts of the tube become equal. Let it is P. If V_1 and V_2 are the volumes of two parts, then

$$V_1 + V_2 = V_0 \quad \text{...(i)}$$

(a) The process in each part is adiabatic, so

$$P_1\left(\frac{V_0}{2}\right)^{\gamma} = PV_1^{\gamma} \quad \text{...(ii)}$$

$$P_2\left(\frac{V_0}{2}\right)^{\gamma} = PV_2^{\gamma} \quad \text{...(iii)}$$

Dividing Eq. (i) by Eq. (iii), we have $\frac{P_1}{P_2} = \frac{V_1^{\gamma}}{V_2^{\gamma}}$

or $$V_1 = \left(\frac{P_1}{P_2}\right)^{1/\gamma} V_2 \quad \text{(iv)}$$

Substituting this value in Eq. (i), we get

$$V_2 = \left[\frac{V_0 P_2^{1/\gamma}}{P_1^{1/\gamma} + P_2^{1/\gamma}}\right] \text{ and } V_1 = \left[\frac{V_0 P_1^{1/\gamma}}{P_1^{1/\gamma} + P_2^{1/\gamma}}\right]$$

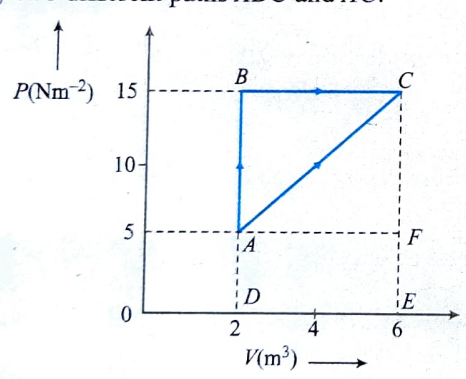
(b) As cylindrical tube and separator both are adiabatic, no heat is given in the process.

(c) Substituting the value of V_1 in Eq. (ii), we get

$$P = \left[\frac{P_1^{1/\gamma} + P_2^{1/\gamma}}{2^{\gamma}}\right]$$

EXAMPLE 4.19

The given figure shows an ideal gas changing its state A to state C by two different paths ABC and AC.



(a) Find the path along which the work done is the least.

(b) The internal energy of the gas at A is 10 J and the amount of heat supplied to change its state to C through the path AC is 200 J. Find the internal energy at C.

(c) The internal energy of the gas at state B is 20 J. Find the amount of heat supplied to the gas to go from state A to state B.

Sol.

(a) We know work done is given by the area below PV curve thus by observing the two PV curves, AC and ABC, we can say that work done in path AC is less than that in path ABC.

(b) Work done in path AC by the gas is

W_{AC} = area of $ACFEDA$

= area of ΔACF + area of quad. $AFED$

$$= \frac{1}{2}\times(15-5)\times(6-2)+(6-2)\times 5 = 20 + 20 = 40 \text{ J}$$

It is given that heat supplied in process AC is $Q_{AC} = 200$ J. Thus change in internal energy of gas in path AC is, from the first law of thermodynamics, given as

$$Q_{AC} = W_{AC} + \Delta U_{AC}$$

or $$\Delta U_{AC} = Q_{AC} - W_{AC} = 200 - 40 = 160 \text{ J}$$

As it is given that at state A, internal energy of gas is 10 J thus at state C, its internal energy is

$$\Delta U_{AC} = U_C - U_A$$

or $$U_C = \Delta U_{AC} + U_A = 16 + 10 = 170 \text{ J}$$

(c) As in process AB, no volume change takes place thus no work is done by or on the gas during path AB. Thus according to the first law of thermodynamics, we have

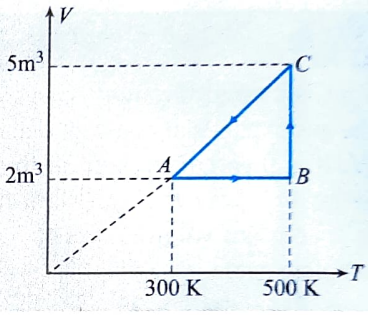
$$Q_{AB} = \Delta U_{AB} + W_{AB}$$

or $$Q_{AB} = U_B - U_A + 0$$

or $$Q_{AB} = 20 - 10 = 10 \text{ J}$$

EXAMPLE 4.20

The given figure shows a process $ABCA$ performed on 1 mole of an ideal gas. Find the net heat supplied to the gaseous system during the process.



Sol. As we know in a cyclic process gas finally returns to its initial state and total change in internal energy of gas is zero thus the total heat supplied to the gas is equal to work done by the gas. Now we find the work done by the gas in different paths of the cycle.

In process *AB*: As shown in graph, during process AB, volume of gas remains constant; thus work done by gas is zero.

$$W_{AB} = 0$$

In Process *BC*: In process BC, volume of gas changes from $V_1 = 2$ m³ to $V_2 = 5$ m³ thus work done can be obtained as

$$W_{BC} = \int_2^5 pdV$$

As in process BC, temperature of gas remains constant at 500 K we can write pressure of gas from gas law as

$$P = \frac{RT}{V} = \frac{500\,R}{V} \quad \text{(as } n = 1 \text{ mol)}$$

Now work done is

$$\therefore \quad W_{BC} = \int_2^5 \frac{500\,R}{V} dV = 500\,R \ln\left(\frac{5}{2}\right)$$

In process *CA*

As in this process, path is a straight line passing through origin thus $V \propto T$ or pressure of gas remains constant and we know if gas pressure is constant work done is given as

$$W_{CA} = nR(T_2 - T_1) = nR\,(300 - 500) = -200R$$

This is negative as gas is being compressed from volume 5 m^3 to 2 m^3 or work is done on the gas.

Now we can find the total work done by the gas in the complete cycle $ABCA$ as

$$W_{ABCA} = W_{AB} + W_{BC} + W_{CA}$$

$$= -0 + 500\,R \ln\left(\frac{5}{2}\right) - 200R - R\left(500 \ln\frac{5}{2} - 200\right)$$

$$= 2146.22 \text{ J} = \text{heat supplied to the gas}$$

EXAMPLE 4.21

One mole of an ideal monatomic gas undergoes the process $P = \alpha T$, where α is a constant.

(a) Find the work done by the gas if its temperature increases by 50 K.

(b) Also, find the molar specific heat of the gas.

Sol.

(a) $P = \alpha T^{1/2}$ ...(i)

where α is constant $n = 1$ mol, monatomic

$$W = \int P\,dV = \int (\alpha T^{1/2})\,dV$$

From Eq. (i), $T = \dfrac{\alpha T^{1/2} V}{nR} \Rightarrow T^{1/2} = \dfrac{\alpha V}{nR}$

Differentiating both sides,

$$\frac{1}{2\sqrt{T}}dT = \frac{\alpha dV}{nR} \Rightarrow dV = \frac{nR}{\alpha \times 2\sqrt{T}}dT$$

$$W = \int \alpha T^{1/2} \frac{nR}{2T^{1/2}}dT$$

$$= \frac{n\alpha R}{2}\int_{T_1}^{T_2} dT = \frac{R}{2} \times 50 = 25\,R = 207.7 \text{ J}$$

(b) $Q = \Delta U + W$

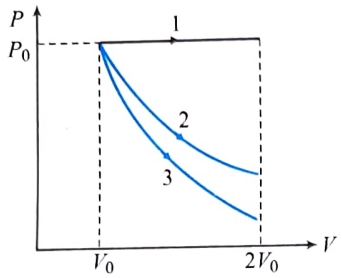
$$C(T_2 - T_1) = \frac{R}{(\gamma - 1)}(T_2 - T_1) + \frac{R}{2}(T_2 - T_1)$$

$$C = \frac{R}{(\gamma - 1)} + \frac{R}{2} = \left(\frac{\gamma + 1}{\gamma - 1}\right)\frac{R}{2}$$

Exercises

Single Correct Answer Type

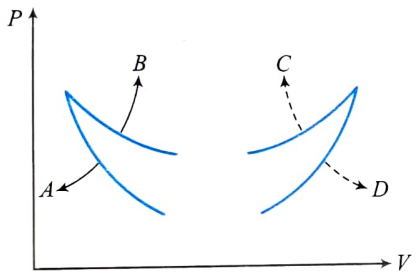
1. A gas is expanded from volume V_0 to $2V_0$ under three different processes. Process 1 is isobaric process, process 2 is isothermal process and process 3 is adiabatic. Let ΔU_1, ΔU_2 and ΔU_3 be the change in internal energy of the gas in these three processes. Then



(1) $\Delta U_1 > \Delta U_2 > \Delta U_3$ (2) $\Delta U_1 < \Delta U_2 < \Delta U_3$

(3) $\Delta U_2 < \Delta U_1 < \Delta U_3$ (4) $\Delta U_2 < \Delta U_3 < \Delta U_1$

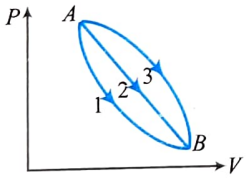
2. Four curves A, B, C and D are drawn in the given figure for a given amount of gas. The curves which represent adiabatic and isothermal changes are



(1) C and D, respectively (2) D and C, respectively

(3) A and B, respectively (4) B and A, respectively

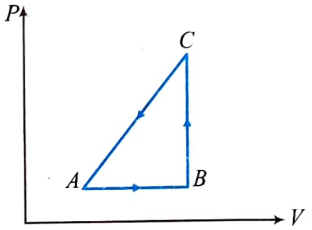
3. An ideal gas of mass m in a state A goes to another state B via three different processes as shown in the given figure. If Q_1, Q_2 and Q_3 denote the heat absorbed by the gas along the three paths, then



(1) $Q_1 < Q_2 < Q_3$ (2) $Q_1 < Q_2 = Q_3$

(3) $Q_1 = Q_2 > Q_3$ (4) $Q_1 > Q_2 > Q_3$

4. The P–V diagram of a system undergoing thermodynamic transformation is shown in the given figure . The work done on the system in going from $A \rightarrow B \rightarrow C$ is 50 J and 20 cal heat is given to the system. The change in internal energy between A and C is



(1) 34 J (2) 70 J

(3) 84 J (4) 134 J

5. If R = universal gas constant, the amount of heat needed to raise the temperature of 2 mol of an ideal monatomic gas from 273 K to 373 K when no work is done is

(1) $100R$ (2) $150R$

(3) $300R$ (4) $500R$

6. A thermally insulated container is divided into two parts by a screen. In one part the pressure and temperature are P and T for an ideal gas filled. In the second part it is vacuum. If now a small hole is created in the screen, then the temperature of the gas will

(1) decrease (2) increase

(3) remain same (4) none of these

7. Two samples A and B of a gas initially at the same pressure and temperature are compressed from volume V to $V/2$ (A isothermally and B adiabatically). The final pressure of A is

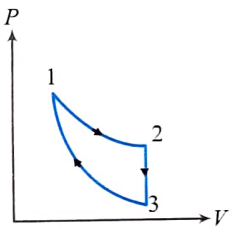
(1) greater than the final pressure of B

(2) equal to the final pressure of B

(3) less than the final pressure of B

(4) twice the final pressure of B

8. Three processes compose a thermodynamic cycle shown in the accompanying P–V, diagram of an ideal gas.



Process 1→2 takes place at constant temperature, during this process 60 J of heat enters the system.

Process 2→3 takes place at constant volume. During this process 40 J of heat leaves the system.

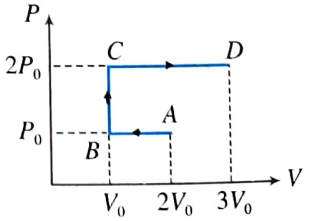
Process 3→1 is adiabatic.

What is the change in internal energy of the system during process 3→1?

(1) –40 J (2) –20 J

(3) +20 J (4) +40 J

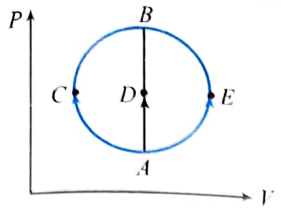
9. P–V diagram of an ideal gas is as shown in the given figure. Work done by the gas in the process $ABCD$ is



(1) $4P_0V_0$ (2) $2P_0V_0$

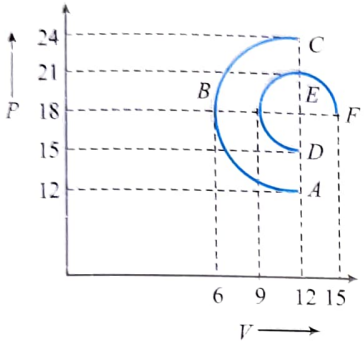
(3) $3P_0V_0$ (4) P_0V_0

10. One mole of an ideal gas is taken from state A to state B by three different processes (a) ACB, (b) ADB and (c) AEB as shown in the P–V diagram. The heat absorbed by the gas is



(1) greater in process (b) than in (a)
(2) the least in process (b)
(3) the same in (a) and (c)
(4) less in (c) than in (b)

11. There are two processes ABC and DEF. In which of the process is the amount of work done by the gas greater?



(1) ABC (2) DEF
(3) Equal in both processes (4) It cannot be predicted

12. Forty calories of heat is needed to raise the temperature of 1 mol of an ideal monatomic gas from 20°C to 30°C at a constant pressure. The amount of heat required to raise its temperature over the same interval at a constant volume ($R = 2\,\text{cal mol}^{-1}\text{K}^{-1}$) is

(1) 20 cal (2) 40 cal
(3) 60 cal (4) 80 cal

13. The temperature of 5 mol of a gas which was held at constant volume was changed from 100°C to 120°C. The change in internal energy was found to be 80 J. The total heat capacity of the gas at constant volume will be equal to

(1) 8 JK^{-1} (2) 0.8 JK^{-1}
(3) 4 JK^{-1} (4) 0.4 JK^{-1}

14. A monatomic gas expands at constant pressure on heating. The percentage of heat supplied that increases the internal energy of the gas and that is involved in the expansion is

(1) 75%, 25% (2) 25%, 75%
(3) 60%, 40% (4) 40%, 60%

15. The specific heat at constant volume for the monatomic argon is 0.075 kcal/kg-K, whereas its gram molecular specific heat is $C_v = 29.8$ cal/mol/K. The mass of the argon atom is (Avogadro's number $= 6.02 \times 10^{23}$ molecules/mol)

(1) 6.60×10^{-23} g (2) 3.30×10^{-23} g
(3) 2.20×10^{-23} g (4) 13.20×10^{-23} g

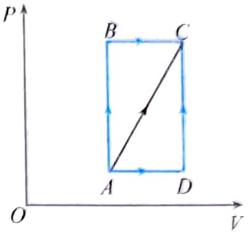
16. The value of $C_p - C_v = 1.00\ R$ for a gas in state A and $C_p - C_v = 1.06\ R$ in another state. If P_A and P_B denote the pressure and T_A and T_B denote the temperatures in the two states, then

(1) $P_A = P_B, T_A > T_B$ (2) $P_A > P_B, T_A = T_B$
(3) $P_A < P_B, T_A > T_B$ (4) $P_A = P_B, T_A < T_B$

17. Twenty-two grams of CO_2 at 27°C is mixed with 16 g of O_2 at 37°C. The temperature of the mixture is about

(1) 31.5°C (2) 27°C
(3) 37°C (4) 30.5°C

18. A thermodynamic process is shown in the given figure. The pressures and volumes corresponding to some points in the figure are: $P_A = 3 \times 10^4$ Pa, $P_B = 8 \times 10^4$ Pa and $V_A = 2 \times 10^{-3}\ \text{m}^3$, $V_D = 5 \times 10^{-3}\ \text{m}^3$.
In process AB, 600 J of heat is added to the system and in process BC, 200 J of heat is added to the system. The change in internal energy of the system in process AC would be

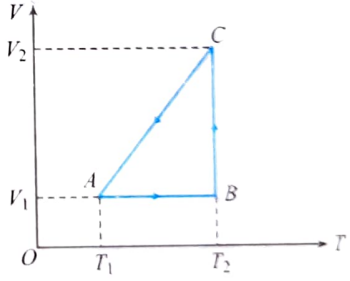


(1) 560 J (2) 800 J
(3) 600 J (4) 640 J

19. When an ideal gas ($\gamma = 5/3$) is heated under constant pressure, what percentage of given heat energy will be utilized in doing external work?

(1) 40% (2) 30%
(3) 60% (4) 20%

20. A cyclic process for 1 mole of an ideal gas is shown in the given figure in the V–T, diagram. The work done in AB, BC and CA, respectively, is



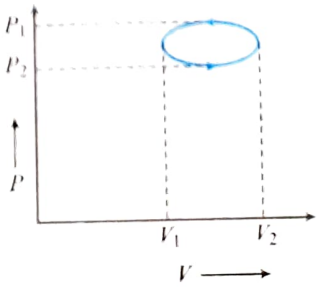
(1) $0, RT_2 \ln\left(\frac{V_1}{V_2}\right), R(T_1 - T_2)$

(2) $R(T_1 - T_2), 0, RT_1 \ln\left(\frac{V_1}{V_2}\right)$

(3) $0, RT_2 \ln\left(\frac{V_2}{V_1}\right), R(T_1 - T_2)$

(4) $0, RT_2 \ln\left(\frac{V_2}{V_1}\right), R(T_2 - T_1)$

21. In the given elliptical P–V diagram,



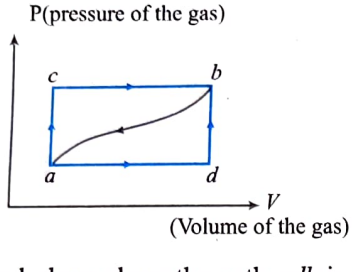
(1) the work done is positive

(2) the change in internal energy is non-zero

(3) the work done $= -(\pi/4)(P_2 - P_1)(V_2 - V_1)$

(4) the work done $= \pi(V_1 - V_1)^2 - \pi(P_1 - P_1)^2$

22. When an ideal gas is taken from state a to b, along a path acb, 84 kJ of heat flows into the gas and the gas does 32 kJ of work. The following conclusions are drawn. Mark the one which is not correct.



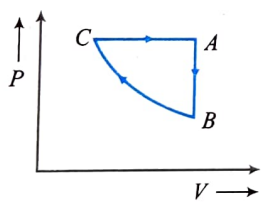
(1) If the work done along the path adb is 10.5 kJ, the heat that will flow into the gas is 62.5 kJ.

(2) When the gas is returned from b to a along the curved path, the work done on the gas is 21 kJ, and the system absorbs 73 kJ of heat.

(3) If $U_a = 0$, $U_d = 42$ kJ, and the work done along the path adb is 10.5 kJ then the heat absorbed in the process ad is 52.5 kJ.

(4) If $U_a = 0$, $U_d = 42$ kJ, heat absorbed in the process db is 10 kJ.

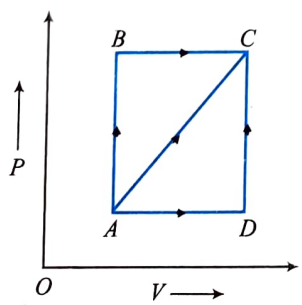
23. A sample of an ideal gas is taken through the cyclic process $ABCA$ shown in the given figure. It rejects 50 J of heat during the part AB, does not absorb or reject the heat during BC, and accepts 70 J of heat during CA. Forty joules of work is done on the gas during the part BC. The internal energy at point A (U_A) is 1500 J. The internal energies at B and C, respectively, will be



(1) 1450 J and 1410 J (2) 1550 J and 1590 J

(3) 1450 J and 1490 J (4) 1550 J and 1510 J

24. A thermodynamic process is shown in the given figure. The pressures and volumes corresponding to some points in the figure are



$P_A = 3 \times 10^4$ Pa $V_A = 2 \times 10^{-3}$ m^3

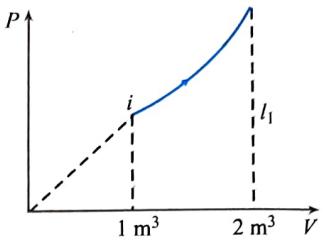
$P_B = 8 \times 10^4$ Pa $V_D = 5 \times 10^{-3}$ m^3

In the process AB, 600 J of heat is added to the system. The change in internal energy of the system in the process AB would be

(1) 560 J (2) 800 J

(3) 60 J (4) 640 J

25. A sample of ideal gas is expanded to twice its original volume of 1 m^3 in a quasi-static process for which $P = \alpha V^2$ with $\alpha = 3 \times 10^5$ Pa/m^6 as shown in the given figure. Work done by the expanding gas is



(1) 8×10^5 J (2) 7×10^5 J

(3) 6×10^5 J (4) 3×10^5 J

26. An ideal gas is taken through a cyclic thermodynamic process through four steps. The amounts of heat involved in these steps are $Q_1 = 5960$ J, $Q_2 = -5585$ J, $Q_3 = -2980$ J, $Q_4 = 3645$ J, respectively. The corresponding works involved are $W_1 = 2200$ J, $W_2 = -825$ J, $W_3 = -1100$ J and W_4, respectively. The value of W_4 is

(1) 1315 J (2) 275 J

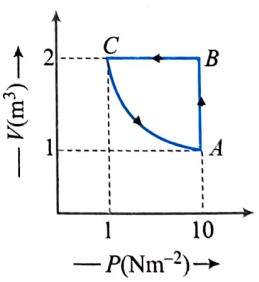
(3) 765 J (4) 675 J

27. One mole of air ($C_V = 5R/2$) is confined at atmospheric pressure in a cylinder with a piston at 0°C. The initial volume occupied by gas is V. After the equivalent of 13200 J of heat is transferred to it, the volume of gas V' is nearly (1 atm $= 10^5$ N/m^3)

(1) 37 L (2) 22 L

(3) 60 L (4) 30 L

28. An ideal gas is taken through $A \to B \to C \to A$, as shown in the given figure. If the net heat supplied to the gas in the cycle is 15 J, the work done by the gas in the process $C \to A$ is



(1) –5 J (2) –10 J

(3) –15 J (4) –20 J

29. Argon gas is adiabatically compressed to half its volume. If P, V and T represent the pressure, volume and temperature of the gaseous system, respectively, at any stage, then the correct equation representing the process is

(1) $TV^{2/5} =$ constant (2) $VP^{5/3} =$ constant

(3) $TP^{-2/5} =$ constant (4) $PT^{2/5} =$ constant

30. A fixed mass of helium gas is made to undergo a process in which its pressure varies linearly from 1 kPa to 2 kPa, in relation to its volume as the latter varies from 0.2 m^3 to 0.4 m^3. The heat absorbed by the gas will be
(1) 300 J (2) 900 J
(3) 1200 J (4) 1500 J

31. The density of a polyatomic gas in standard conditions is 0.795 kg m^{-3}. The specific heat of the gas at constant volume is
(1) 930 J-kg^{-1} K^{-1}
(2) 1400 J-kg^{-1} K^{-1}
(3) 1120 J-kg^{-1} K^{-1}
(4) 1600 J-kg^{-1} K^{-1}

32. Oxygen gas is made to undergo a process in which its molar heat capacity C depends on its absolute temperature T as $C = \alpha T$. Work done by it when heated from an initial temperature T_0 to a final temperature $2T_0$, will be
(1) $4\alpha T_0^2$ (2) $(\alpha T_0 - R)\frac{3T_0}{2}$
(3) $(3\alpha T_0 - 5R)\frac{T_0}{2}$ (4) none of these

33. Two moles of an ideal gas is contained in a cylinder fitted with a frictionless movable piston, exposed to the atmosphere, at an initial temperature T_0. The gas is slowly heated so that its volume becomes four times the initial value. The work done by the gas is
(1) zero (2) $2RT_0$
(3) $4RT_0$ (4) $6RT_0$

34. If for hydrogen $C_P - C_V = m$ and for nitrogen $C_P - C_V = n$, where C_P and C_V refer to specific heats per unit mass respectively at constant pressure and constant volume, the relation between m and n is (molecular weight of hydrogen = 2 and molecular weight of nitrogen = 14)
(1) $n = 14m$ (2) $n = 7m$
(3) $m = 7n$ (4) $m = 14n$

35. An ideal gas expands isothermally from volume V_1 to V_2 and is then compressed to original volume V_1 adiabatically. Initial pressure is P_1 and final pressure is P_3. The total work done is W. Then
(1) $P_3 > P_1, W > 0$
(2) $P_3 < P_1, W < 0$
(3) $P_3 > P_1, W < 0$
(4) $P_3 = P_1, W = 0$

36. An ideal gas ($\gamma = 1.5$) is expanded adiabatically. How many times has the gas to be expanded to reduce the root-mean-square velocity of molecules becomes half?
(1) 4 times (2) 16 times
(3) 8 times (4) 2 times

37. The equation of state for a gas is given by $PV = \eta RT + \alpha V$, where η is the number of moles and α a positive constant. The initial pressure and temperature of 1 mol of the gas contained in a cylinder is P_0 and T_0, respectively. The work done by the gas when its temperature doubles isobarically will be
(1) $\frac{P_0 T_0 R}{P_0 - \alpha}$ (2) $\frac{P_0 T_0 R}{P_0 + \alpha}$
(3) $P_0 T_0 R \ln 2$ (4) none of these

38. The molar specific heat of oxygen at constant pressure $C_P = 7.03$ cal/mol °C and $R = 8.31$ J/mol °C. The amount of heat taken by 5 mol of oxygen when heated at constant volume from 10° C to 20°C will be approximately
(1) 25 cal
(2) 50 cal
(3) 250 cal
(4) 500 cal

39. In an adiabatic process, $R = \frac{2}{3}C_V$. The pressure of the gas will be proportional to:
(1) $T^{5/3}$ (2) $T^{5/2}$
(3) $T^{5/4}$ (4) $T^{5/6}$

40. In an isobaric process, $\Delta Q = \frac{K\gamma}{\gamma - 1}$ where $\gamma = C_P/C_V$. What is K?
(1) Pressure (2) Volume
(3) ΔU (4) ΔW

41. Four moles of hydrogen, 2 moles of helium and 1 mole of water vapour form an ideal gas mixture. What is the molar specific heat at constant pressure of mixture?
(1) $\frac{16}{7}R$ (2) $\frac{7R}{16}$
(3) R (4) $\frac{23}{7}R$

42. Two cylinders A and B fitted with pistons contain equal amounts of an ideal diatomic gas at 300 K. The piston of A is free to move while that of B is held fixed. The same amount of heat is given to the gas in each cylinder. If the rise in temperature of the gas in A is 30 K, then the rise in temperature of the gas in B is
(1) 30 K (2) 18 K
(3) 50 K (4) 42 K

43. A gas under constant pressure of 4.5×10^5 Pa, when subjected to 800 kJ of heat, changes the volume from 0.5 m^3 to 2.0 m^3. The change in internal energy of the gas is
(1) 6.75×10^5 J
(2) 5.25×10^5 J
(3) 3.25×10^5 J
(4) 1.25×10^5 J

44. Five moles of hydrogen ($\gamma = 7/5$), initially at STP, is compressed adiabatically so that its temperature becomes 400°C. The increase in the internal energy of the gas in kilojoules is ($R = 8.30$ J/mol-K):
(1) 21.55 (2) 41.50
(3) 65.55 (4) 80.55

45. Three samples of the same gas A, B and C ($\gamma = 3/2$) initially have equal volume. Now the volume of each sample is doubled. The process is adiabatic for A, isobaric for B and isothermal for C. If the final pressures are equal for all the three samples, the ratio of their initial pressures is

(1) $2\sqrt{2}:2:1$ (2) $2\sqrt{2}:1:2$

(3) $\sqrt{2}:1:2$ (4) $2:1:\sqrt{2}$

46. One mole of an ideal gas at temperature T_1 expands according to the law (P/V) = constant. Find the work done when the final temperature becomes T_2.

(1) $R(T_2 - T_1)$

(2) $(R/2)(T_2 - T_1)$

(3) $(R/4)(T_2 - T_1)$

(4) $PV(T_2 - T_1)$

47. Two moles of an ideal gas at 300 K were cooled at constant volume so that the pressure is reduced to half the initial value. Then as a result of heating at constant pressure, the gas expands till it attains the original temperature. Find the total heat absorbed by the gas, if R is the gas constant.

(1) $150R$ J (2) $300R$ J

(3) $75R$ J (4) $100R$ J

48. Two different ideal diatomic gases A and B are initially in the same state. A and B are then expanded to same final volume through adiabatic and isothermal process respectively. If P_A, P_B and T_A, T_B represents the final pressure and temperatures of A and B respectively then:

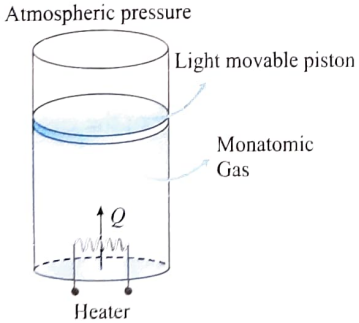
(1) $P_A < P_B$ and $T_A < T_B$

(2) $P_A > P_B$ and $T_A > T_B$

(3) $P_A > P_B$ and $T_A < T_B$

(4) $P_A < P_B$ and $T_A > T_B$

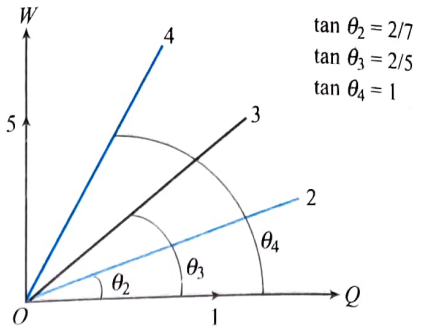
49. If 50 cal of heat is supplied to the system containing 2 mol of an ideal monatomic gas, the rise in temperature is ($R = 2$ cal/mol-K)



(1) 50 K (2) 5 K

(3) 10 K (4) 20 K

50. The given figure shows an isochore, an isotherm, an adiabatic and two isobars of two gases on a work done versus heat supplied curve. The initial states of both gases are the same and the scales for the two axes are same.



Which of the following statements is incorrect?

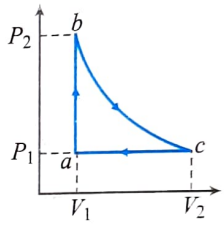
(1) Straight line 1 corresponds to an isochoric process.

(2) Straight line 2 corresponds to an isobaric process for diatomic gas.

(3) Straight line 4 corresponds to an isothermal process.

(4) Straight line 5 corresponds to an isothermal process.

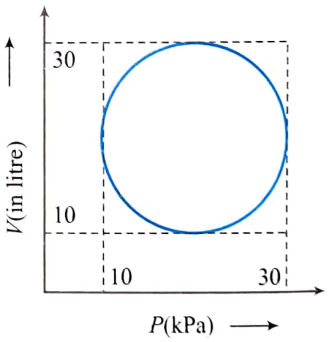
51. Carbon monoxide is carried around a closed cyclic process abc, in which bc is an isothermal process, as shown in the given figure. The gas absorbs 7000 J of heat as its temperature is increased from 300 K to 1000 K in going from a to b. The quantity of heat ejected by the gas during the process ca is



(1) 4200 J (2) 500 J

(3) 9000 J (4) 9800 J

52. Heat energy absorbed by a system in going through a cyclic process shown in the given figure is



(1) $10^7\pi$ J (2) $10^4\pi$ J

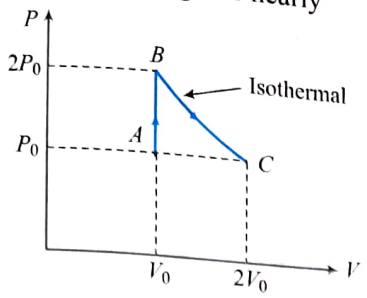
(3) $10^2\pi$ J (4) $10^{-3}\pi$ J

53. A diatomic ideal gas is heated at constant volume until the pressure is doubled and again heated at constant pressure until the volume is doubled. The average molar heat capacity for the whole process is

(1) $\frac{13R}{6}$ (2) $\frac{19R}{6}$

(3) $\frac{23R}{6}$ (4) $\frac{17R}{6}$

54. A diatomic ideal gas undergoes a thermodynamic change according to the P–V diagram shown in the given figure. The total heat given to the gas is nearly



(1) $2.5P_0V_0$ (2) $1.4P_0V_0$

(3) $3.9P_0V_0$ (4) $1.1P_0V_0$

55. The relation between internal energy U, pressure P and volume V of a gas in an adiabatic process is

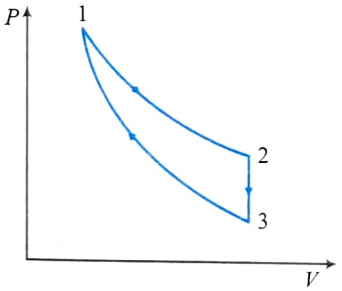
$U = a + bPV$

where a and b are constants. What is the effective value of adiabatic constant γ?

(1) $\frac{a}{b}$ (2) $\frac{b+1}{b}$

(3) $\frac{a+1}{a}$ (4) $\frac{b}{a}$

56. Three processes compose a thermodynamic cycle shown in the accompanying P–V diagram. Process 1→2 takes place at constant temperature, process 2→3 takes place at constant volume and process 3→1 is adiabatic. During the complete cycle the total amount of work done is 10 J. During process 3→1, 20 J of work is done on the system. Which of the following statements is incorrect?



(1) $(\Delta U)_{cycle} = 0$

(2) $(\Delta Q)_{cycle} = 10$ J

(3) $(\Delta Q)_{1\to 2} = 30$ J

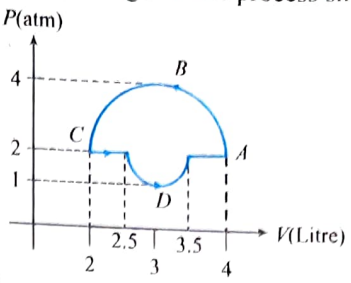
(4) During process 1→2, work is done on the system.

57. A certain balloon maintains an internal gas pressure of $P_0 = 100$ kPa until the volume reaches $V_0 = 20$ m^3. Beyond a volume of 20 m^3, the internal pressure varies as $P = P_0 + 2k(V - V_0)^2$ where P is in kPa, V is in m^3 and k is a constant ($k = 1$ kPa/m^3). Initially the balloon contains helium gas at 20°C, 100 kPa with a 15 m^3 volume. The balloon is then heated until the volume becomes 25 m^3 and the pressure is 150 kPa. Assume ideal gas behaviour for helium. The work done by the balloon for the entire process in kJ is

(1) 1256 (2) 1414

(3) 1083 (4) 1512

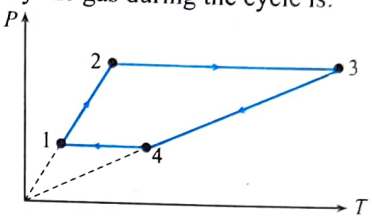
58. Find work done by the gas in the process shown in figure:



(1) $\frac{5}{2}\pi$ atm L (2) $\frac{5}{2}$ atm L

(3) $-\frac{3}{2}\pi$ atm L (4) $-\frac{5}{4}\pi$ atm L

59. Three moles of an ideal monatomic gas perform a cycle shown in figure. The gas temperatures in different states are $T_1 = 400$ K, $T_2 = 800$ K, $T_3 = 2400$ K, and $T_4 = 1200$ K. The work done by the gas during the cycle is:



(1) 5 kJ (2) 10 kJ

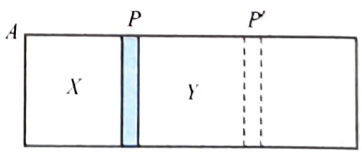
(3) 15 kJ (4) 20 kJ

60. A certain ideal gas undergoes a polytropic process PV^n = constant such that the molar specific heat during the process is negative. If the ratio of the specific heats of the gas be γ, then the range of values of n will be

(1) $0 < n < \gamma$ (2) $1 < n < \gamma$

(3) $n = \gamma$ (4) $n > \gamma$

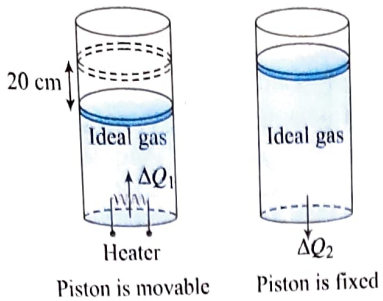
61. A cylindrical chamber A of uniform cross section is divided into two parts X and Y by a movable piston P which can slide without friction inside the chamber. Initially part X contains 1 mol of a monochromatic gas and Y contains 2 mol of a diatomic gas, and the volumes of X and Y are in the ratio 1:2 with both parts X and Y being at the same temperature T. Assuming the gases to be ideal, the work W that will be done in moving the piston slowly to the position where the ratio of the volumes of X and Y is 2:1 will be



(1) $-5.8T$ (2) $8.3T$

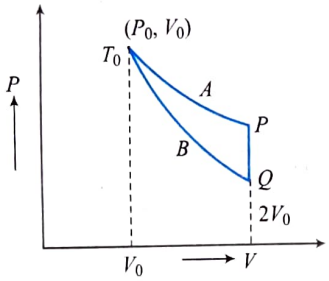
(3) $12.3T$ (4) zero

62. A cylinder of ideal gas is closed by an 8 kg movable piston (area 60 cm^2) as shown in the figure. Atmospheric pressure is 100 kPa. When the gas is heated from 30°C to 100°C, the piston rises by 20 cm. The piston is then fixed in its placed and the gas is cooled back to 30°C. Let ΔQ_1 be the heat added to the gas in the heating process and $|\Delta Q_2|$ the heat lost during cooling. Then the value of $[\Delta Q_1 - |\Delta Q_2|]$ will be



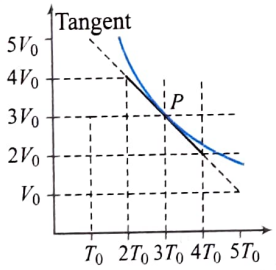
(1) zero
(2) 136 J
(3) –136 J
(4) –68 J

63. An ideal gas (1 mol, monatomic) is in the initial state P (see the given figure) on an isothermal A at temperature T_0. It is brought under a constant volume ($2V_0$) process to Q which lies on an adiabatic B intersecting the isothermal A at (P_0, V_0, T_0). The change in the internal energy of the gas during the process is (in terms of T_0) ($2^{2/3} = 1.587$)



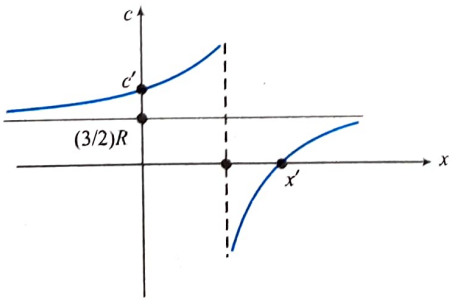
(1) $2.3T_0$
(2) $-4.6T_0$
(3) $-2.3T_0$
(4) $4.6T_0$

64. The given figure shows the adiabatic curve for n moles of an ideal gas; the bulk modulus for the gas corresponding to the point P will be



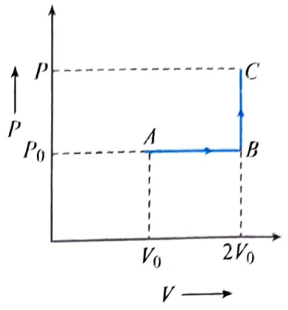
(1) $\frac{5nRT_0}{3V_0}$
(2) $nR\left(2+\frac{T_0}{V_0}\right)$
(3) $nR\left(1+\frac{T_0}{V_0}\right)$
(4) $\frac{2nRT_0}{V_0}$

65. One mole of an ideal gas is taken along the process in which PV^x = constant. The graph shown represents the variation of molar heat capacity of such a gas with respect to 'x'. The values of c' and x', respectively, are given by



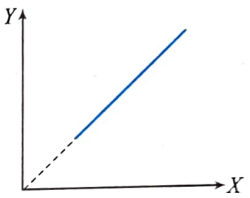
(1) $\frac{5}{2}R, \frac{5}{2}$
(2) $\frac{5}{2}R, \frac{5}{3}$
(3) $\frac{7}{2}R, \frac{7}{2}$
(4) $\frac{5}{2}R, \frac{7}{5}$

66. Find the pressure P, in the diagram as shown, of monatomic gas of one mole in the process ABC if $\Delta U/\Delta W = 6/7$.



(1) $\frac{5}{2}P_0$
(2) $\frac{3}{2}P_0$
(3) $2P_0$
(4) $\frac{7}{2}P_0$

67. The ratio of heat absorbed and work done by the gas in the process, as shown is 5/2. Find the parameters representing x and y axes:



(1) Temperature and pressure
(2) Temperature and volume
(3) Volume and pressure
(4) Volume and density

68. In previous problem, find the value of $\gamma = C_P/C_V$:
(1) 1.4
(2) 1.5
(3) 1.60
(4) 1.66

Multiple Correct Answers Type

1. At ordinary temperatures, the molecules of an ideal gas have only translational and rotational kinetic energies. At high temperatures they may also have vibrational energy. As a result of this, at higher temperature

(1) $C_v = \frac{3R}{2}$ for a monatomic gas
(2) $C_v > \frac{3R}{2}$ for a monatomic gas
(3) $C_v < \frac{5R}{2}$ for a diatomic gas
(4) $C_v > \frac{5R}{2}$ for a diatomic gas

2. The molar heat capacity for an ideal gas cannot
(1) be negative
(2) be equal to either C_v or C_P

(3) lie in the range $C_v \le C \le C_p$

(4) it may have any value between $-\infty$ and $+\infty$

3. Select the correct alternatives for an ideal gas:

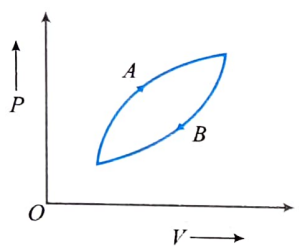
(1) The change in internal energy in a constant pressure process from temperature T_1 to T_2 is equal to $nC_v(T_2 - T_1)$, where C_v is the molar specific heat at constant volume and n the number of moles of the gas.

(2) The change in internal energy of the gas and the work done by the gas are equal in magnitude in an adiabatic process.

(3) The internal energy does not change in an isothermal process.

(4) No heat is added or removed in an adiabatic process.

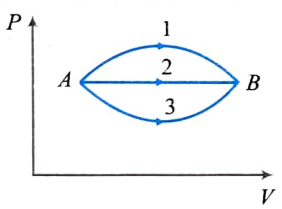
4. In the cyclic process shown in the given figure, ΔU_1 and ΔU_2 represent the change in internal energy in process A and B, respectively. If ΔQ be the net heat given to the system in the process and ΔW be the work done by the system in the process, then



(1) $\Delta U_1 + \Delta U_2 = 0$ (2) $\Delta U_1 - \Delta U_2 = 0$

(3) $\Delta Q - \Delta W = 0$ (4) $\Delta Q + \Delta W = 0$

5. A gas undergoes change in its state from position A to position B via three different paths as shown in the given figure. Select the correct alternatives:



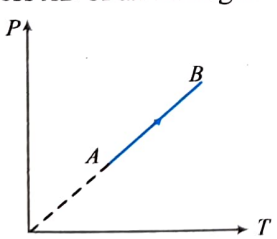
(1) Change in internal energy in all the three paths is equal.

(2) In all the three paths heat is absorbed by the gas.

(3) Heat absorbed/released by the gas is maximum in path (1).

(4) Temperature of the gas first increases and then decreases continuously in path (1).

6. During the process AB of an ideal gas



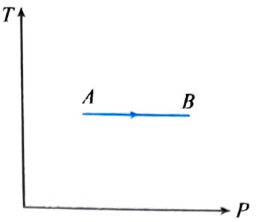
(1) work done on the gas is zero

(2) density of the gas is constant

(3) slope of line AB from the T-axis is inversely proportional to the number of moles of the gas

(4) slope of line AB from the T-axis is directly proportional to the number of moles of the gas

7. Temperature versus pressure graph of an ideal gas is shown in the given figure. During the process AB,



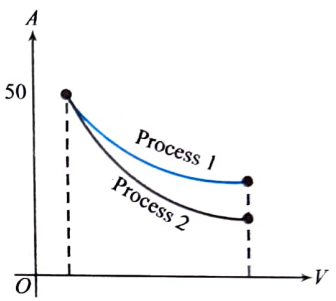
(1) internal energy of the gas remains constant

(2) volume of the gas is increased

(3) work done by the atmosphere on the gas is positive

(4) pressure is inversely proportional to volume

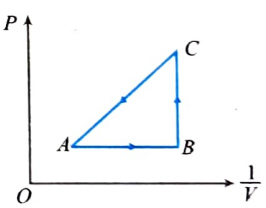
8. The indicator diagram for two processes 1 and 2 carrying on an ideal gas is shown in the given figure. If m_1 and m_2 be the slopes (dP/dV) for Process 1 and Process 2, respectively, then



(1) $m_1 = m_2$ (2) $m_1 > m_2$

(3) $m_1 < m_2$ (4) $m_2 C_V = m_1 C_P$

9. An ideal gas undergoes a thermodynamic cycle as shown in the given figure. Which of the following statements are correct?



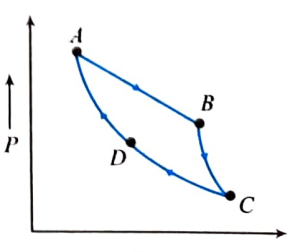
(1) Straight line AB cannot pass through O.

(2) During process AB, temperature decreases while during process BC it increases.

(3) During process BC, work is done by the gas against external pressure and temperature of the gas increases.

(4) During process CA, work is done by the gas against external pressure and heat supplied to the gas is exactly equal to this work.

10. The given figure shows the P–V diagram for a Carnot cycle. In this diagram:



(1) curve AB represents isothermal process and BC adiabatic process

(2) curve AB represents adiabatic process and BC isothermal process

(3) curve CD represents isothermal process and DA adiabatic process

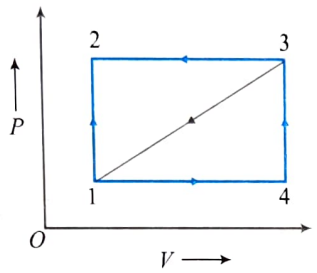
(4) curve CD represents adiabatic process and DA isothermal process

11. The given figure shows an indicator diagram. During path 1–2–3, 100 cal is given to the system and 40 cal worth work is done. During path 1–4–3, the work done is 10 cal. Then

(1) heat given to the system during path 1–4–3 is 70 cal

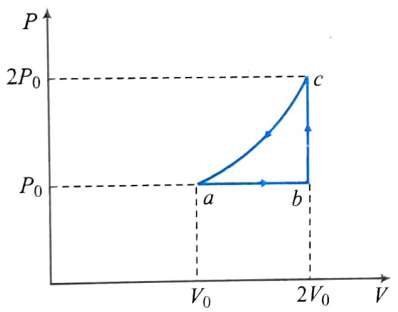
(2) if the system is brought from 3 to 1 along straight line path 3–1, work done is worth 25 cal

(3) along straight line path 3–1, the heat ejected by the system is 85 J



(4) the internal energy of the system in state 3 is 140 cal above that in state 1

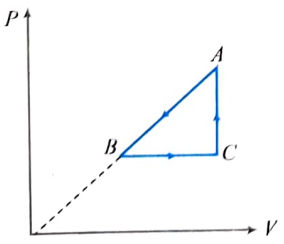
12. One mole of an ideal monatomic gas has initial temperature T_0, is made to go through the cycle $abca$ shown in the given figure. If U denotes the internal energy, then choose the correct alternative.



(1) $U_c > U_b > U_a$ (2) $U_c - U_b = 3RT_0$

(3) $U_c - U_a = \frac{9RT_0}{2}$ (4) $U_b - U_a = \frac{3RT_0}{2}$

13. P–V diagram of a cyclic process $ABCA$ is as shown in the given figure. Choose the correct alternative.



(1) $\Delta Q_{A\to B}$ is negative (2) $\Delta U_{B\to C}$ is positive

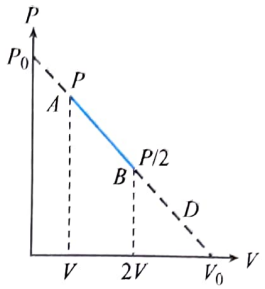
(3) $\Delta U_{C\to A}$ is negative (4) ΔW_{CAB} is negative

14. An insulated 0.2 m^3 tank contains helium at 1200 kPa and 47°C. A valve is now opened, allowing some helium to escape. The valve is closed when one-half of the initial mass has escaped. The temperature of the gas is $\left(\sqrt[3]{4} = 1.6\right)$.

(1) 100 K (2) 200 K

(3) 73°C (4) –73°C

15. An ideal gas is taken from the state A (pressure P, volume V) to the state B (pressure $P/2$, volume $2V_0$ along a straight line path in the P–V diagram. Select the correct statement(s) from the following:



(1) The work done by the gas in process AB is greater than the work that would be done if the system were taken from A to B along the isotherm

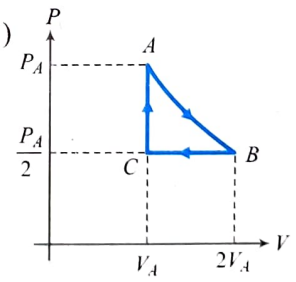
(2) In the T–V diagram, the path AB becomes a part of parabola.

(3) In the P–T diagram, the path AB becomes a part of hyperbola

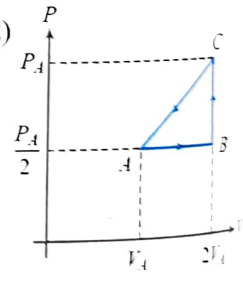
(4) In going from A to B, the temperature T of the gas first increases to a maximum value and then decreases

16. Three moles of an ideal gas $C_P = 7/2R$ at pressure P_A and temperature T_A is isothermally expanded to twice its initial volume. It is then compressed at constant pressure to its original volume. Finally the gas is compressed at constant volume to the original pressure P_A. The correct P–V and P–T diagrams indicating the process are

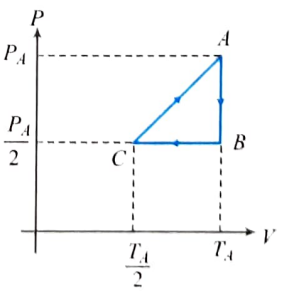
(1)



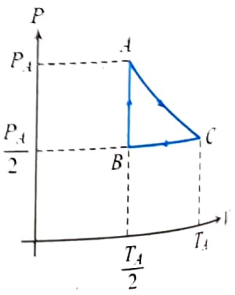
(2)



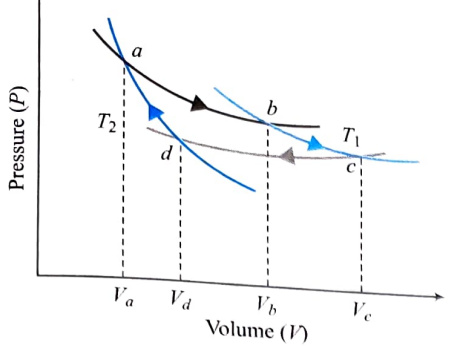
(3)



(4)



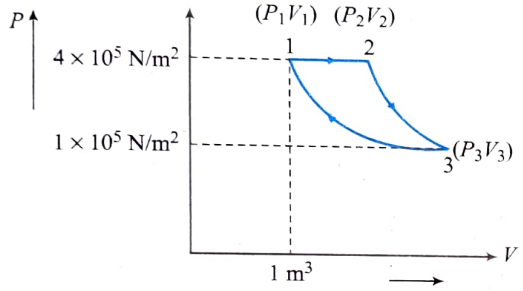
17. An ideal gas undergoes the cyclic process shown in a graph below:



(1) $T_1 = T_2$ (2) $T_1 > T_2$

(3) $V_a V_c = V_b V_d$ (4) $V_a V_b = V_c V_d$

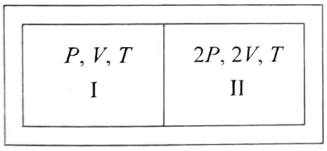
18. A system undergoes three quasi-static processes sequentially as indicated in the given figure. 1–2 is an isobaric process, 2–3 is a polytropic process with $\gamma = 4/3$ and 3–1 is a process in which PV = constant. $P_2 = P_1 = 4 \times 10^5$ N/m^2, $P_3 = 1 \times 10^5$ N/m^2 and $V_1 = 1$ m^3. The heat transfer for the cycle is ΔQ, the change in internal energy is ΔU and the work done is ΔW. Then



(1) $\Delta W = 0$ (2) $\Delta Q = 1.08 \times 10^5$ J

(3) $\Delta U = 0$ (4) $\Delta Q > \Delta W$

19. A partition divides a container having insulated walls into two compartments I and II. The same gas fills the two compartments whose initial parameters are given. The partition is a conducting wall which can move freely without friction. Which of the following statements is/are correct, with reference to the final equilibrium position?



(1) The pressure in the two compartments are equal

(2) Volume of compartment I is $\frac{3V}{5}$

(3) Volume of compartment II is $\frac{12V}{5}$

(4) Final pressure in compartment I is $\frac{5P}{3}$

20. Three identical adiabatic containers A, B and C contain helium, neon and oxygen respectively at equal pressure. The gases are pushed to half their original volumes:

(1) The final temperature in the three containers will be the same

(2) The final pressures in the three containers will be the same

(3) The pressure of helium and neon will be the same but that of oxygen will be different

(4) The temperature of helium and neon will be the same but that of oxygen will be different

21. A thermally insulated chamber of volume 2 V_0 is divided by a frictionless piston of area 5 into two equal parts A and B. Part A has an ideal gas at pressure P_0 and temperature T_0 and in part B is vacuum. A massless spring of force constant K is connected with the piston of cross sectional area S and the wall of the container as shown in figure. Initially the spring is unstretched. The gas in chamber A is allowed to expand. Let in equilibrium the spring is compressed by x_0. Then:

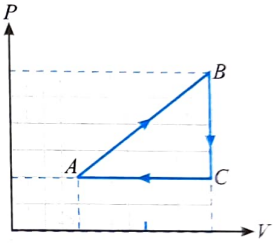
(1) Final pressure of the gas is $\frac{kx_0}{S}$

(2) Work done by the gas is $-kx$

(3) Change in internal energy of the gas is $\frac{1}{2}kx_0^2$

(4) Temperature of the gas is decreased

22. Consider a thermodynamic cycle in a PV diagram shown in the figure performed on one mole of a monatomic gas. The temperature at A is T_0 and volume at A and B are related as $V_B = V_C = 2V_A$. Choose the correct option(s) from the following.



(1) The maximum temperature during the cycle is $4\ T_0$

(2) Net work done by the gas during the cycle is $0.5\ RT_0$

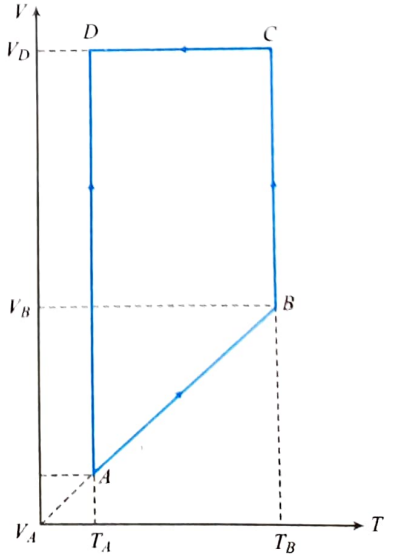
(3) The heat capacity of the process AB is $2R$

(4) The efficiency of the cycle is 8.33%

Linked Comprehension Type

For Problems 1–6

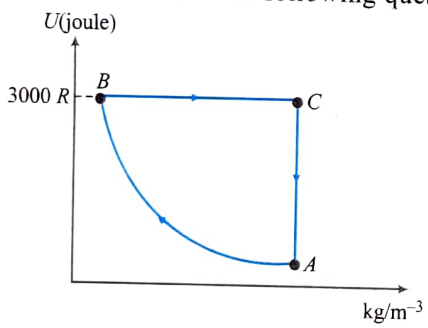
A monatomic ideal gas of 2 mol is taken through a cyclic process starting from A as shown in the given figure. The volume ratios are $V_B/V_A = 2$ and $V_D/V_A = 4$. If the temperature T_A at A is 27°C, and gas constant is R. Calculate



1. The temperature of the gas at point B
   (1) 500 K (2) 700 K
   (3) 600 K (4) 300 K
2. Heat absorbed or released by the gas in process $A \to B$.
   (1) $1500R$, added (2) $1200R \ln 2$, added
   (3) $900R$, rejected (4) $1200R \ln 2$, rejected
3. Heat absorbed or released by the gas in process $B \to C$
   (1) $1500R$, added (2) $1200R \ln 2$, added
   (3) $900R$, rejected (4) $1200R \ln 2$, rejected
4. Heat absorbed or released by the gas in process $C \to D$.
   (1) $1500R$, added (2) $1200R \ln 2$, added
   (3) $900R$, rejected (4) $1200R \ln 2$, rejected
5. Heat absorbed or released by the gas in process $D \to A$.
   (1) $1500R$, added (2) $1200R \ln 2$, added
   (3) $900R$, rejected (4) $1200R \ln 2$, rejected
6. The total work done by the gas during the complete cycle.
   (1) $500R$ J (2) $600R$ J
   (3) $700R$ J (4) $300R$ J

For Problems 7–9

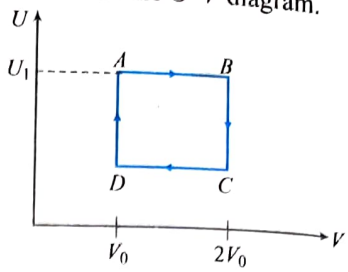
The given figure shows the variation of potential energy (U) of 2 mol of Argon gas with its density in a cyclic process $ABCA$. The gas was initially in the state A whose pressure and temperature are $P_A = 2$ atm, $T_A = 300$ K, respectively. It is also stated that the path AB is a rectangular hyperbola and the internal energy of the gas at state C is $3000\,R$. Based on the above information answer the following questions:



7. (1) The process AB is isobaric, BC is adiabatic and CA is isochoric.
   (2) The process AB is adiabatic, BC is isothermal and CA is isochoric.
   (3) The process AB is isochoric, BC is isothermal and CA is isobaric
   (4) The process AB is isobaric, BC is isothermal and CA is isochoric.
8. The heat supplied to the gas in the process AB is
   (1) $700R$ (2) $3500R$
   (3) $4400R$ (4) $1600R$
9. Heat supplied in the process CA is
   (1) $-1400R$ (2) $1400R$
   (3) $2100R$ (4) $-2100R$

For Problems 10–12

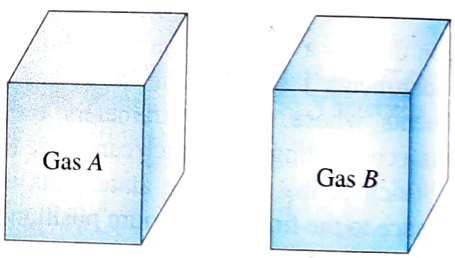
One mole of an ideal gas has an internal energy given by $U = U_0 + 2PV$, where P is the pressure and V the volume of the gas. U_0 is a constant. This gas undergoes the quasi-static cyclic process $ABCD$ as shown in the U–V diagram.



10. The molar heat capacity of the gas at constant pressure is
    (1) $2R$ (2) $3R$
    (3) $\frac{5}{2}R$ (4) $4R$
11. The work done by the ideal gas in the process AB is
    (1) zero (2) $\frac{U_1 - U_0}{2}$
    (3) $\frac{U_0 - U_1}{2}$ (4) $\frac{U_1 - U_0}{2}\log_e 2$
12. The gas must be
    (1) monatomic
    (2) diatomic
    (3) a mixture of mono and diatomic gases
    (4) a mixture of di- and tri-atomic gases

For Problems 13–15

Two closed identical conducting containers are found in the laboratory of an old scientist. For the verification of the gas some experiments are performed on the two boxes and the results are noted.



Experiment 1: When the two containers are weighed $W_A = 225$ g, $W_B = 160$ g and mass of evacuated container $W_C = 100$ g.

Experiment 2: When the two containers are given same amount of heat, same temperature rise is recorded. The pressure changes found are $\Delta P_A = 2.5$ atm, $\Delta P_B = 1.5$ atm.

Required data for unknown gas:

Mono (molar mass)	He 4 g	Ne 20 g	Ar 40 g	Kr 84 g	Xe 131 g	Rd 222 g
Di (molar mass)	H_2 2 g	F_2 19 g	N_2 28 g	O_2 32 g	Cl_2 71 g	

13. Identify the type of gas filled in containers A and B, respectively.
    (1) Monatomic, Monatomic (2) Diatomic,, Diatomic
    (3) Monatomic, Diatomic (4) Diatomic, Monatomic
14. Identify the gas filled in the container A and B.
    (1) N_2, Ne (2) He, H_2
    (3) O_2, Ar (4) Ar, O_2

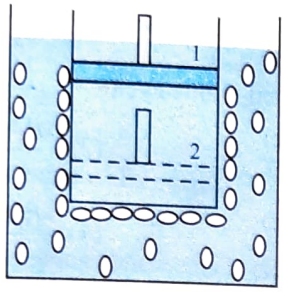
15. The gases have initial temperature 300 K and they are mixed in an adiabatic container having the same volume as the previous container. Now, if the temperature of the mixture is T and pressure is P, then

(1) $P > P_A$, $T > 300$ K (2) $P > P_B$, $T > 300$ K

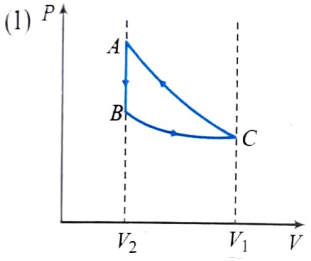
(3) $P > P_A$, $T = 300$ K (4) $P > P_A$, $T < 300$ K

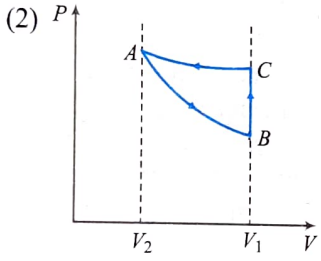
For Problems 16–18

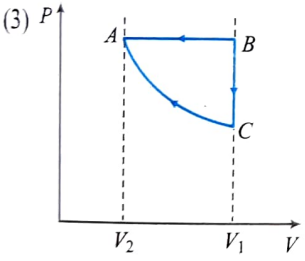
A cylinder containing an ideal gas (see figure) and closed by a movable piston is submerged in an ice–water mixture. The piston is quickly pushed down from position (1) to position (2) (process AB). The piston is held at position (2) until the gas is again at 0°C (process BC). Then the piston is slowly raised back to position (1) (process CA).

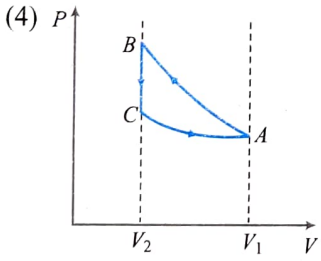


16. Which of the following P–V diagrams will correctly represent the processes AB, BC and CA and the cycle $ABCA$?

(1) 

(2) 

(3) 

(4) 

17. If 100 g of ice is melted during the cycle $ABCA$, how much work is done on the gas?

(1) 8 kcal (2) 5 kcal

(3) 2.1 kJ (4) 4.2 kJ

18. If the change in the volume is $(V_1 - V_2) = V\,m^3$, the work done (in N/m^2) during the cycle is, with P being the atmospheric pressure acting on the piston.

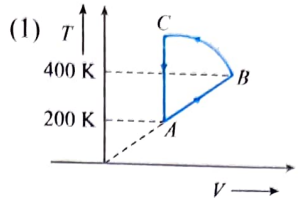
(1) $\frac{PV}{2}$ (2) $\frac{2PV}{3}$

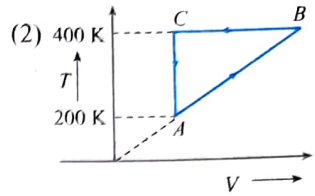
(3) PV (4) None of these

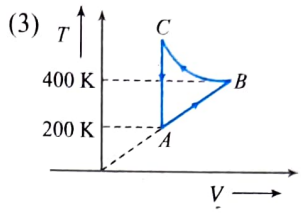
For Problems 19–21

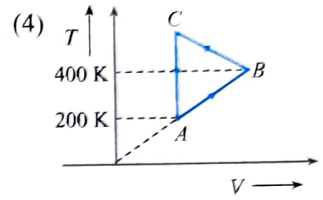
Four moles of an ideal gas is initially in state A having pressure $2 \times 10^5\ N/m^2$ and temperature 200 K. Keeping the pressure constant the gas is taken to state B at temperature of 400 K. The gas is then taken to a state C in such a way that its temperature increases and volume decreases. Also from B to C, the magnitude of dT/dV increases. The volume of gas at state C is equal to its volume at state A. Now gas is taken to initial state A keeping volume constant. A total of 1000 J of heat is withdrawn from the sample in the cyclic process. Take $R = 8.3$ J/K/mol.

19. Which graph between temperature T and volume V for the cyclic process is correct.

(1) 

(2) 

(3) 

(4) 

20. The work done in path B to C is

(1) 1000 J (2) –1000 J

(3) –7640 J (4) 5640 J

21. The volume of gas at state C is

(1) 0.0332 m^3 (2) 0.22 m^3

(3) 0.332 m^3 (4) 3.32 m^3.

For Problems 22–24

An ideal diatomic gas is expanded so that the amount of heat transferred to the gas is equal to the decrease in its internal energy.

22. The molar specific heat of the gas in this process is given by C whose value is

(1) $-\frac{5R}{2}$ (2) $-\frac{3R}{2}$

(3) $2R$ (4) $\frac{5R}{2}$

23. The process can be represented by the equation TV^n = constant, where the value of n is

(1) $n = \frac{7}{5}$ (2) $n = \frac{1}{5}$

(3) $n = \frac{3}{2}$ (4) $n = \frac{3}{5}$

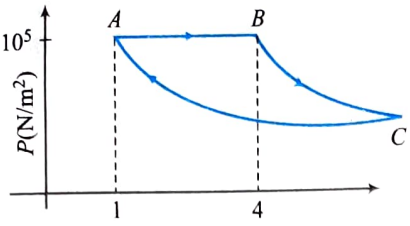
24. If in the above process, the initial temperature of the gas be T_0 and the final volume be 32 times the initial volume, the work done (in joules) by the gas during the process will be

(1) RT_0 (2) $\frac{5RT_0}{2}$

(3) $2RT_0$ (4) $\frac{RT_0}{2}$

For Problems 25–27

A fixed mass of gas is taken through a process $A \to B \to C \to A$. Here $A \to B$ is isobaric, $B \to C$ is adiabatic and $C \to A$ is isothermal.



25. Find pressure at C.

(1) $\frac{10^5}{64}\,N/m^2$ (2) $\frac{10^5}{32}\,N/m^2$

(3) $\frac{10^5}{12}\,N/m^2$ (4) $\frac{10^5}{6}\,N/m^2$

26. Find volume at C.

(1) 32 m^3 (2) 100 m^3

(3) 64 m^3 (4) 25 m^3

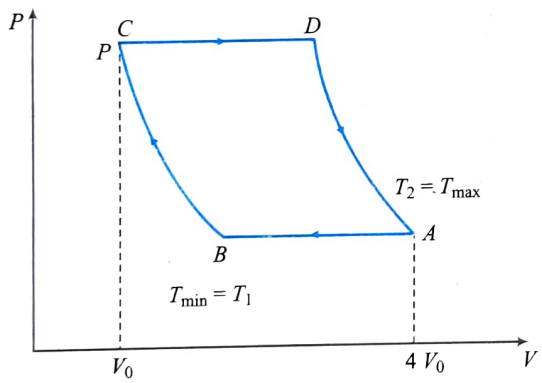
27. Find work done in the process (take $\gamma = 1.5$).

(1) 4.9×10^5 J (2) 3.2×10^5 J

(3) 1.2×10^5 J (4) 7.2×10^5 J

For Problems 28–31

A monatomic gas undergoes a cycle consisting of two isothermals and two isobarics. The minimum and maximum temperatures of the gas during the cycle are $T_1 = 400$ K and $T_2 = 800$ K, respectively, and the ratio of maximum to minimum volume is 4.



28. The volume at B is

(1) $1.5V_0$ (2) $2V_0$

(3) $3V_0$ (4) $2.5V_0$

29. The volume at D is

(1) $1.5V_0$ (2) $2V_0$

(3) $3V_0$ (4) $2.5V_0$

30. The heat is extracted from the system in process

(1) $A \to B, B \to C$ (2) $C \to D, D \to A$

(3) $A \to B, C \to D$ (4) $B \to C, C \to D$

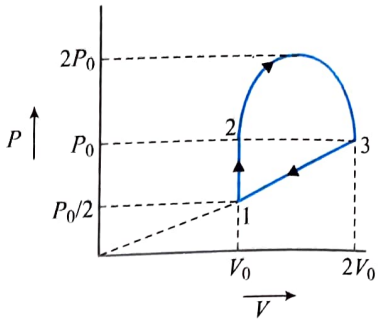
31. The efficiency of the cycle is

(1) $\frac{3\ln 2}{5+4\ln 2} \times 100\%$ (2) $\frac{2\ln 2}{5+4\ln 2} \times 100\%$

(3) $\frac{3\ln 3}{5+4\ln 2} \times 100\%$ (4) $\frac{2\ln 2}{3+4\ln 2} \times 100\%$

For Problems 32–33

A monatomic ideal gas undergoes the shown cyclic process in which path of the process 2→3 is a semicircle. If 2 mol of gas is taken, find



32. Heat given to the system in semicircular process

(1) $\frac{P_0V_0}{2}\left(7-\frac{\pi}{4}\right)$ (2) $\frac{P_0V_0}{3}\left(7+\frac{\pi}{4}\right)$

(3) $\frac{P_0V_0}{2}\left(7+\frac{\pi}{4}\right)$ (4) $\frac{P_0V_0}{3}\left(7-\frac{\pi}{4}\right)$

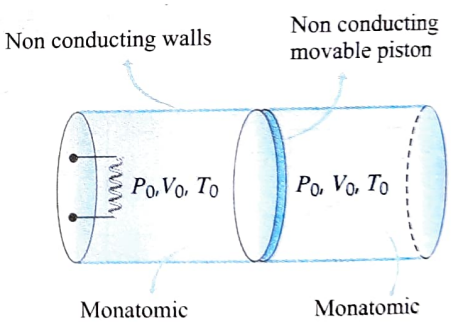
33. Total heat rejected in one cycle

(1) $\frac{P_0V_0(32-\pi)}{4}$ (2) $\frac{P_0V_0(32+\pi)}{4}$

(3) $\frac{P_0V_0(32+\pi)}{8}$ (4) $\frac{P_0V_0(32-\pi)}{8}$

For Problems 34–35

The insulated box shown in the given figure has an insulated partition which can slide without friction along the length of the box. Initially each of the two chambers of the box has 1 mol of a monatomic ideal gas ($\gamma = 5/3$) at a pressure P_0, volume V_0 and temperature T_0. The chamber on the left is slowly heated by an electric heater so that its gas, pushing the partition, expands until the final pressure in both the chambers becomes $243P_0/32$. Determine



34. Final temperature of the gas in each chamber.

(1) $4/9T_0$ (2) $4/7T_0$

(3) $1/5T_0$ (4) $9/4T_0$

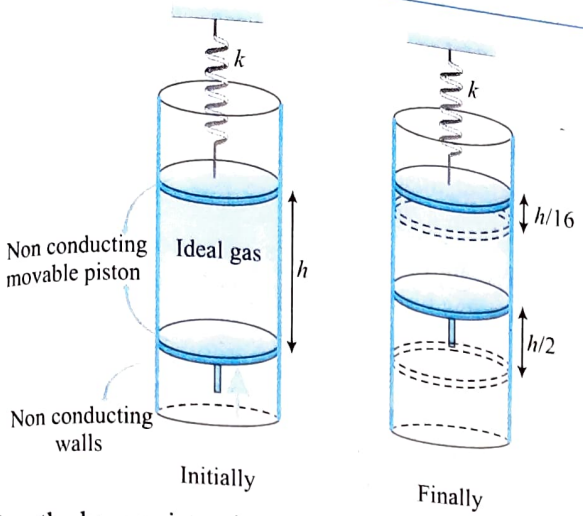
35. Work done on the gas in the right chamber

(1) $\frac{15}{8}RT_0$ (2) $-\frac{15}{8}RT_0$

(3) $\frac{7}{4}RT_0$ (4) $-\frac{7}{4}RT_0$

For Problems 36–38

An ideal gas at NTP is enclosed in an adiabatic vertical cylinder having an area of cross section $A = 27$ cm² between two light movable pistons as shown in the given figure. Spring with force constant $k = 3700$ N/m is in a relaxed state initially. Now the lower position is moved upwards a distance $h/2$, h being the initial length of gas column. It is observed that the upper piston moves up by a distance $h/16$. Final temperature of gas = $4/3 \times 273$ K. Take γ for the gas to be 3/2.



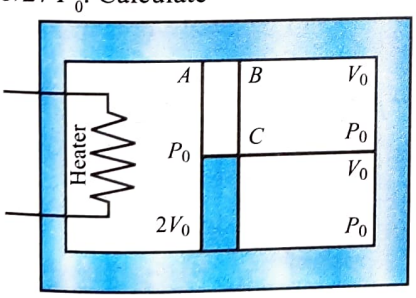
36. When the lower piston is moved upwards a distance $h/2$, the compression
(1) leads to cooling
(2) takes place isothermally
(3) takes place adiabatically
(4) leads to heating

37. Which of the following statements is correct?
(1) Heat given in compression is used to do work against elastic force only.
(2) Heat given in compression is used to do work against elastic force and atmospheric force.
(3) No work is done during compression.
(4) There is no change in internal energy.

38. The value of h is
(1) 1 m (2) 1.4 m
(3) 1.6 m (4) 2 m

For Problems 39–41

A container of volume $4V_0$ made of a perfectly non-conducting material is divided into two equal parts by a fixed rigid wall whose lower half is non-conducting and upper half is purely conducting. The right side of the wall is divided into equal parts (initially) by means of a massless non-conducting piston free to move as shown. Section A contains 2 mol of a gas while the section B and C contain 1 mol each of the same gas ($\gamma=1.5$) at pressure P_0. The heater in left part is switched on till the final pressure in section C becomes $125/27\ P_0$. Calculate



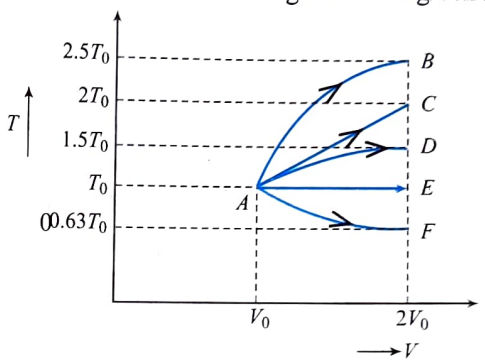
39. Final temperature in part A.
(1) $\frac{205P_0V_0}{27R}$ (2) $\frac{P_0V_0}{R}$
(3) $\frac{105P_0V_0}{13R}$ (4) $\frac{12P_0V_0}{13R}$

40. Final temperature in part C.
(1) $\frac{P_0V_0}{R}$ (2) $\frac{5P_0V_0}{3R}$
(3) $\frac{P_0V_0}{3R}$ (4) $\frac{5P_0V_0}{R}$

41. The heat supplied by the heater.
(1) $\frac{368}{9}P_0V_0$ (2) $\frac{113}{5}P_0V_0$
(3) $\frac{316}{9}P_0V_0$ (4) $\frac{405}{8}P_0V_0$

Matrix Match Type

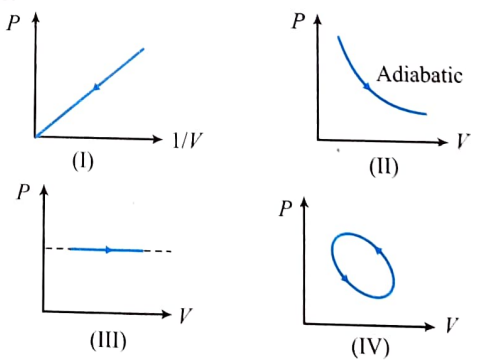
1. An ideal monatomic gas at initial temperature T_0 expands from initial volume V_0 to volume $2V_0$ by each of the five processes indicated in the T–V diagram in the given figure.



Match Column I with Column II.

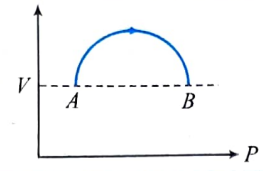
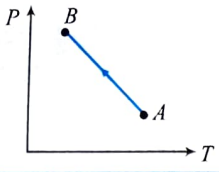
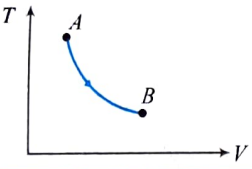
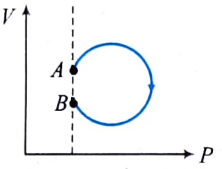
Column I	Column II
i. Isothermal	a. AB
ii. Isobaric	b. AC
iii. Adiabatic	c. AD
	d. AE
	e. AF

2. The figures given below show different processes (relating pressure P and volume V) for a given amount for an ideal gas. ΔW is work done by the gas and ΔQ is heat absorbed by the gas.

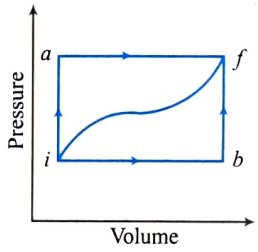


Column I	Column II
i. In Fig. (I)	a. $\Delta Q > 0$
ii. In Fig. (II)	b. $\Delta W < 0$
iii. In Fig. (III)	c. $\Delta Q < 0$
iv. In Fig. (IV)	d. $\Delta W > 0$

3. A sample of gas goes from state A to state B in four different manners, as shown by the graphs. Let W be the work done by the gas and ΔU be change in internal energy along the path AB. Correctly match the graphs with the statements provided.

Column I	Column II
i. 	a. Both W and ΔU are positive
ii. 	b. Both W and ΔU are negative
iii. 	c. W is positive whereas ΔU is negative
iv. 	d. W is negative whereas ΔU is positive

4. When a sample of a gas is taken from state f along path iaf, heat supplied to the gas is 50 cal and work done by the gas is 20 cal. If it is taken by path ibf, then heat supplied is 36 cal.



Column I	Column II
i. Work done by the gas along path ibf is	a. 6 cal
ii. If work done upon the gas is 13 cal for the return path fi, then heat rejected by the gas along path ft is	b. 18 cal
iii. If internal energy of the gas at state i is 10 cal, then internal energy at state f is	c. 40 cal
iv. If internal energy at state b is 22 cal and at i is 10 cal then heat supplied to the gas along path ib is	d. 43 cal

5. Match the change in the internal energy of the system given in column II for each situation in given Column I.

Column I	Column II
i. A system absorbs 500 cal of heat and at the same time does 400 J of work.	a. −5000 J
ii. A system absorbs 300 cal and at the same time 420 J of work is done on it.	b. 1700 J
iii. Twelve hundred calories is removed from a gas held at constant volume.	c. 1680 J

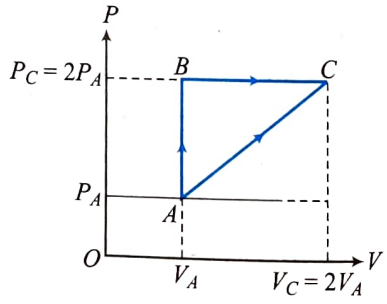
6. The heat capacity of a certain amount of a particular gas at constant pressure is greater than that at constant volume by 29.1 J/K. Match the items given in Column I with the items given in Column II.

Column I	Column II
i. If the gas is monatomic, heat capacity at constant volume	a. 131 J/K
ii. If the gas is monatomic, heat capacity at constant pressure	b. 43.7 J/K
iii. If the gas is rigid diatomic, heat capacity at constant pressure	c. 72.7 J/K
iv. If the gas is vibrating diatomic, heat capacity at constant pressure	d. 102 J/K

7. An ideal monatomic gas undergoes different types of processes which are described in Column I. Match the corresponding effects in Column II. The letters have their usual meanings.

Column I	Column II
i. $P = 2V^2$	a. If volume increases then temperature will also increase
ii. PV^2 = constant	b. If volume increases then temperature decreases
iii. $C = C_V + 2R$	c. For expansion, heat will have to be supplied to the gas
iv. $C = C_V - 2R$	d. If temperature increases then work done by gas is positive

8. n kmol of a monochromatic ideal gas is taken quasi-statically from state A to state C along the straight line shown in the given figure. Alternatively, the same gas is taken quasi-statically from A to C along the path ABC.



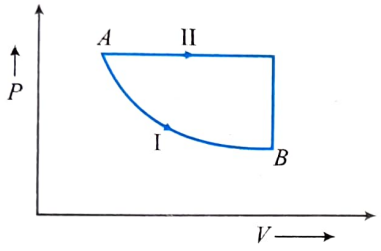
Match Column I with Column II on the basis of given information.

Column I	Column II
i. The heat ΔH added to the gas along the straight line path AC	a. $2P_AV_A$

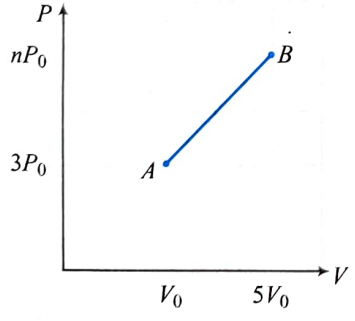
ii. Change in internal energy of the gas along the straight line path AC	b. $13/2P_AV_A$
iii. The work done by the gas along the path ABC	c. $6P_AV_A$
iv. The heat $\Delta H'$ added to the gas along the path ABC	d. $9/2P_AV_A$

Numerical Value Type

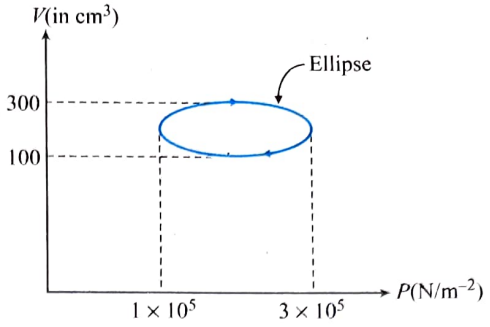
1. A certain mass of gas is taken from an initial thermodynamics state A to another state B by process I and II. In process I for the gas does 5 J of work and absorbs 4 J of heat energy. In process II, the gas absorbs 5 J of heat. The work done by the gas in process II is



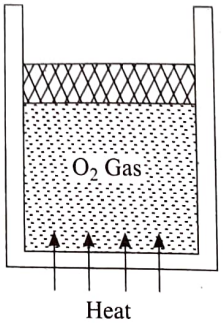
2. A vessel contains helium, which expands at constant pressure when 15 kJ of heat is supplied to it. The variation of the internal energy of the gas (in kJ) will be
3. A vessel of volume 0.2 m^3 contains hydrogen gas at temperature 300 K and pressure 1 bar. The molar heat capacity of hydrogen at constant volume is 5 cal/mol K. The heat (in kcal) required to raise the temperature to 400 K is
4. If P–V diagram of a diatomic gas is plotted, it is a straight line passing through origin. The molar heat capacity of the gas in the process is nR where n is an integer. The value of n is
5. The value of $\gamma = C_p/C_v$ is 4/3 for an adiabatic process of an ideal gas for which internal energy $U = K + nPV$. The value of n (K is a constant) is
6. One mole of a monatomic ideal gas follows a process $A \rightarrow B$, as shown. The specific heat of the process is $\frac{13R}{6}$. The value of n on P-axis (see following figure) is



7. Heat $Q = \frac{3}{2}RT$ is supplied to 4 moles of an ideal diatomic gas at temperature T, which remains constant. Number of moles of the gas dissolved into atoms is
8. A gaseous mixture enclosed in a vessel consists of 1 g mole of a gas A with $\gamma = 5/3$ and another B with $\gamma = 7/5$ at a temperature T. The gases A and B do not react with each other and are assumed to be ideal. Find the number of gram moles of gas if γ of the gaseous mixture is 19/13.
9. Calculate the heat absorbed by a system in going through the cyclic process shown in the given figure.



10. A gas undergoes a change of state during which 100 J of heat is supplied to it and it does 20 J of work. The system is brought back to its original state through a process during which 20 J of heat is rejected by the gas. Find the work done by the gas in the second process in joules.
11. Temperature of 1 mole of an ideal gas is increased from 300 K to 310 K under isochoric process. Heat supplied to the gas in this process is 25 R. What amount of work(in joules) has to be done by the gas if temperature of the gas decreases from 310 K to 300 K adiabatically?
(Given: Gas constant $R = 8.3$ J/mol-K)
12. A sample of an ideal gas is taken through a process $ABCD$. It absorbs 50 J of energy during the process AB which is an isometric process, no heat during BC and it rejects 70 J of heat during isobaric process CD. It is also given that 40 J of work is done on the gas during the process BC. Internal energy of gas in state A is 1500 J. Find the internal energy of gas in state C in joules.
13. A gas consisting of monatomic molecules (degrees of freedom = 3) was expanded in a polytropic process so that the rate of collisions of the molecules against the vessel's wall did not change. Find the molar heat capacity of the gas in the process in J/mol-K. (Given: Gas constant $R = 8.3$ J/mol-K)
14. A gas consisting of rigid diatomic molecules was expanded *ma* polytropic process so that the rate of collisions of the molecules against the vessel's wall did not change. Find the molar heat capacity of the gas in this process in J/mol-K.
(Given: Gas constant $R = 8.3$ J/mol-K)
15. A vertically oriented cylindrical vessel containing oxygen gas and closed by a piston of mass 50 kg. Piston can slide smoothly in the cylinder. Its cross-sectional area is 100 cm^2 and atmospheric pressure is 10^5 pa. Some heat is supplied to the cylinder so that the piston is slowly displaced up by 20 cm. Find the amount of heat supplied to the gas (in Joules).

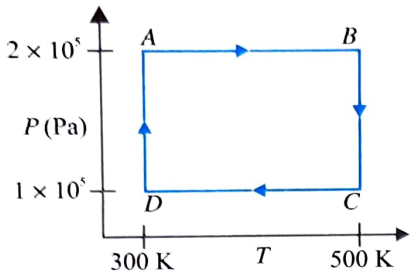


JEE Main

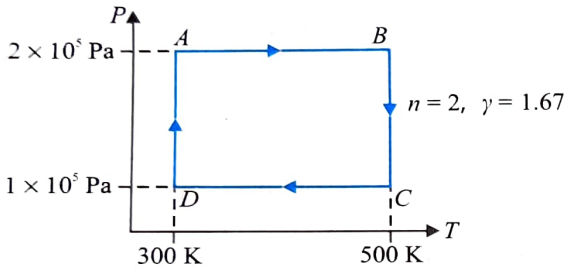
Single Correct Answer Type

For Problems 1–3

Two moles of helium gas is taken over the cycle $ABCDA$, as shown in the P–T diagram. **(AIEEE 2009)**



1. Assuming the gas to be ideal, the work done on the gas in taking it from A to B is



(1) $200R$ (2) $300R$
(3) $400R$ (4) $500R$

2. The work done on the gas in taking it from D to A is
(1) $-414R$ (2) $+414R$
(3) $-690R$ (4) $+690R$

3. The net work done on the gas in the cycle $ABCDA$ is
(1) zero (2) $276R$
(3) $1076R$ (4) $1904R$

4. A diatomic ideal gas is used in a Carnot engine as the working substance. If during the adiabatic expansion part of the cycle, the volume of the gas increases from V to $32V$, the efficiency of the engine is
(1) 0.5 (2) 0.75
(3) 0.99 (4) 0.25

(AIEEE 2010)

5. A thermally insulated vessel contains an ideal gas of molecular mass M and ratio of specific heats γ. It is moving with speed v and is suddenly brought to rest. Assuming no heat is lost to the surroundings, its temperature increases by

(1) $\frac{(\gamma-1)}{2(\gamma+1)R}Mv^2$ K (2) $\frac{(\gamma-1)}{2\gamma}Mv^2$ K

(3) $\frac{\gamma Mv^2}{2R}$ K (4) $\frac{(\gamma-1)}{2R}Mv^2$ K

(AIEEE 2011)

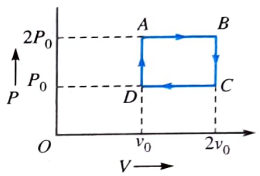
6. A Carnot engine operating between temperatures T_1 and T_2 has efficiency 1/6. When T_2 is lowered by 62 K, its efficiency increases to 1/3. Then T_1 and T_2 are, respectively,
(1) 372 K and 310 K (2) 372 K and 330 K
(3) 330 K and 268 K (4) 310 K and 248 K

(AIEEE 2011)

7. A Carnot engine, whose efficiency is 40%, takes in heat from a source maintained at a temperature of 500 K. It is desired to have an engine of efficiency 60%. Then, the intake temperature for the same exhaust (sink) temperature must be
(1) efficiency of carnot engine cannot be made larger than 50%
(2) 1200 K
(3) 750 K
(4) 600 K

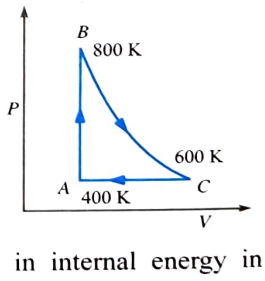
(AIEEE 2012)

8. The above P–V diagram represents the thermodynamic cycle of an engine, operating with an ideal monoatomic gas. The amount of heat extracted from the source in a single cycle is



(1) $\left(\frac{13}{2}\right)P_0V_0$ (2) $\left(\frac{11}{2}\right)P_0V_0$
(3) $4P_0V_0$ (4) P_0V_0 **(JEE Main 2013)**

9. One mole of diatomic ideal gas undergoes a cyclic process ABC as shown in the figure. The process BC is adiabatic. The temperatures at A, B and C are 400 K, 800 K and 600 K respectively. Choose the correct statement:



(1) The change in internal energy in the process AB is $-350R$.
(2) The change in internal energy in the process BC is $-500R$.
(3) The change in internal energy in whole cyclic process is $250R$.
(4) The change in internal energy in the process CA is $700R$.

(JEE Main 2014)

10. A solid body of constant heat capacity 1 J/°C is being heated by keeping it in contact with reservoirs in two ways:
(i) Sequentially keeping in contact with 2 reservoirs such that each reservoir supplies same amount of heat.

(ii) Sequentially keeping in contact with 8 reservoirs such that each reservoir supplies same amount of heat.

In both the cases, body is brought from initial temperature 100°C to final temperature 200°C. Entropy changes of the body in the two cases, respectively, is

(1) ln 2, 4 ln 2 (2) ln 2, ln 2

(3) ln 2, 2 ln 2 (4) 2 ln 2, 8 ln 2

(JEE Main 2015)

11. Consider an ideal gas confined in an isolated closed chamber. As the gas undergoes an adiabatic expansion, the average time of collision between molecules increases as V^q, where V is the volume of the gas. The value of q is $\left(\gamma = \frac{C_P}{C_V}\right)$

(1) $\frac{3\gamma+5}{6}$ (2) $\frac{3\gamma-5}{6}$

(3) $\frac{\gamma+1}{2}$ (4) $\frac{\gamma-1}{2}$

(JEE Main 2015)

12. An ideal gas undergoes a quasi-static, reversible process in which its molar heat capacity C remains constant. If during this process the relation of pressure P and volume V is given by PV^n = constant, then n is given by (Here C_P and C_V are molar specific heat at constant pressure and constant volume, respectively):

(1) $n = \frac{C_P}{C_V}$ (2) $n = \frac{C - C_P}{C - C_V}$

(3) $n = \frac{C_P - C}{C - C_V}$ (4) $n = \frac{C - C_V}{C - C_P}$

(JEE Main 2016)

13. C_p and C_v are specific heats at constant pressure and constant volume respectively. It is observed that

$C_p - C_v = a$ for hydrogen gas

$C_p - C_v = b$ for nitrogen gas

The correct relation between a and b is

(1) $a = 14b$ (2) $a = 28b$

(3) $a = \frac{1}{14}b$ (4) $a = b$ **(JEE Main 2017)**

JEE ADVANCED

Single Correct Answer Type

1. 5.6 litres of helium gas at STP is adiabatically compressed to 0.7 litre. Taking the initial temperature to be T_1, the work done in the process is

(1) $\frac{9}{8}RT_1$ (2) $\frac{3}{2}RT_1$

(3) $\frac{15}{8}RT_1$ (4) $\frac{9}{2}RT_1$ **(IIT-JEE 2011)**

2. Two moles of ideal gas are in a rubber balloon at 30°C. The balloon is fully expandable and can be assumed to require no energy in its expansion. The temperature of the gas in the balloon is slowly changed to 35°C. The amount of heat required in raising the temperature is nearly (take R = 8.31 J/mol.K)

(1) 62 J (2) 104 J

(3) 124 J (4) 208 J **(IIT-JEE 2012)**

3. A gas is enclosed in a cylinder with a movable frictionless piston. Its initial thermodynamic state at pressure $P_i = 10^5$ Pa and volume $V_i = 10^{-3}$ m^3 changes to a final state at $P_f = (1/32) \times 10^5$ pa and $V_f = 8 \times 10^{-3}$ m^3 in an adiabatic quasi-static process, such that P^3V^5 = constant. Consider another thermodynamic process that brings the system from the same initial state to the same final state in two steps : an isobaric expansion at P_i followed by an isochoric (isovolumetric) process at volume V_f. The amount of heat supplied to the system in the two-step process is approximately.

(1) 112 J (2) 294 J

(3) 588 J (4) 813 J

(JEE Advanced 2016)

Multiple Correct Answers Type

1. C_V and C_P denote the molar specific heat capacities of a gas at constant volume and constant pressure, respectively. Then

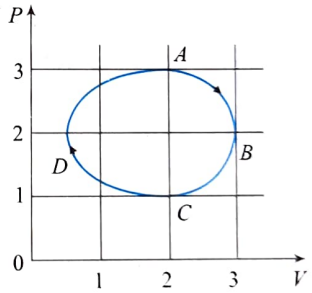
(1) $C_P - C_V$ is larger for a diatomic ideal gas than for a monatomic ideal gas.

(2) $C_P + C_V$ is larger for a diatomic ideal gas than for a monatomic ideal gas.

(3) C_P/C_V is larger for a diatomic ideal gas than for a monatomic ideal gas.

(4) $C_P \cdot C_V$ is larger for a diatomic ideal gas than for a monatomic ideal gas. **(IIT-JEE 2009)**

2. Figure shows the P–V plot of an ideal gas taken through a cycle $ABCDA$. The part ABC is a semi-circle and CDA is half of an ellipse. Then



(1) the process during the path $A \to B$ is isothermal

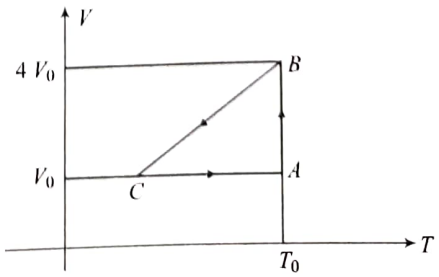
(2) heat flows out of the gas during the path $B \to C \to D$

(3) work done during the path $A \to B \to C$ is zero

(4) positive work is done by the gas in the cycle $ABCDA$

(IIT-JEE 2009)

3. One mole of an ideal gas in initial state A undergoes a cyclic process $ABCA$, as shown in the figure. Its pressure at A is P_0. Choose the correct option(s) from the following



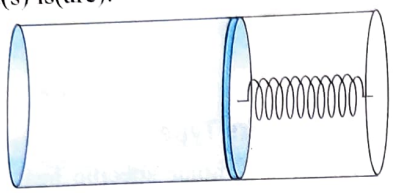
(1) Internal energies at A and B are the same

(2) Work done by the gas in process AB is $P_0V_0 \ln 4$

(3) Pressure at C is $P_0/4$

(4) Temperature at C is $T_0/4$ **(IIT-JEE 2010)**

4. An ideal monatomic gas is confined in a horizontal cylinder by a spring loaded piston (as shown in the figure). Initially the gas is at temperature T_1, pressure P_1 and volume V_1 and the spring is in its relaxed state. The gas is then heated very slowly to temperature T_2, pressure P_2 and volume V_2. During this process the piston moves out by a distance x. Ignoring the friction between the piston and the cylinder, the correct statement(s) is(are):



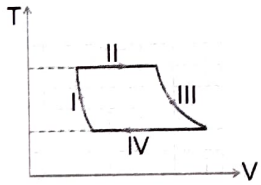
(1) If $V_2 = 2V_1$ and $T_2 = 3T$, then the energy stored in the spring is $\frac{1}{4}P_1V_1$.

(2) If $V_2 = 2V_1$ and $T_2 = 3T_1$, then the change in internal energy is $3P_1V_1$.

(3) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the work done by the gas is $\frac{7}{3}P_1V_1$.

(4) If $V_2 = 3V_1$ and $T_2 = 4T_1$, then the heat supplied to the gas is $\frac{17}{6}P_1V_1$. **(JEE Advanced 2015)**

5. One mole of a monatomic ideal gas undergoes a cyclic process as shown in the figure (where V is the volume and T is the temperature). Which of the statements below is (are) true?



(1) Process I is an isochoric process

(2) In process II, gas absorbs heat

(3) In process IV, gas releases heat

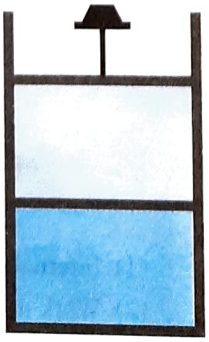
(4) Processes I and II are not isobaric

(JEE Advanced 2018)

Linked Comprehension Type

For Problems 1–2

In the figure, a container is shown to have a movable (without friction) piston on top. The container and the piston are all made of perfectly insulating material allowing no heat transfer between outside and inside the container. The container is divided into two compartments by a rigid partition made of a thermally conducting material that allows slow transfer of heat. The lower compartment of the container is filled with 2 moles of an ideal monatomic gas at 700 K and the upper compartment is filled with 2 moles of an ideal diatomic gas at 400 K. The heat capacities per mole of an ideal monatomic gas are $C_V = \frac{3}{2}R, C_P = \frac{5}{2}R$ and those for an ideal diatomic gas are $C_V = \frac{5}{2}R, C_P = \frac{7}{2}R$. **(JEE Advanced 2014)**



1. Consider the partition to be rigidly fixed so that it does not move. When equilibrium is achieved, the final temperature of the gases will be

(1) 550 K (2) 525 K

(3) 513 K (4) 490 K

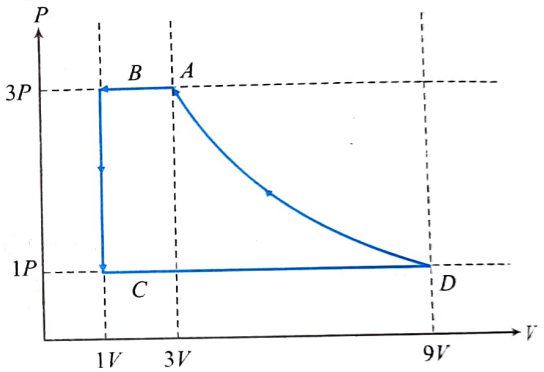
2. Now consider the partition to be free to move without friction so that the pressure of gases in both compartments is the same. Then total work done by the gases till the time they achieve equilibrium will be

(1) 250 R (2) 200 R

(3) 100 R (4) −100 R

Matrix Match Type

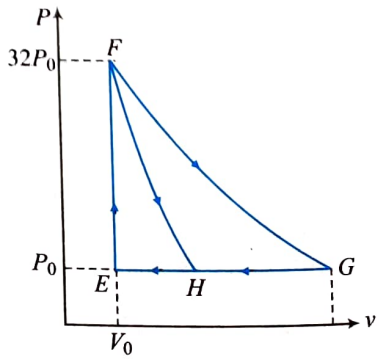
1. One mole of a monatomic ideal gas is taken through a cycle $ABCDA$ as shown in the P–V diagram. Column II gives the characteristics involved in the cycle. Match them with each of the processes given in Column I.



Column I	Column II
i. Process $A \to B$	a. Internal energy decreases
ii. Process $B \to C$	b. Internal energy increases
iii. Process $C \to D$	c. Heat is lost
iv. Process $D \to A$	d. Heat is gained
	e. Work is done on the gas

(IIT-JEE 201

2. One mole of mono-atomic ideal gas is taken along two cyclic processes $E \to F \to G \to E$ and $E \to F \to H \to E$ as shown in the PV diagram. The processes involved are purely isochoric, isobaric, isothermal or adiabatic.



Match the paths in Column I with the magnitudes of the work done in Column II and select the correct answer using the codes given below the lists.

Column I	Column II
i. $G \to E$	a. $160\ P_0 V_0 \ln 2$
ii. $G \to H$	b. $36\ P_0 V_0$
iii. $F \to H$	c. $24\ P_0 V_0$
iv. $F \to G$	d. $31\ P_0 V_0$

Codes:

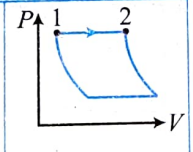
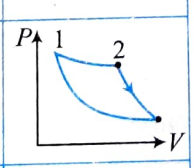
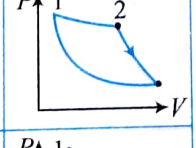
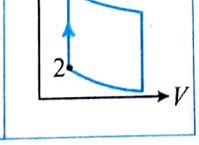
	i.	ii.	iii.	iv.
(1)	d	c	b	a
(2)	d	c	a	b
(3)	c	a	b	d
(4)	a	c	b	d

(JEE Advanced 2013)

For Problems 3–5

Answer Q.3, Q.4 and Q.5 by appropriately matching the information given in the three columns of the following table.

An ideal gas is undergoing a cyclic thermodynamic process in different ways as shown in the corresponding P–V diagrams in column III of the table. Consider only the path from state 1 to state 2. W denotes the corresponding work done on the system. The equations and plots in the table have standard notations as used in thermodynamic processes. Here γ is the ratio of heat capacities at constant pressure and constant volume. The number of moles in the gas in n.

Column I		Column II		Column III	
(I)	$W_{1\to2} = \frac{1}{\gamma-1}(P_2V_2 - P_2V_1)$	(i)	Isothermal	(P)	
(II)	$W_{1\to2} = -PV_2 + PV_1$	(ii)	Isochoric	(Q)	
(III)	$W_{1\to2} = 0$	(iii)	Isobaric	(R)	
(IV)	$W_{1\to2} = nRT \ln\left(\frac{V_2}{V_1}\right)$	(iv)	Adiabatic	(S)	

(JEE Advanced 2017)

3. Which of the following options is the only correct representation of a process in which $\Delta U = \Delta Q - P\Delta V$?

(1) (II) (iv) (R) (2) (III) (iii) (P)

(3) (II) (iii) (P) (4) (II) (iii) (S)

4. Which one of the following options correctly represents a thermodynamic process that is used as a correction in the determination of the speed of sound in an ideal gas?

(1) (I) (iv) (Q) (2) (III) (iv) (R)

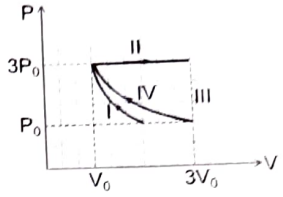
(3) (I) (ii) (Q) (4) (IV) (ii) (R)

5. Which one of the following options is the correct combination?

(1) (IV) (ii) (S) (2) (III) (ii) (S)

(3) (II) (iv) (R) (4) (II) (iv) (P)

6. One mole of a monatomic ideal gas undergoes four thermodynamic processes as shown schematically in the PV-diagram below. Among these four processes, one is isobaric, one is isochoric, one is isothermal and one is adiabatic. Match the processes mentioned in List-I with the corresponding statements in List-II.



List-I		List-II	
P	In process I	1.	Work done by the gas is zero
Q	In process II	2.	Temperature of the gas remains unchanged
R	In process III	3.	No heat is exchanged between the gas and its surroundings
S	In process IV	4.	Work done by the gas is P_0V_0

(1) P → 4; Q → 3; R → 1; S → 2

(2) P → 1; Q → 3; R → 2; S → 4

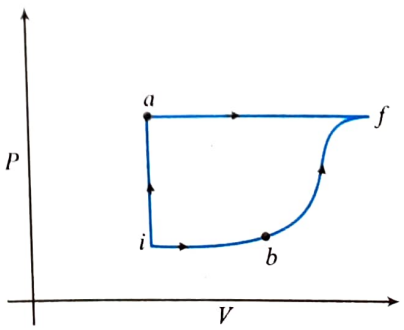
(3) P → 3; Q → 4; R → 1; S → 2

(4) P → 3; Q → 4; R → 2; S → 1

(JEE Advanced 2018)

Numerical Value Types

1. A thermodynamic system is taken from an initial state i with internal energy $U_i = 100$ J to the final state f along two different paths iaf and ibf, as schematically shown in the figure. The work done by the system along the paths af, ib and bf are $W_{af} = 200$ J, $W_{ib} = 50$ J and $W_{bf} = 100$ J respectively. The heat supplied to the system along the path iaf, ib and bf are Q_{iaf}, Q_{ib} and Q_{bf} respectively. If the internal energy of the system in the state b is $U_b = 200$ J and $Q_{iaf} = 500$ J, the ratio Q_{bf} / Q_{ib} is



(JEE Advanced 2014

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	**2.** (3)	**3.** (1)	**4.** (4)	**5.** (3)
6. (3)	**7.** (3)	**8.** (4)	**9.** (3)	**10.** (4)
11. (2)	**12.** (1)	**13.** (3)	**14.** (3)	**15.** (1)
16. (3)	**17.** (1)	**18.** (1)	**19.** (1)	**20.** (4)
21. (3)	**22.** (2)	**23.** (3)	**24.** (1)	**25.** (2)
26. (3)	**27.** (3)	**28.** (1)	**29.** (3)	**30.** (3)
31. (2)	**32.** (3)	**33.** (4)	**34.** (3)	**35.** (3)
36. (2)	**37.** (1)	**38.** (3)	**39.** (2)	**40.** (4)
41. (4)	**42.** (4)	**43.** (4)	**44.** (2)	**45.** (2)
46. (2)	**47.** (2)	**48.** (1)	**49.** (2)	**50.** (4)
51. (4)	**52.** (3)	**53.** (2)	**54.** (3)	**55.** (2)
56. (4)	**57.** (3)	**58.** (4)	**59.** (4)	**60.** (2)
61. (1)	**62.** (2)	**63.** (2)	**64.** (4)	**65.** (2)
66. (1)	**67.** (2)	**68.** (4)		

Multiple Correct Answers Type

1. (1),(4)	**2.** (1),(2),(3)	**3.** (1),(2),(3),(4)
4. (1),(3)	**5.** (1),(2),(3)	**6.** (1),(2),(4)
7. (1),(3),(4)	**8.** (3),(4)	**9.** (2),(4)
10. (1),(3)	**11.** (1),(2),(3)	**12.** (1),(2),(3),(4)
13. (1),(2),(4)	**14.** (2),(4)	**15.** (1),(2),(4)
16. (1),(3)	**17.** (2),(3)	**18.** (2),(3)
19. (1),(2),(3),(4)	**20.** (3),(4)	**21.** (1),(2),(3),(4)
22. (1),(2),(3),(4)		

Linked Comprehension Type

1. (3)	**2.** (1)	**3.** (2)	**4.** (3)	**5.** (4)
6. (2)	**7.** (4)	**8.** (2)	**9.** (4)	**10.** (2)
11. (4)	**12.** (3)	**13.** (3)	**14.** (4)	**15.** (3)
16. (4)	**17.** (1)	**18.** (4)	**19.** (3)	**20.** (3)
21. (1)	**22.** (1)	**23.** (2)	**24.** (2)	**25.** (1)
26. (3)	**27.** (1)	**28.** (2)	**29.** (2)	**30.** (1)
31. (2)	**32.** (3)	**33.** (4)	**34.** (4)	**35.** (2)
36. (3)	**37.** (2)	**38.** (3)	**39.** (1)	**40.** (2)
41. (1)				

Matrix Match Type

1. i. → d.; ii. → b.; iii. → c.
2. i. → a., d.; ii. → d.; iii. → a., d.; iv. → b., c.
3. i. → d.; ii. → b.; iii. → c.; iv. → b.
4. i. → a.; ii. → d.; iii. → c.; iv. → b.
5. i. → b.; ii. → c.; iii. → a.
6. i. → b.; ii. → c.; iii. → d.; iv. → a.
7. i. → a., c., d.; ii. → b.; iii. → a., c., d.; iv. → b., c.
8. i. → c.; ii. → d.; iii. → a.; iv. → b.

Numerical Value Type

1. (6)	**2.** (9)	**3.** (4)	**4.** (3)	**5.** (3)
6. (6)	**7.** (3)	**8.** (2)	**9.** (3.14)	**10.** (60)
11. (207.5)	**12.** (1590)	**13.** (16.6)	**14.** (24.9)	**15.** (1050)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3)	**2.** (1)	**3.** (2)	**4.** (2)	**5.** (4)
6. (1)	**7.** (3)	**8.** (1)	**9.** (2)	**10.** (2)
11. (3)	**12.** (2)	**13.** (1)		

JEE Advanced

Single Correct Answer Type

1. (1) **2.** (4) **3.** (3)

Multiple Correct Answers Type

1. (2),(4) **2.** (2),(4) **3.** (1),(2)
4. (1),(2),(3) **5.** (2),(3),(4)

Linked Comprehension Type

1. (4) **2.** (4)

Matrix Match Type

1. i. → a., c., e.; ii. → a., c.; iii. → b., d.; iv. → c., e.
2. i. → d.; ii. → c.; iii. → b.; iv. → a.
3. (3) **4.** (1) **5.** (2) **6.** (3)

Numerical Value Type

1. (2)

5 Linear and Angular Simple Harmonic Motion

PERIODIC MOTION

Periodic motion is motion of an object that regularly returns to a given position after a fixed time interval. With a little thought, we can identify several types of periodic motions in everyday life. Your car returns to the driveway each afternoon. You return to the dinner table each night to eat. A bumped chandelier swings back and forth, returning to the same position at a regular rate. The earth returns to the same position in its orbit around the sun each year, resulting in the variation among the four seasons.

In addition to these everyday examples, numerous other systems exhibit periodic motion. The molecules in a solid oscillate about their equilibrium positions; electromagnetic waves such as light waves, radar and radio waves are characterized by oscillating electric and magnetic field vectors; in an alternating current electrical circuit, voltage, current and electric charge vary periodically with time.

If T is the period of motion after which it repeats itself, then the frequency f of the periodic motion is the number of cycles performed in 1 s and it is given as $f = 1/T$.

The units of f are s^{-1} or per second. A special name is given to the unit of frequency, hertz (Hz) after the discoverer of radio waves.

1 Hz = 1 cycle per second

OSCILLATIONS

An oscillation is a special type of periodic motion in which a particle moves to and fro about a fixed point called mean position of particle. Oscillations are commonly seen in general life in our surroundings. As discussed, in all types of oscillations, there is always a mean position about which the particle can oscillate. This is the position where the particle is in equilibrium, that is, net force on the particle at this position is zero. If particle is displaced from the mean position, due to this displacement some forces appear on it which act on the particle in a direction directed toward its equilibrium position, these forces are called restoring forces as these forces tend the particle to move towards its equilibrium position. Due to restoring forces, particle starts moving toward the mean position and when it reaches the mean position, it gains some KE due to work done by the restoring forces and it will overshoot from this point with some velocity in other direction; again restoring forces appear on the particle toward mean position and now the particle is retarded and will stop after travelling some distance. It will return toward the mean position and start accelerating and in such a way motion is continued which we call oscillation. The maximum displacement of particle from its mean position, where it will come to rest or from where it started with zero initial speed, is called as amplitude of oscillations.

PERIODIC FUNCTION

If a particle moves along x-axis, its position depends upon time t. We express this fact mathematically by writing

$x = f(t)$ or $x(t)$

There are certain motions that are repeated at equal intervals of time. By this we mean that particle is found at the same position moving in the same direction with the same velocity and acceleration, after each period of time. Let T be the interval of time in which motion is repeated. Then

$x(t) = x(t + T)$

where T is the minimum change in time. The function that repeats itself is known as a periodic function. During the period, its values may remain finite. Such functions are bound functions. Periodic motion of a particle is also bound because it must not go to infinity and return back in one finite period.

Periodic motions may be oscillatory or non-oscillatory. Uniform circular motion, the motion of a planet around the sun, etc. are periodic but not oscillatory. Also, an oscillatory motion may not repeat its position with the old velocity due to friction and will be non-periodic.

ILLUSTRATION 5.1

What is the time-period of $x = A \sin(\omega t + \alpha)$?

Sol. It is known from trigonometry that $\sin\theta = \sin(\theta + 2\pi)$

Hence $x = A\sin(\omega t + \alpha + 2\pi) = A\sin\left\{\omega\left(t + \frac{2\pi}{\omega}\right) + \alpha\right\}$

$$x = A\sin(\omega t' + \alpha), \quad \text{where } t' = t + \frac{2\pi}{\omega}.$$

This shows that the function at t coincides with the function at t'. The interval $(t' - t)$ is the period of x.

This period is $\frac{2\pi}{\omega}$.

ILLUSTRATION 5.2

Find the period of the function, $y = \sin\omega t + \sin 2\omega t + \sin 3\omega t$

Sol. The given function can be written as $y = y_1 + y_2 + y_3$

Here $y_1 = \sin\omega t, T_1 = 2\pi/\omega$

$$y_2 = \sin 2\omega t, \; T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$$

and $y_3 = \sin 3\omega t$, $T_3 = 2\pi/3\omega$

$\therefore \quad T_1 = 2T_2$ and $T_1 = 3T_3$

So, the time period of the given function is T_1 or $2\pi/\omega$.

Because in time $T = 2\pi/\omega$, first function completes one oscillation, the second function two oscillations and the third, three.

Note:

- Period is not changed by multiplying or dividing by (or by adding) a constant:

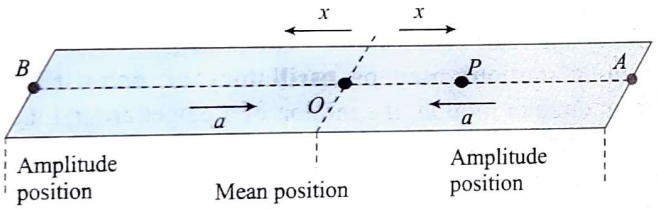
  If $x(t) = x(t+T)$, then $\dfrac{x(t)+x_0}{m}$ has the same period T.

  Here, x_0 and m are constant.
- Period is reduced α times if t be multiplied by α:

  If $x(t) = x(t+T)$ then $x(\alpha t)$ will have a period $\dfrac{T}{\alpha}$.
- The sum of periodic functions is also periodic.

SIMPLE HARMONIC MOTION

It is a special case of oscillatory motion in which the acceleration of the vibrating particle (or body), at any position directly as its displacement from a fixed point (which may or may not lie along the line of motion) and is always directed towards that fixed point. Thus, simple harmonic motion (abbreviated as SHM) is a case of variable acceleration; however, the variation takes place in a regular and periodic fashion.



Suppose a particle P performs oscillatory motion between two fixed points A and B. Let O be the mid-point of A and B. If the particle P oscillates about point O in such a way that its acceleration 'a' at any position when its displacement from O is x can be mathematically expressed as $a \propto x$ and is directed towards O, then its motion will be SHM. Here, the point 'O' is known as mean or stable or equilibrium or neutral position, and in case of 'simple' harmonic motion, the maximum displacement of the particle on either side of the mean position is the same, i.e., $OA = OB$.

Now, taking the motion of the particle to be along the x-axis, SHM can be mathematically expressed as

$$a \propto x \quad \text{or} \quad \vec{a} = (-\omega^2)\vec{x} \qquad \ldots(\text{i})$$

Here, the negative sign stands for the fact that the direction of acceleration is opposite to that of displacement, and ω^2 is a positive constant.

Again, if m is the mass of the oscillating particle (or body), then multiplying both side of Eq. (i) by m, we have

$$m\vec{a} = (-m\omega^2)\vec{x} \quad \text{or} \quad F = (-k)\vec{x} \qquad \ldots(\text{ii})$$

where $m\vec{a} = \vec{F}$ is the force acting on the particle and $k = m\omega^2$ is a positive constant.

Eqs. (i) and (ii) can equally be used to show that a given motion is SHM.

From Eq. (ii) it is evident that at $x = 0$ and $F = 0$, which shows that the particle experiences no force at mean position. Thus, the particle in itself will not oscillate. However, if it is disturbed even slightly by external force, then new forces (restoring forces) will be set up in the system which will tend to bring the particle to its mean position. Depending upon the nature of restoring forces, we have several types of SHM. The restoring forces can be electrical, gravitational, magnetic, elastic, etc.

In this section, we discussed a special type of oscillation called simple harmonic motion. A general oscillation can be regarded as SHM if it satisfies the following basic conditions:

Note:

- The oscillation amplitude of the particle must be very small as compared to its surrounding dimensions (dimensions of bodies with which it can interact).
- During oscillation the acceleration of the particle toward mean position, due to net restoring forces, must be directly proportional to its displacement for mean position.

A FEW TERMS RELATED TO SHM

(a) **Amplitude:** It is maximum displacement of the particle executing SHM from its mean position.
Thus, Amplitude $= |x_{max}| = A$

(b) **Time period or (period of oscillation):** We have discussed earlier that all SHMs are periodic motions which repeat themselves in equal time intervals. This minimum time interval is known as time period for the oscillations.

(c) **Angular frequency:** The number of revolutions (expressed in radian) performed per unit time is known as angular frequency.
(Each oscillation corresponds to one revolution; see more in the next section.)
Therefore, in a time T (time period), number of revolutions covered = 1
Therefore, in a time of 1 s, no. of revolutions covered = $1/T$
Therefore, angle described in each revolution = 2π
Therefore, angle described in $1/T$ revolution = $\dfrac{2\pi}{T}$
Thus, angular frequency = $\dfrac{2\pi}{T} = \omega$
The number of oscillations described per unit time is known as the frequency of oscillations. Evidently,
Frequency $n = \dfrac{1}{T}$

EQUATION OF SIMPLE HARMONIC MOTION

The differential equation of simple harmonic motion is given by $\dfrac{d^2x}{dt^2} = -\omega^2 x$.

The necessary and sufficient condition for a motion to be simple harmonic is that the net restoring force (or torque) must be linear, i.e., $F = ma = -kx$ (where k is a constant). Thus,

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

where x is the instantaneous displacement. Multiplying both sides by dx/dt and integrating with respect to t,

$$\left[\frac{d}{dt}\left(\frac{dx}{dt}\right)\right]\frac{dx}{dt} = \left[\frac{-kx}{m}\right]\frac{dx}{dt}$$

$$\Rightarrow \quad \int v\frac{dv}{dt} = \int \frac{-k}{m}\frac{x\,dx}{dt} \qquad \left[\because\ v = \frac{dx}{dt}\right]$$

$$\frac{v^2}{2} = \frac{-k}{m}\frac{x^2}{2} + c$$

We get $$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = -\frac{kx^2}{2m} + C \qquad ...(i)$$

where C is a constant of integration. Now when x is maximum dx/dt will be zero. The maximum displacement x_{max} of the particle from the mean position is called amplitude and is represented by A, then the value of C comes out to be

$$C = \frac{k}{m}\frac{A^2}{2}$$

Here $$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{k}{2m}(A^2 - x^2)$$

Putting $k/m = \omega^2$, we get $$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \qquad ...(ii)$$

This equation gives the velocity of the particle in SHM

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega\,dt \qquad ...(iii)$$

Integrating this equation with respect to t, we get

$$\sin^{-1}\frac{x}{A} = \omega t + \phi$$

where ϕ is another constant of integration which depends on initial conditions. Thus

$$x = A\sin(\omega t + \phi) \qquad ...(iv)$$

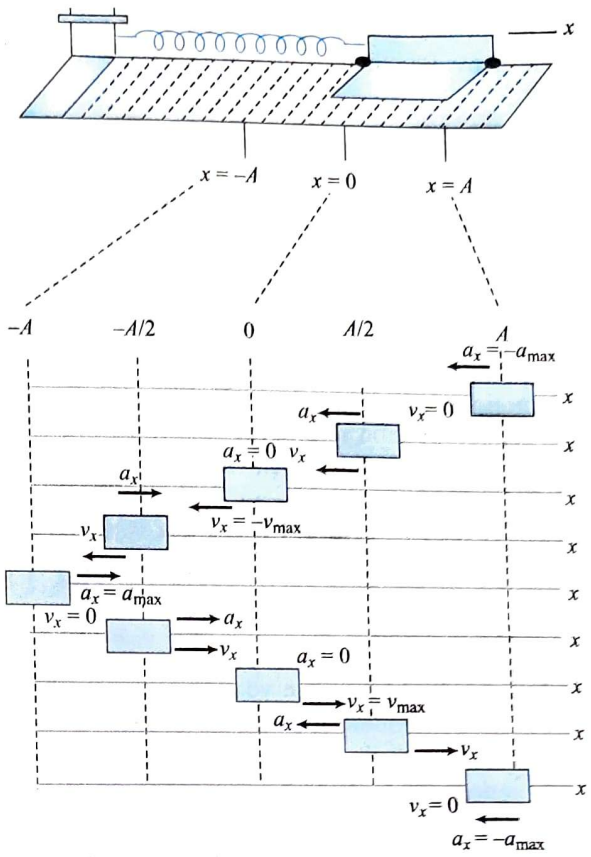
Here ω is called angular frequency and ϕ is called initial phase or epoch constant, whose value depends upon initial conditions.

MOTION OF AN OBJECT ATTACHED TO A SPRING

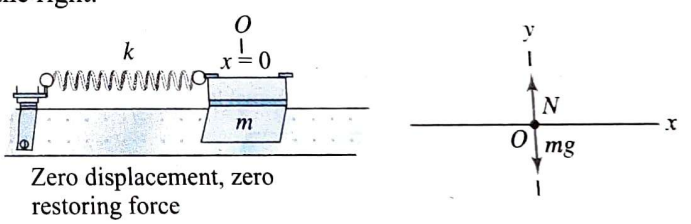
As a model for simple harmonic motion, consider a block of mass m attached to the end of a spring, with the block free to move on a horizontal, frictionless surface. When the spring is neither stretched nor compressed, the block is at rest at the position called the equilibrium position of the system, which we identify as $x = 0$. We know from experience that such a system oscillates back and forth if disturbed from its equilibrium position.

We can understand the oscillating motion of the block in following figure qualitatively by first recalling that when the block is displaced to a position x, the spring exerts on the block a force that is proportional to the position and given by Hooke's law

$$F_s = -kx \qquad ...(i)$$

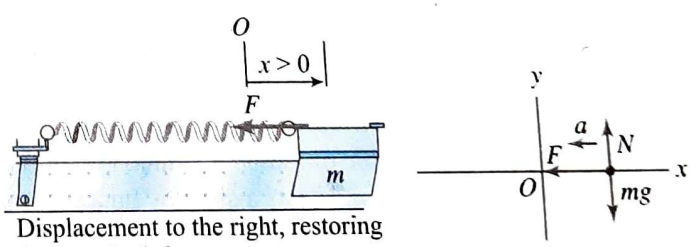


We call F_s a restoring force because it is always directed toward the equilibrium position and therefore opposite the displacement of the block from equilibrium. That is, when the block is displaced to the right of x = [Fig. (a)] the position is positive and the restoring force is directed to the left. Figure (b) shows the block at $x = 0$, where the force on the block is zero. When the block is displaced to the left of $x = 0$ as in Fig. (c), the position is negative and the restoring force is directed to the right.



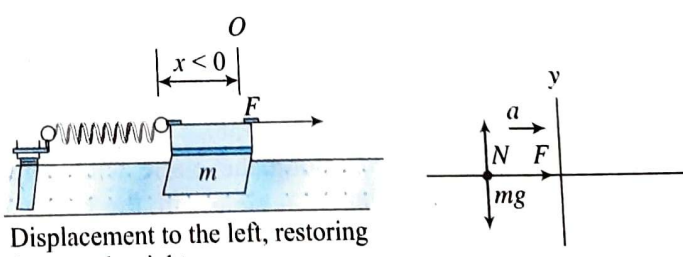
Zero displacement, zero restoring force

(a)



Displacement to the right, restoring force to the left

(b)



Displacement to the left, restoring force to the right

(c)

Applying Newton's second law to the motion of the block, with Eq. (i) providing the net force in the x direction, we obtain

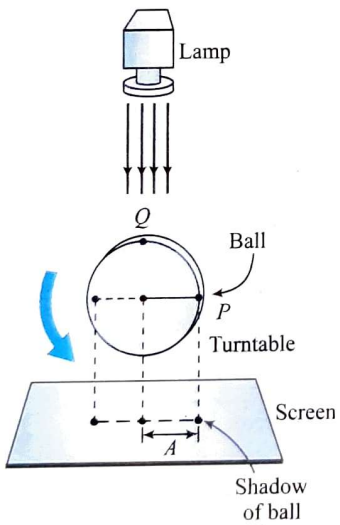
$$-kx = ma_x \quad \text{or} \quad a_x = -\frac{k}{m}x = -\omega^2 x \qquad \text{...(ii)}$$

Here ω is a positive constant, $\omega = \sqrt{\frac{k}{m}}$

That is, the acceleration of the block is proportional to its position, and the direction of the acceleration is opposite the direction of the displacement of the block from equilibrium. Systems that behave in this way are said to exhibit simple harmonic motion. An object moves with simple harmonic motion whenever its acceleration is proportional to its position and is oppositely directed to the displacement from equilibrium.

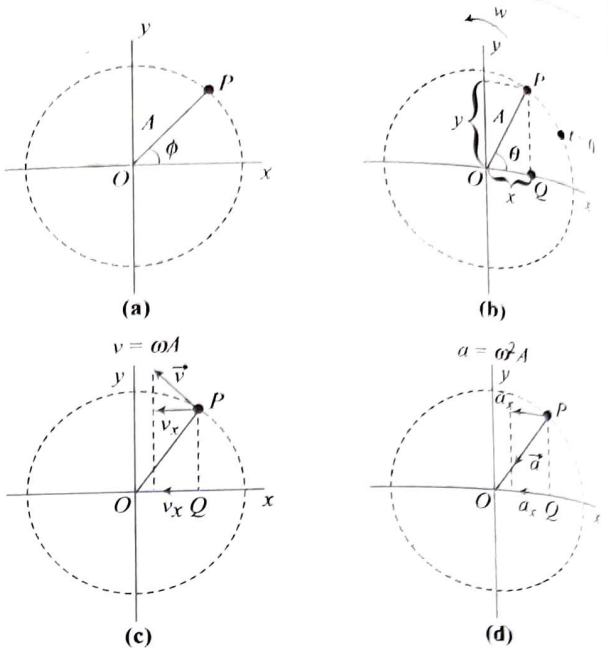
COMPARING SIMPLE HARMONIC MOTION WITH UNIFORM CIRCULAR MOTION

Some common devices in our everyday life exhibit a relationship between oscillatory motion and circular motion. For example, the pistons in an automobile engine go up and down—oscillatory motion—yet the net result of this motion is circular motion of the wheels. In an old-fashioned locomotive , the drive shaft goes back and forth in oscillatory motion, causing a circular motion of the wheels.



In this section, we explore this interesting relationship between these two types of motion. The given figure is a view of an experimental arrangement that shows this relationship. A ball is attached to the rim of a turntable of radius A, which is illuminated from the side by a lamp. The ball casts a shadow on a screen. As the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.

Consider a particle located at point P on the circumference of a circle of radius A as in Fig. (a) with the line OP making an angle ϕ with the x-axis at $t = 0$; we call this circle a reference circle for comparing simple harmonic motion with uniform circular motion, and we choose the position of P at $t = 0$ as our reference position. If the particle moves along the circle with constant angular speed ω until OP makes an angle ϕ with the x-axis as in Fig. (b), at some time $t > 0$ the angle between OP and the x-axis is $\theta = \omega t + \phi$.



As the particle moves along the circle, the projection of P on the x-axis, labelled as point Q, moves back and forth along the x-axis between the limits $x = \pm a$. Notice that points P and Q always have the same x-coordinate. From the right triangle OPQ we see that this x-coordinate is

$$x(t) = A\cos(\omega t + \phi)$$

This expression is the same as Eq. (i) and shows that the point Q moves with simple harmonic motion along the x-axis. Therefore, simple harmonic motion along a straight line can be represented by the projection of uniform circular motion along a diameter of a reference circle.

If we take projection of point P on y-axis, the y-coordinate is given by

$$y(t) = A\sin(\omega t + \phi)$$

This geometric interpretation shows that the time interval for one complete revolution of point P on the reference circle is equal to the period of motion T for simple harmonic motion between $x = \pm a$. That is, the angular speed ω of P is the same as the angular frequency ω of simple harmonic motion along the x-axis (which is why we use the same symbol).

We may choose any position of the particle as initial position by pressing the stop watch on. It means, from any position of the particle you may start calculating time; for this, we can use $y = A\sin(\omega t + \phi)$ or $x = A\cos(\omega t + \phi)$.

The displacement is always measured from mean position whereas time t can be calculated from any position of the oscillating particle.

If we differentiate Eq. (i) or (ii) twice, we get differential equation of SHM.

Differentiating Eq. (i), we get

$$\frac{dy}{dt} = A\omega\cos(\omega t + \phi)$$

Again differentiating Eq. (iii),

$$\frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 y, \text{ i.e., } \frac{d^2y}{dt^2} + \omega^2 y = 0$$

This is the basic equation of SHM.

ILLUSTRATION 5.3

Identify which of the following functions represent simple harmonic motion.

(a) $y = Ae^{i\omega t}$ (b) $y = ae^{-\omega t}$
(c) $y = a\sin^2\omega t$ (d) $y = a\sin\omega t + b\cos\omega t$
(e) $y = \sin\omega t + \cos 2\omega t$

Sol.

(a) According to given equation in problem, differentiating with respect to time, we get $\frac{dy}{dt} = iA\omega e^{i\omega t}$.

Differentiating again with respect to time, we get

$$\frac{d^2y}{dt^2} = -\omega^2 Ae^{i\omega t} = -\omega^2 y \qquad [\text{as } y = Ae^{i\omega t}]$$

Thus we have $\frac{d^2y}{dt^2} + \omega^2 y = 0$

This is the basic differential equation of SHM.

(b) The function $y = ae^{-\omega t}$ is not harmonic as it is not expressed in terms of sine and cosine functions. So, it cannot be simple harmonic.
Moreover, this function is not periodic.

(c) $y = a\sin^2\omega t$
The function $y = a\sin^2\omega t$ is harmonic.
To become simple harmonic $\frac{d^2y}{dt^2} \propto y$

Here, $\frac{dy}{dt} = 2a\omega\sin\omega t\cos\omega t$

$$\frac{d^2y}{dt^2} = 2a\omega^2[\cos^2\omega t - \sin^2\omega t] = 2a\omega^2[1 - 2\sin^2\omega t]$$

$$= 2a\omega^2\left[1 - \frac{2y}{a}\right]$$

The function is not simple harmonic.

(d) $y = a\sin\omega t + b\cos\omega t$
The function $y = a\sin\omega t + b\cos\omega t$ is simple harmonic.

Because $\frac{dy}{dt} = \omega a\cos\omega t - \omega b\sin\omega t$

$$\frac{d^2y}{dt^2} = -\omega^2 a\sin\omega t - \omega^2 b\cos\omega t \quad \Rightarrow \frac{d^2y}{dt^2} = -\omega^2 y$$

This is the basic differential equation of SHM.

(e) $y = \sin\omega t + \cos 2\omega t$
The function $y = \sin\omega t + \cos 2\omega t$ is not simple harmonic.

Because $\frac{dy}{dt} = \omega\cos\omega t - 2\omega\sin 2\omega t$

$$\frac{d^2y}{dt^2} = -\omega^2\sin\omega t - 4\omega^2\cos 2\omega t$$

$$= -\omega^2[\sin\omega t + 4\cos 2\omega t]$$

$$\frac{d^2y}{dt^2} \neq -\omega^2 y$$

The function is not simple harmonic.

VELOCITY AND ACCELERATION IN SHM

We can write the equation of motion of a particle performing SHM is

$$y = A\sin(\omega t + \phi) \qquad ...(i)$$

Differentiating y with respect to 't' given velocity of the particle

$$v = \frac{dy}{dt} = A\omega\cos(\omega t + \phi) \qquad ...(ii)$$

From Eq. (i), we can write $\sin(\omega t + \phi) = \frac{y}{A}$ and we know that

$$\cos(\omega t + \phi) = \sqrt{1 - \sin^2(\omega t + \phi)} = \sqrt{1 - \frac{y^2}{A^2}} = \frac{1}{A}\sqrt{A^2 - y^2}$$

Substituting the value of $\cos(\omega t + \phi)$ in Eq. (i), we get

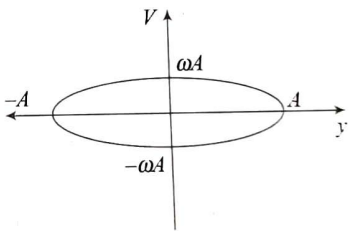
$$v = A\omega\left[\frac{1}{A}\sqrt{A^2 - y^2}\right] = \omega\sqrt{A^2 - y^2} \qquad ...(iii)$$

Differentiating Eq. (ii) again w.r.t. time.

$$a = \frac{dv}{dt} = -A\omega^2\sin\omega t = -\omega^2 y$$

Equation (iii) can be rearranged in the form $\frac{v^2}{\omega^2A^2} + \frac{y^2}{A^2} = 1$

This shows that velocity–position graph is an ellipse. This is also true for momentum–position graph. The momentum–position graph is known as phase–space graph of the particle.



Note:

For a particle performing SHM

- Velocity of a particle is maximum at mean position and is zero at amplitude positions.
- Acceleration of a particle is zero at mean position and maximum at amplitude positions.
- We have differential equation of SHM as

$$a = -\omega^2 y \Rightarrow \frac{d^2y}{dt^2} = -\omega^2 y$$

$$v\frac{dv}{dy} = -\omega^2 y \Rightarrow v\,dv = -\omega^2 y\,dy$$

Integrating both sides $\int v\,dv = -\omega^2\int y\,dy$

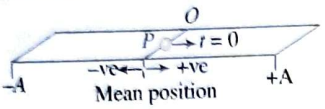
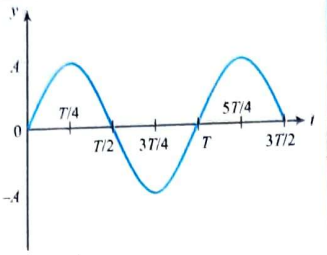
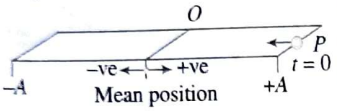
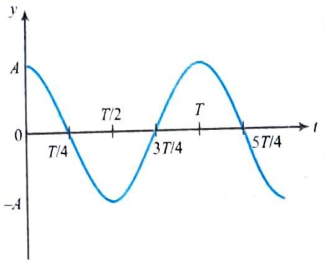
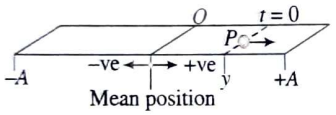
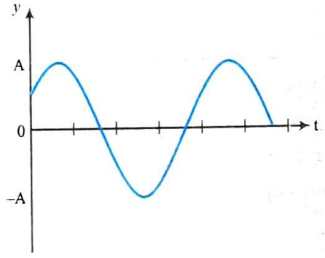
$$\frac{v^2}{2} = -\omega^2\frac{y^2}{2} + \text{constant}$$

$$v^2 + \omega^2y^2 = \text{constant} \qquad ...(i)$$

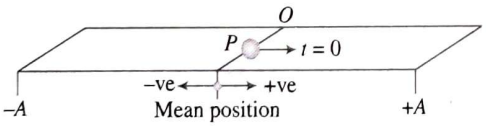
The above equation is also an important equation to express SHM.

DIFFERENT GRAPHS IN SHM

(a) Displacement–time graphs: The displacement of the particle is always measured from the mean position P whereas, time t can be calculated from any position of the oscillating particle.

Situations	Graphs
Particle starts SHM from mean position.  Displacement equation, $y = A \sin \omega t$	
Particle starts SHM from amplitude position.  Displacement equation $y = A \cos \omega t$	
Particle starts SHM from any general position.  Displacement equation, $y = A \sin(\omega t + \phi_0)$	

(b) Velocity–time and acceleration–time graphs: If particle starts SHM from mean position, then



Displacement equation, $y = A \sin \omega t$ ...(i)

Differentiation of equation (i) gives velocity of the particle.

$$v = \frac{dy}{dt} = A\omega \cos \omega t \qquad \text{...(ii)}$$

Differentiation of Eq. (ii) again gives acceleration of the particle.

$$a = \frac{dv}{dt} = -A\omega^2 \sin \omega t$$

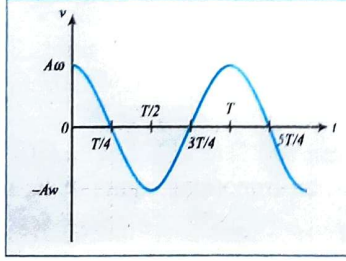
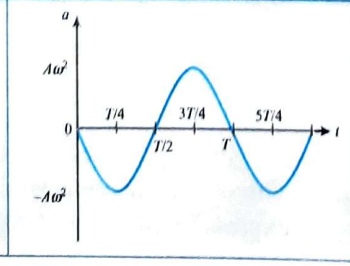
Velocity–time graph	Acceleration–time graph
 	

ILLUSTRATION 5.4

A particle executes SHM with an angular frequency ω = 4π rad/s. If it is at its extreme position initially, then find the instants when it is at a distance $\sqrt{3}/2$ times its amplitude from the mean position.

Sol. Let a be the amplitude and ϕ the phase constant for the oscillation.

The displacement equation can be written as

$x = a \sin(4\pi t + \phi)$ [$\because \omega = 4\pi$ rad/s]

Since at $t = 0$; $x = +a$, therefore $4\pi(0) + \phi = \frac{\pi}{2} \Rightarrow \phi = \frac{\pi}{2}$

$\therefore \quad x = a \sin(4\pi t + \pi/2) = a \cos(4\pi t)$

Now let t_n be the instant when the particle's displacement is $\sqrt{3}a/2$, nth

Evidently $\pm\frac{\sqrt{3}a}{2} = a\cos(4\pi t_n)$

$$\Rightarrow \quad 4\pi t_n = n\pi \pm \frac{\pi}{6} \quad \Rightarrow t_n = \frac{n}{4} \pm \frac{1}{24}$$

Taking only the positive values for 't',

$$t_1 = \frac{1}{24}\text{ s}, t_2 = \frac{5}{24}\text{ s}, t_3 = \frac{7}{24}\text{ s}; t_4 = \frac{11}{24}\text{ s}, t_5 = \frac{13}{24}\text{ s, etc.}$$

ILLUSTRATION 5.5

A particle executes SHM with an amplitude 8 cm and a frequency 10 s^{-1}. Assuming the particle to be at a displacement 4 cm initially, in the positive direction, determine its displacement equation and the maximum velocity and acceleration.

Sol. Given $a = 8$ cm and $\omega = 2\pi n = 20\pi$ rad/s

Let the phase constant be ϕ.

The displacement equation can be written as $x = 8 \sin(20\pi t + \phi)$

Given, at $t = 0$; $x = 4$ cm, therefore

$$4 = 8 \sin(20\pi(0) + \phi) \quad \Rightarrow \sin(\phi) = \frac{1}{2} \quad \Rightarrow \phi = \frac{\pi}{6}$$

$\therefore$ Displacement equation $x = 8 \sin\left(20\pi t + \frac{\pi}{6}\right)$

Differentiating the above equation w.r.t. time 't'

$$\frac{dx}{dt} = v = 160\pi \cos\left(20\pi t + \frac{\pi}{6}\right)$$

$\therefore \quad v_{max} = \pm 160\pi$ cm/s [when $\cos(20\pi t + \pi/6) = \pm 1$]

Differentiating once again w.r.t. time t.

$$\frac{d^2x}{dt^2} = f = -3200\pi^2 \sin\left(20\pi t + \frac{\pi}{6}\right)$$

$$f_{max} = \pm 3200\pi^2$$

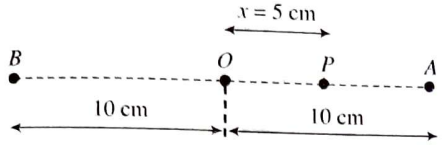
ILLUSTRATION 5.6

A particle executing SHM oscillates between two fixed points separated by 20 cm. If its maximum velocity be 30 cm/s. Find its velocity when its displacement is 5 cm from its mean position.

Sol. Evidently, the maximum displacement = 20/2 = 10 cm. For a particle executing SHM, the speed at a displacement 'x' is given by

$$v = \omega\sqrt{a^2 - x^2} \quad ...(i)$$

and $v_{max} = \omega a$ ...(ii)



Dividing Eq. (i) by Eq. (ii), we get $\frac{v_{max}}{v} = \frac{a}{\sqrt{a^2 - x^2}}$

$$\Rightarrow \quad \frac{v_{max}}{v} = \frac{10}{\sqrt{10^2 - 5^2}} = \frac{10}{5\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\therefore \quad v = 30 \times \frac{3}{2\sqrt{3}} = 15\sqrt{3} \text{ cm/s}$$

ILLUSTRATION 5.7

A particle executes SHM with time period of 2 s; find the time taken by it to move from one amplitude position to half the amplitude position.

Sol. Let 'a' be the amplitude of oscillation then evidently

$$\omega = \frac{2\pi}{T} = \pi \text{ rad/s}$$

The displacement equation can be written as $x = a \sin(\pi t + \phi)$
Let, for simplicity, the time be reckoned from the instant when the particle was at its extreme position.
Thus, at $t = 0$; $x = a$. Therefore,

$$a = a\sin(\phi) \quad \Rightarrow \phi = \frac{\pi}{2}$$

Let at $t = t_1$, $x = a/2$,

$$\therefore \quad \frac{a}{2} = a \sin\left(\pi t_1 + \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \left(\pi t_1 + \frac{\pi}{2}\right) = \frac{5\pi}{6} \Rightarrow \pi t_1 = \frac{\pi}{3} \Rightarrow t_1 = \frac{1}{3} \text{ s}$$

PHASE AND PHASE DIFFERENCE IN SIMPLE HARMONIC MOTION

Phase: We have seen earlier that the displacement, velocity and acceleration of a particle executing SHM vary periodically with the angle $(\omega t + \phi)$ associated with the sine or cosine term. Knowing this angle, we can be sure of its position as well as state of motion as to how and where it is oscillating. This angle $(\omega t + \phi)$ is known as the 'phase' of the oscillating particle. Since the phase is a time dependent factor, it will be more worthwhile to speak in terms of instantaneous phase (i.e., phase at any instant).

Taking the displacement equation as $x = A \sin(\omega t + \phi)$, we have

at $\omega t + \phi = 0$; $x = 0$ (mean position)

at $\omega t + \phi = \pi/2$; $x =$ A (amplitude position)

at $\omega t + \phi = \pi$; $x = 0$ mean position

at $\omega t + \phi = 3\pi/2$; $x = -A$ and so on

Notice that as time varies indefinitely, phase keeps on changing, 0 to 2π; the reason being $(\omega t + \phi) = (\omega t + \phi) + 2n\pi$, where $n \in I$.

Phase constant (Epoch): The phase of a particle executing SHM, initially (i.e., at the instant when time was reckoned), is known as the initial phase or phase constant or epoch.

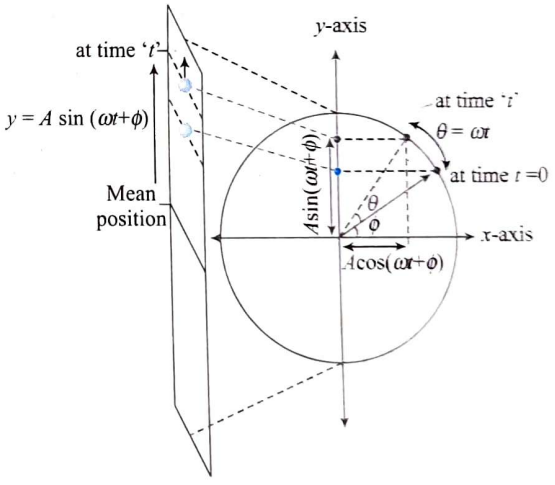
Since instantaneous phase = $\omega t + \phi$, so initial phase can be obtained by putting $t = 0$

$\therefore$ Phase constant = ϕ

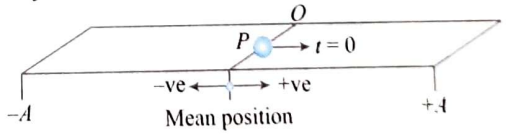
ANALYSING SIMPLE HARMONIC MOTION THROUGH PHASOR DIAGRAMS

Phasor diagram is the visual representation of a particle executing simple harmonic motion. It is represented by a circle in x-y plane centered at the origin and having radius A (amplitude of the body executing SHM). The radius makes an angle ϕ (initial phase) with positive direction of x-axis at $t = 0$. This radius rotates with constant angular velocity of ω in anti-clockwise direction (ω is same as the angular frequency of SHM).

This implies that the projection of the tip of the radius on y-axis will be equal to the displacement of the body executing SHM from its mean position. Let us learn how to write equation of SHM from phasor diagrams in different situations.



Situation I: A particle starts SHM from mean position and it is moving away from mean position.

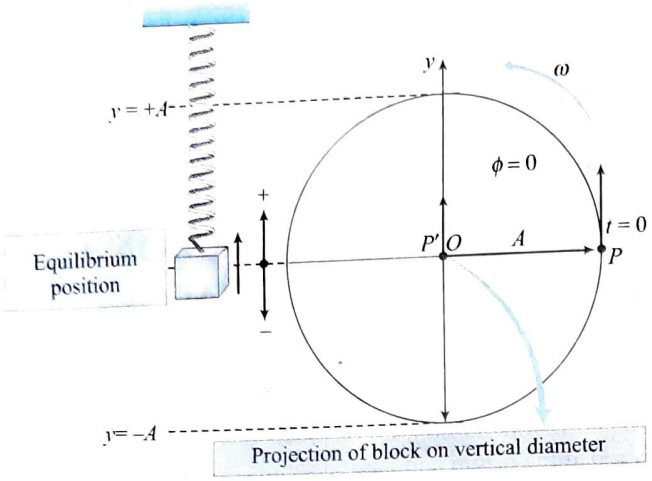


In the figure below, a block connected with a spring is executing SHM with angular frequency ω about its equilibrium position. The amplitude of oscillation is A. Let us take upward direction positive and downward direction negative. In this situation, the block is moving at $t = 0$ in positive direction. Here, we draw a circle called phasor circle of radius A. The corresponding position of the particle on phasor circle should be P as shown in the figure. The angle made by radius through P should represent, the initial phase of the block performing SHM. In this situation,

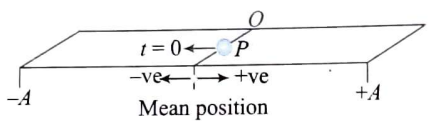
it is clear from phasor diagram that the initial phase of the block should be zero.

The equation of SHM is $y = A \sin(\omega t + \phi)$

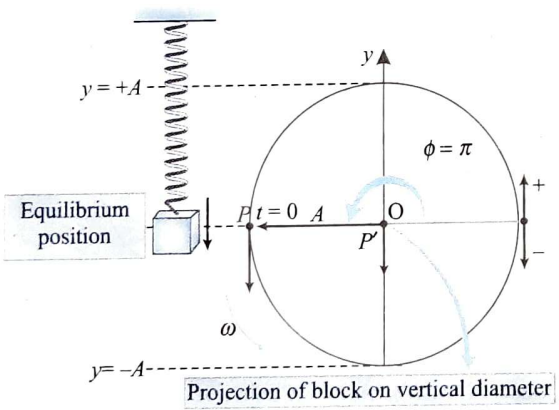
$\Rightarrow \quad y = A \sin(\omega t + 0) = A \sin \omega t$



Situation II: A particle starts SHM from equilibrium position and it is moving away from equilibrium position in negative direction.



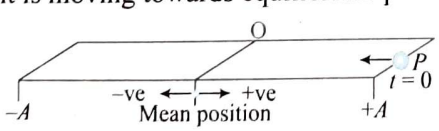
We can draw the phasor circle corresponding to this situation as shown in the figure below. In this situation, it is clear from phasor diagram that the initial phase of the block should be π.



The equation of SHM is $y = A \sin(\omega t + \phi)$

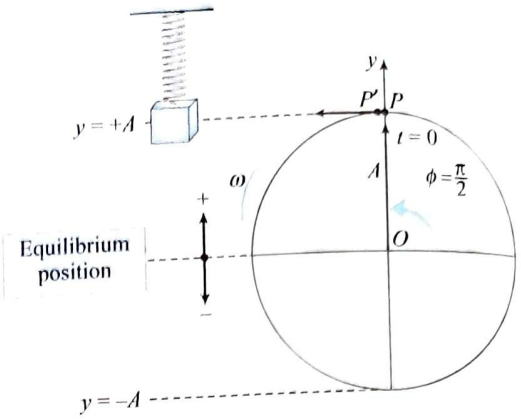
$\Rightarrow \quad y = A \sin(\omega t + \pi) = -A \sin \omega t$

Situation III: A particle starts SHM from positive amplitude position and it is moving towards equilibrium position.

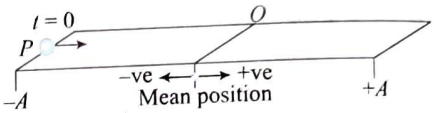


We can draw the phasor circle corresponding to this situation as shown in the figure below. In this situation, it is clear from phasor diagram that the initial phase of the block should be $\pi/2$. The equation of SHM is $y = A \sin(\omega t + \phi)$

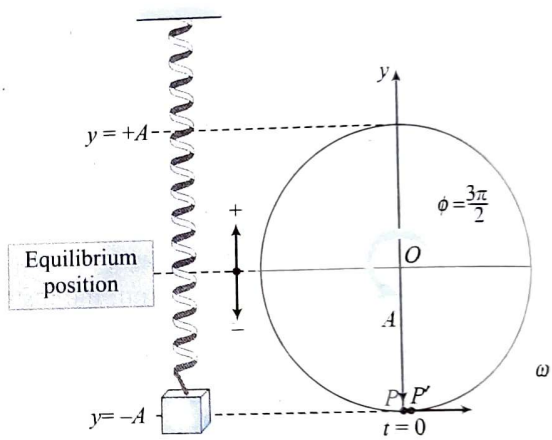
$$\Rightarrow \quad y = A \sin\left(\omega t + \frac{\pi}{2}\right) = A \cos \omega t$$



Situation IV: A particle starts SHM from negative amplitude position and it is moving towards equilibrium position.



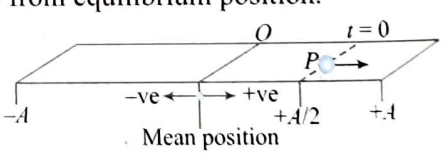
We can draw the phasor circle corresponding to this situation as shown in the figure below. In this situation, it is clear from phasor diagram that the initial phase of the block should be $3\pi/2$.



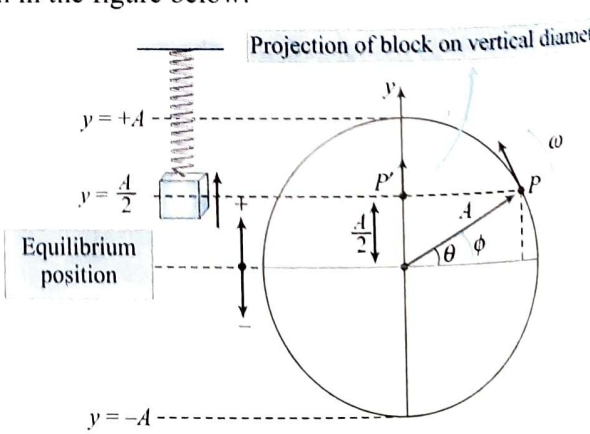
Equation of motion is $y = A \sin(\omega t + \phi)$

$$\Rightarrow \quad y = A \sin\left(\omega t + \frac{3\pi}{2}\right)$$

Situation V: A particle starts SHM from $y = +\frac{A}{2}$ and it is moving away from equilibrium position.



We can draw the phasor circle corresponding to this situation as shown in the figure below.

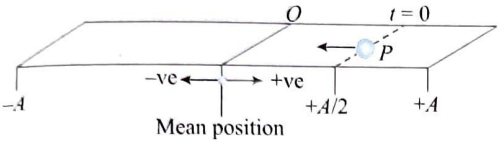


In the figure, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

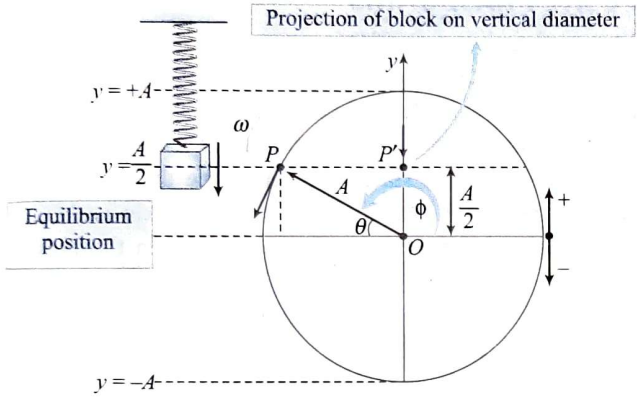
Hence, initial phase $\phi = \frac{\pi}{6}$

Equation of SHM is $y = A\sin\left(\omega t + \frac{\pi}{6}\right)$

Situation VI: A particle starts SHM from $y = +\frac{A}{2}$ and it is moving towards equilibrium position.



We can draw the phasor circle corresponding to this situation as shown in figure below.

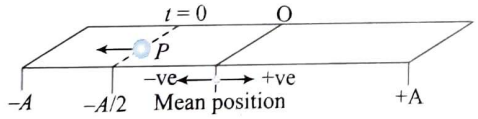


In the figure, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

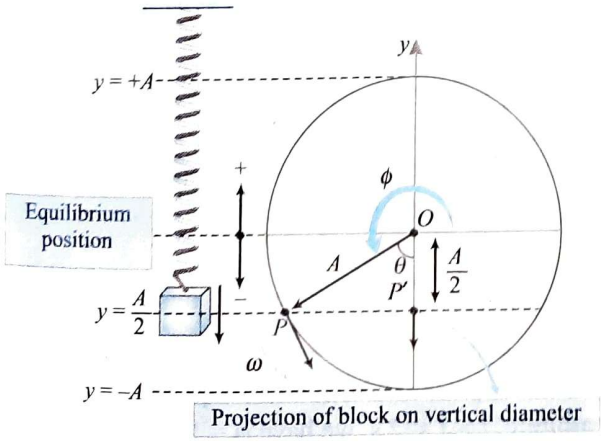
Hence, initial phase $\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

Equation of SHM is $x = A\sin\left(\omega t + \frac{5\pi}{6}\right)$

Situation VII: A particle starts SHM from $y = -\frac{A}{2}$ and it is moving away from equilibrium position.



We can draw the phasor circle corresponding to this situation as shown in the figure below.

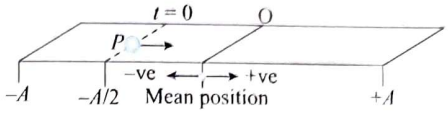


In the figure, $\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

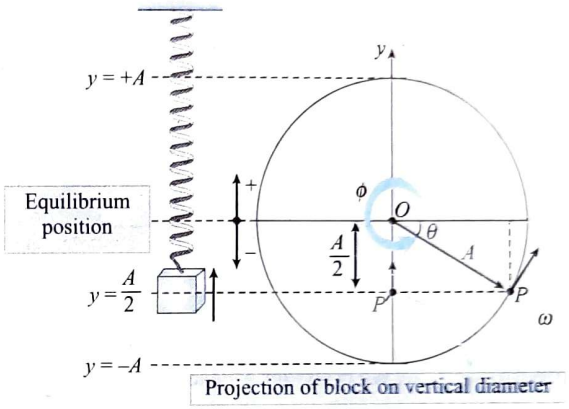
Hence, initial phase $\phi = \frac{3\pi}{2} - \frac{\pi}{3} = \frac{7\pi}{6}$

Equation of SHM is $y = A\sin\left(\omega t + \frac{7\pi}{6}\right)$

Situation VIII: A particle starts SHM from $y = -\frac{A}{2}$ and it is moving towards equilibrium position.



We can draw the phasor circle corresponding to this situation as shown in the figure below.



In the figure, $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

Hence, initial phase $\phi = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

Equation of SHM is $y = A\sin\left(\omega t + \frac{11\pi}{6}\right)$

or $y = \sin(\omega t - \pi/6)$

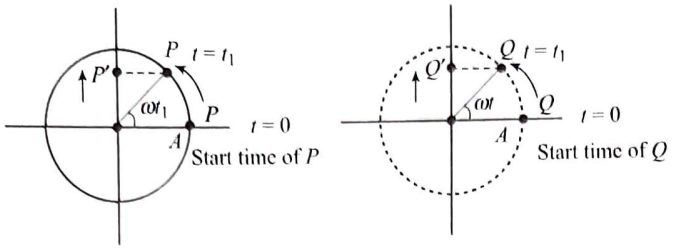
PHASE DIFFERENCE IN TWO SIMPLE HARMONIC MOTIONS

We can easily analyse phase or phase difference using reference circle.

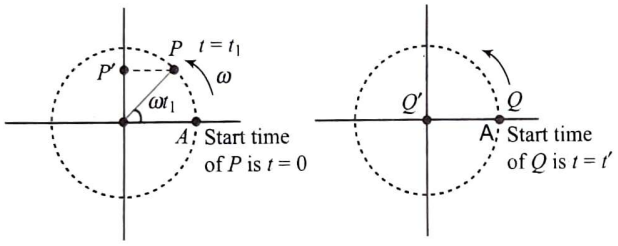
Figure shows two particles P' and Q' in SHM with same angular frequency ω. P and Q are the corresponding particles in circular motion for SHM of P' and Q'. ***P'* and *Q'* are the projection of images of *P* and *Q* on *y*-axis.**

To visualize same phase: Let P and Q both start their circular motion at the same time $t = 0$, then at the same instant P' and Q' start their motion in upward directions as shown. As the frequency of both are equal, both will reach their amplitude positions (topmost point) at the same time and will again reach their mean position simultaneously at time $t = T/2$. (T = Time period of SHM $= 2\pi/\omega$) and will move in downward direction together or we can state that the oscillations of P' and Q' are exactly parallel and at every instant the phases of both P' and Q' are equal, thus phase

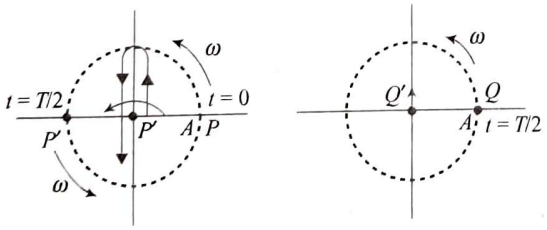
difference in these two SHMs is zero. These SHMs are called same phase SHMs.



Difference in phase: Now consider the reference circle figure where we assume if P' starts its motion at $t = 0$ but Q' starts at time t_1. In this duration from $t = 0$ to $t = t_1$, P' will move ahead in phase by ωt_1 radians while Q' was at rest. Now Q' starts at time t_1 and move with same angular frequency ω. It can never catch P' as both are oscillating at same angular frequency. Thus here Q' will always lag in phase by ωt_1 than P' or we say P' is leading in phase by ωt_1 than Q' and as ω of both are constant, their phase difference will also remain constant. So in two SHMs of same angular frequency, if they have same phase difference, it will always remain constant.



Opposite phase: Now we consider a case when time lag between the starting of two SHMs is $T/2$, i.e., half of time period. Consider the figure below, here we assume that particle P' starts at $t = 0$ and Q' at $t = T/2$ when P' completes its half oscillation. Here we can see that the phase difference in the two SHM is π by which Q' is lagging. Here when Q' starts its oscillation in upward direction, P' moves in downward direction. As angular velocity of P and Q are same, both complete their oscillation in same time. Thus when P' reaches its bottom extreme position, Q' will reach its upper extreme position and then after Q' starts moving downward, P' starts moving upward and both of these will reach their mean position simultaneously but in opposite directions; P' has completed its one oscillation where as Q' is at half of its oscillation due to a phase lag of π.



Thus if we observe both oscillations simultaneously, we can see that oscillations of the two particles P' and Q' are exactly anti-parallel, i.e., when P' goes up, Q' comes down and at all instants of time their displacements from mean position are equal but in opposite directions, if there amplitudes are equal. Such SHMs are called opposite phase SHMs.

SAME PHASE SHMS

As discussed earlier, two particles execute SHM in such a way that their oscillations are exactly parallel to each other or their phase difference during oscillation is zero, i.e., they are said to be in same phase. We have seen that this happens when both SHMs are started at same time with same angular frequency. This can happen also when the time lag between starting of the two SHMs is T or an integral multiple of time period of SHM. Because if one starts at $t = 0$ and other starts at $t = T$, in this duration the first particle will complete its the first oscillation and is going to start its second oscillations and the second particle will start in synchronization with the first. Hence, the two oscillation will still be parallel or in same phase. The phase difference in the two SHMs will be 2π. Not only this even if the time lag in starting of the two SHMs is $2T$, $3T$ ... nT or the phase difference in the two SHMs is 4π, 6π, 8π ... $2n\pi$, then also these SHMs can be treated in same phase.

Thus phase difference in two SHMs of same phase is

$$\phi = 2\pi, 4\pi, 6\pi \ldots 2n\pi$$

OPPOSITE PHASE SHMS

As discussed earlier, two SHMs are said to be in opposite phase when their oscillations are antiparallel. This happens when two SHMs of same angular frequency start with a time lag of $T/2$ and phase difference among the two SHMs is π. By analyzing the situation it can also be stated that the same thing also happens when the time lag in starting of the two SHMs is $3T/2$, $5T/2$ ... $(2n + 1)T/2$ or the phase difference between the two is 3π, 5π ...$(2n + 1)\pi$.

Thus phase difference in two SHMs of opposite phase is

$$\phi = \pi, 3\pi\, 5\pi \ldots, (2n + 1)\pi$$

RELATION BETWEEN PHASE DIFFERENCE AND TIME DIFFERENCE

Let the phase of a particle at time t_1 be ϕ_1, then

$$\phi_1 = \omega t_1 + \phi \quad \ldots(i)$$

Let the phase of the particle at time t_2 changes to ϕ_2, then

$$\phi_2 = \omega t_2 + \phi \quad \ldots(ii)$$

Subtracting Eq. (i) from Eq. (ii), phase difference becomes

$$\phi_2 - \phi_1 = \omega(t_2 - t_1) \quad \text{or} \quad \Delta\phi = \omega\,\Delta t = \frac{2\pi}{T}\cdot\Delta t$$

Putting $\Delta t = T$ and $\omega = 2\pi/T$, we have $\Delta\phi = 2\pi$

Thus, a phase difference of 2π is equivalent to a time difference of T.

Similarly, a phase difference of π is equivalent to a time difference of $T/2$, and so on.

ILLUSTRATION 5.8

A particle of mass $m = 1$ kg oscillates simple harmonically with angular frequency 1 rad/s. Find the phase of the particle at $t = 1$ s and 2 s. Start calculating time when the particle moves up passing through the mean position.

Sol. We need to find $\phi = \omega t + \phi_0$, where ω is the angular frequency of SHM.

Since, the particle moves up at the equilibrium position at $t = 0$, we have $\phi_0 = 0$. Then, $\phi = t$ rad.

Substituting, $t = 1$ s, 2 s, we have $\phi = 1$ rad, 2 rad

ILLUSTRATION 5.9

If $x = A/2$ at $t = 0$, find phase constant α in $x = A\sin(\omega t + \alpha)$. At $t = 0$, a particle executing SHM is going along x-axis.

Sol. Here $\frac{A}{2} = A\sin(\omega \times 0 + \alpha)$

$$\frac{A}{2A} = \sin\alpha \quad \text{or} \quad \sin\alpha = \frac{1}{2}$$

$\alpha = 30°$ or $5\pi/6$

Now we use the other condition: $V_x < 0$ at $t = 0$

We have $V_x = \omega A\cos(\omega t + \alpha)$

Putting $t = 0$ and $\alpha = \pi/6$,

$$V_x = \omega A\cos(\omega \times 0 + \pi/6) = \frac{\sqrt{3}}{2}\omega A$$, which is positive.

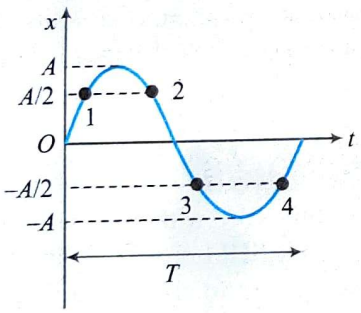
Now $\alpha = 5\pi/6$

$$V_x = \omega A\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}\omega A$$

which is negative and satisfies the given condition. Hence, the required $\alpha = \frac{5\pi}{6}$.

ILLUSTRATION 5.10

Figure shows the displacement–time graph of a particle executing SHM with a time period T. Four points 1, 2, 3 and 4 are marked on the graph where the displacement is half that of the amplitude.



(a) Identify the points of same displacement but with opposite direction of motion. Find the time difference between them.

(b) Identify the points where the particles move in the same direction. Find the time difference between them.

Sol.

(a) Points 1 and 2 have same displacements $x = +A/2$ with equal but opposite velocities.

Similarly points 3 and 4 have same displacements, $x = -A/2$ with equal but opposite velocities.

The phase difference between 1 and 2 and 3 and 4 is

$$\Delta\phi = \frac{2}{3}\pi$$

Therefore, the time difference is

$$\Delta t = \frac{\Delta\phi}{\omega} = -\frac{T}{2\pi}\left(\frac{2\pi}{3}\right) = \frac{T}{3}$$

(b) Points 1 and 4 have same direction of motion. Similarly, points 2 and 3 have same direction of motion.

The phase difference between 1 and 4 is

$$\Delta\phi = 2\pi - 2\left(\frac{\pi}{6}\right) = \frac{5\pi}{3}$$

Therefore, the time difference is

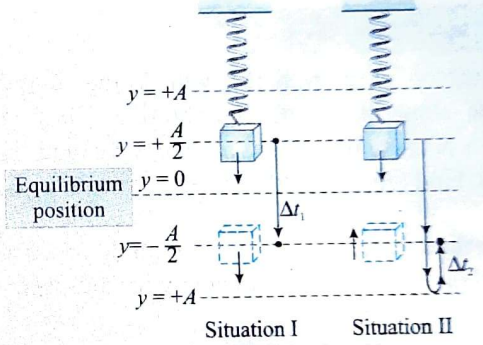
$$\Delta t = \frac{\Delta\phi}{\omega} = \frac{T}{2\pi}\left(\frac{5\pi}{3}\right) = \frac{5T}{6}$$

The phase difference between 2 and 3 is $\Delta\phi = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$

Thus, the time difference is $\Delta t = \frac{\Delta\phi}{\omega} = \frac{T}{2\pi}\left(\frac{\pi}{3}\right) = \frac{T}{6}$

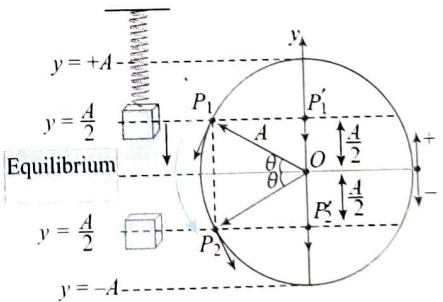
ILLUSTRATION 5.11

A block is connected with a spring executing SHM with time period T in vertical plane. The amplitude of oscillation is A. Find the time taken by the particle to move from position $y = +\frac{A}{2}$ to $y = -\frac{A}{2}$ in the following situation.



Sol.

Situation I: Let us draw phasor circle corresponding to this situation as shown below.



The block moves from position P_1 to P_2 on the phasor circle. From P_1 to P_2 the block turns through on angle 2θ. It is also equal to change in phase of the block.

Hence, $\sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$

Hence, angle rotated, $2\theta = \frac{\pi}{3} = \Delta\phi$ (change in phase)

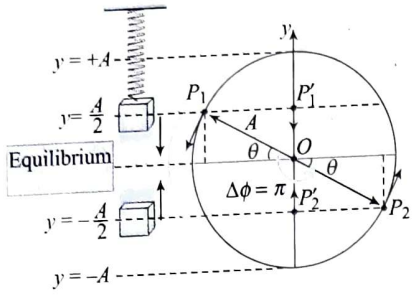
Thus, time taken for block in reaching from P_1 to P_2,

$$\Delta t = \frac{\Delta\phi}{\omega} = \frac{\pi}{3\omega}$$

Here $\omega = \dfrac{2\pi}{T}$, hence $\Delta t = \dfrac{\pi}{3\times(2\pi/T)} = \dfrac{T}{6}$

$$\Delta t = \frac{T}{2\pi}\times\Delta\phi = \frac{T}{2\pi}\times\frac{\pi}{3} = \frac{T}{6}$$

Situation II: We can draw the phasor circle corresponding to this situation as shown below.



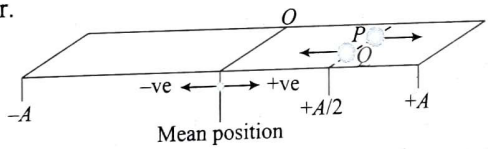
We can observe that the change in phase during the motion of the block from P_1 to P_2 is π or $\Delta\phi = \pi$.

Hence, time taken by the block, $\Delta t = \dfrac{T}{2\pi}\times\Delta\phi = \dfrac{T}{2\pi}\times\pi = \dfrac{T}{2}$

ILLUSTRATION 5.12

Two particles execute SHM with same frequency and amplitude along the same straight line. They cross each other, at a point midway between the mean and the extreme positions. Find the phase difference between them.

Sol. **Method 1:** Let 'A' be the amplitude of particles and ϕ_1 and ϕ_2 be their respective phases at any instant, when they cross each other.



Then their displacement equations can be written as:

$$y = A\sin(\omega t + \phi_0) = A\sin(\text{phase})$$

For the two particles, we have $\dfrac{A}{2} = A\sin\phi_1$ and $\dfrac{A}{2} = A\sin\phi_2$

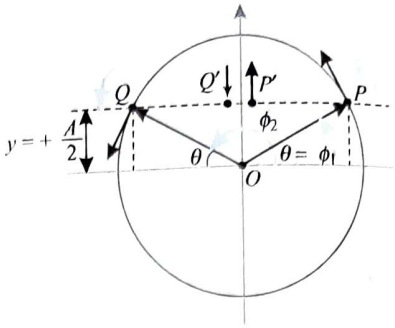
Taking the least value, $\phi_1 = \dfrac{\pi}{6}$ and taking the next possible value $\phi_2 = \dfrac{5\pi}{6}$, we get

The velocity of the particle, $v = \dfrac{dy}{dt}$

$$\Rightarrow \quad v = A\omega\cos(\omega t + \phi_0) = A\omega\cos(\text{phase})$$

Let $\vec{v}_P$ and $\vec{v}_Q$ be the respective velocities of the two particles. The particles cross each other, so their velocities should be in opposite directions. If we take velocity of the particle 'P' positive then velocity of the particle 'Q' should be negative. The velocity of the particle 'Q' will be negative when it has phase $\phi_2 = \dfrac{5\pi}{6}$.

Method 2: Figure shows two respective particles P and Q in SHM along with their corresponding particles in circular motion. Let P' moves in upward direction when crossing Q' at $A/2$ as shown in the figure.



For particle P, $\sin\theta = \sin\phi = \dfrac{1}{2}$

$$\Rightarrow \quad \theta = \phi_1 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad \ldots(i)$$

Similarly, the particle Q' is moving in downward direction (opposite P') at $A/2$, this implies that its circular motion particle is in second quadrant and thus, its phase angle is

$$\phi_2 = \pi - \theta = \pi - \sin^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \qquad \ldots(ii)$$

As both particles are oscillating at same angular frequency, their phase difference remains constant which can be given from Eqs. (i) and (ii).

Phase difference, $\Delta\phi = \phi_2 - \phi_1 = \dfrac{5\pi}{6} - \dfrac{\pi}{6} = \dfrac{2\pi}{3}$

Note: In a particular position, we have one displacement, one acceleration but two velocities; one is away from mean position other is towards mean position. It means that we have two different phases at a position. Hence, the phase is not only decided by the position but it is also decided by the velocity of the particle.

ILLUSTRATION 5.13

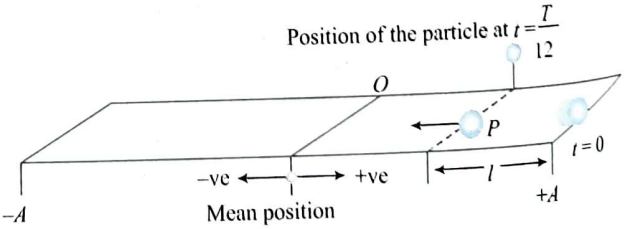
A particle is executing SHM with time period T and amplitude A. The particle starts SHM from amplitude position. Find distance travelled by the particle in time $T/12$.

Sol. **Method 1:** Initially, the particle is at extreme position $x = +A$. Equation of SHM, $x(t) = A\cos\omega t$ ...(i)

Position of the particle at time $t = \dfrac{T}{12}$

Substituting the value of t in Eq. (i),

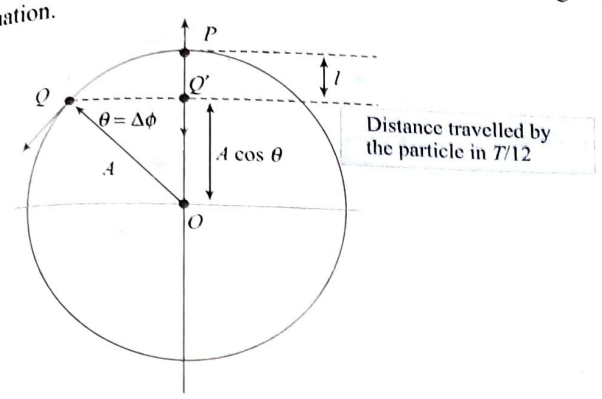
$$x(t) = A\cos\left(\frac{2\pi}{T}\times\frac{T}{12}\right) = \frac{A\sqrt{3}}{2}$$



Thus, distance travelled during this interval $= A - x(t)$

$$l = A - \frac{A\sqrt{3}}{2} = \frac{A(2-\sqrt{3})}{2}$$

Method 2: Let us draw a phasor circle corresponding to this situation.



Phase difference $\Delta\phi$ corresponding to this time interval, $\Delta t = \frac{T}{12}$.

$$\theta = \Delta\phi = \frac{2\pi}{T}\cdot\Delta t = \frac{2\pi}{T}\times\frac{T}{12} = \frac{\pi}{6}$$

Hence, distance travelled in this time,

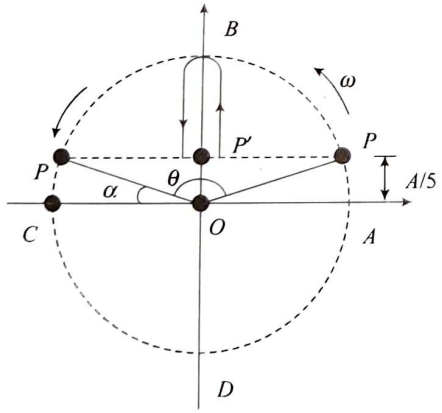
$$l = A - A\cos\theta = A - A\cos\frac{\pi}{6} = A - \frac{A\sqrt{3}}{2} = A\left(1-\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad l = \frac{A(2-\sqrt{3})}{2}$$

ILLUSTRATION 5.14

A particle executes SHM with amplitude A and angular frequency ω. At an instant the particle is at a distance $A/5$ from the mean position and is moving away from it. Find the time after which it will come back to this position again and also find the time after which it will pass through the mean position.

Sol. Figure shows the circular motion representation for the particle P' given in problem. The initial situation of particle is shown in the figure. As P moves, its projection P' will go up and then come back to its initial position when P reaches to the corresponding position in the second quadrant as shown. In this process P traversed an angular displacement θ with angular velocity ω, thus time taken in the process is



$$t = \frac{\theta}{\omega} = \frac{1}{\omega}[2\cos^{-1}(1/5)] = \frac{2\cos^{-1}(1/5)}{\omega}$$

Now when P reaches point C, P' will reach its mean position. Thus time taken by P from its initial position to point C is

$$t = \frac{(\pi-\alpha)}{\omega} = \frac{\pi - \sin^{-1}(1/5)}{\omega}$$

The same time P' will take from $A/5$ position to mean position through its extreme position.

ILLUSTRATION 5.15

Two particles executing SHM with same angular frequency and amplitudes A and $2A$ on same straight line with same mean position cross each other in opposite direction at a distance $A/3$ from mean position. Find the phase difference in the two SHMs.

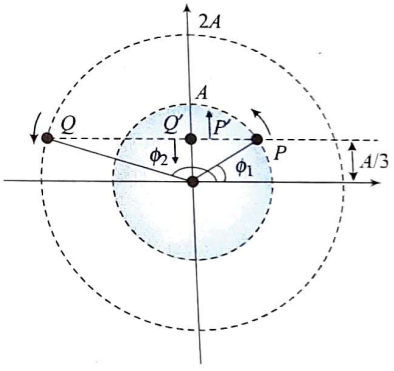
Sol. Figure shows the two corresponding particles of circular motion for the two mentioned particles in SHM. Let particle P be going up and particle Q be going down. From the figure shown, the respective phase differences of particles P' and Q' are

$$\phi_1 = \sin^{-1}\left(\frac{1}{3}\right) \qquad \text{[Phase angle of } P'\text{]}$$

And $\phi_2 = \pi - \sin^{-1}\left(\frac{1}{6}\right)$ [Phase angle of Q']

Thus phase difference in the two SHMs is

$$\Delta\phi = \phi_2 - \phi_1 = \pi - \sin^{-1}\left(\frac{1}{6}\right) - \sin^{-1}\left(\frac{1}{3}\right)$$



CONCEPT APPLICATION EXERCISE 5.1

1. Which of the following functions of time represent **(a)** Simple harmonic, **(b)** Periodic but not simple harmonic, and **(c)** Non-periodic motion? Give period for each case of periodic motion (ω is a positive constant):

(i) $\sin\omega t - \cos\omega t$ **(ii)** $\sin^3\omega t$ **(iii)** $3\cos(\pi/4 - 2\omega t)$

(iv) $\cos\omega t + \cos 3\omega t + \cos 5\omega t$ **(v)** $1 + \omega t + \omega^2 t^2$

2. A particle is in linear simple harmonic motion between two points A and B, 10 cm apart. Take the direction from A to B as the positive direction and give the signs of velocity, acceleration and force on the particle when it is:

(a) at the end A,

(b) at the end B,

(c) at the mid-point of AB going towards A,

(d) 2 cm away from B going towards A

(e) 3 cm away from A going towards B, and

(f) 4 cm away from B going towards A.

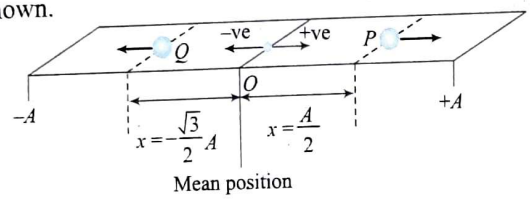
3. The motion of a particle executing simple harmonic motion is described by the displacement function.

$$x(t) = A\cos(\omega t + \phi)$$

If the initial ($t = 0$) position of the particle is 1 cm and its initial velocity is π cm/s, **(i)** what are its amplitude and initial phase angle? The angular frequency of the particle is π per second. **(ii)** If instead of the cosine function, we choose the sine function to describe the SHM: $x = B \sin(\omega t + \alpha)$, what are the amplitude and initial phase of the particle with the above initial conditions?

4. Write equation of SHM for a situation with angular frequency ω and amplitude A, if the particle is situated at $\frac{A}{\sqrt{2}}$ at $t = 0$ and is going towards mean position.

5. Two particles P and Q are executing SHM across same straight line whose equations are given by $y_P = A\sin(\omega t + \phi_1)$ and $y_Q = A\sin(\omega t + \phi_2)$. An observer, at $t = 0$, observes the particle 'P' at distance $\frac{A}{2}$ moving to the right from mean position O while Q at $\frac{-\sqrt{3}}{2}A$ moving to the left from mean position O as shown.



Find $(\phi_2 - \phi_1)$.

6. A particle slides back and forth between two inclined frictionless planes.

(a) If h is the initial height of the particle, the period of oscillation _____.

(b) Is the motion oscillatory? Is it SHM?

7. Suppose a tunnel is dug along the diameter of earth. A particle is dropped from a point at a distance h directly above the tunnel. The motion of the particle as seen from the earth is _____.

8. The equation of motion of a particle started at $t = 0$ is given by $x = 5\sin(20t + \pi/3)$ where x is in centimetres and t is in second. When does the particle

(a) first come to rest _____.

(b) first have zero acceleration _____.

(c) first have maximum speed _____.

9. If the maximum speed and acceleration of a particle executing SHM is 20 cm/s and 100π cm/s^2, find the time period of oscillation.

10. A particle is performing SHM of amplitude 'A' and time period 'T'. Find the time taken by the particle to go from 0 to $A/2$.

11. A particle of mass 2 kg is moving on a straight line under the action force $F = (8 - 2x)$ N. It is released at rest from $x = 6$ m.

(a) Is the particle moving simple harmonically.

(b) Find the equilibrium position of the particle.

(c) Write the equation of motion of the particle.

(d) Find the time period of SHM.

12. A particle executing simple harmonic motion has amplitude of 1 m and time period 2 s. At $t = 0$, net force on the particle is zero. Find the equation of displacement of the particle.

13. In the previous question, find maximum velocity and maximum acceleration.

14. A particle in SHM has a period of 4 s. It takes time t_1 to start from mean position and reach half the amplitude. In another case it takes a time t_2 to start from extreme position and reach half the amplitude. Find the ratio t_1/t_2.

15. A body executing SHM has its velocity 10 cm/sec and 7 cm/sec when its displacement from the mean position are 3 cm and 4 cm, respectively. Calculate the length of the path.

ANSWERS

1. (i) SHM (ii) Periodic but not SHM (iii) SHM (iv) Periodic but not SHM (v) Non-periodic

2. (a) (0, +, +) (b) (0, –, –) (c) (–, 0, 0) (d) (–, –, –) (e) (+, +, +) (f) (–, –, –)

3. (i) $\sqrt{2}$ cm, $\frac{7\pi}{4}$, (ii) $\sqrt{2}$ cm, $\frac{\pi}{4}$ **4.** $x = A\sin\left(\omega t + \frac{3\pi}{4}\right)$

5. $\frac{7}{6}\pi$ **6.** (a) $\frac{4}{\sin\alpha}\sqrt{\frac{2h}{g}}$ (b) Oscillatory **7.** Periodic

8. (a) $\frac{\pi}{120}$ s (b) $\frac{\pi}{30}$ s (c) $\frac{\pi}{30}$ s **9.** 0.4 s **10** $\frac{T}{12}$

11. (a) Yes (b) $x = 4$ m (c) $x = 4 + 2\cos t$ (d) 2π s **12.** $x = \sin \omega t$

13. π m/s, π^2 m/s^2 **14.** $\frac{1}{2}$ **15.** 9.5 cm

ENERGY OF A BODY IN SIMPLE HARMONIC MOTION

The total energy of a harmonic oscillator consists of two parts potential energy (PE) and kinetic energy (KE). The former being due to its displacement from the mean position and later due to its velocity. Since the position and velocity of the harmonic oscillator are continuously changing, PE and KE also change but their sum i.e., the total energy (TE) must have the same value at all times.

Simple harmonic motion is defined by the equation $F = -kx$ The work done by the force F during the displacement from x to $x + dx$ is $dW = Fdx = -kxdx$.

The work done in the displacement from $x = 0$ to x is

$$W = \int_0^x (-kx)\,dx = -\frac{1}{2}kx^2$$

Let $U(x)$ be the potential energy of the system when the displacement is x. The change in potential energy corresponding to a force is negative of the work done by this force,

$$U(x) - U(0) = -W = \frac{1}{2}kx^2$$

Let us choose the potential energy to be zero when the particle is at the centre of oscillation $x = 0$. Then

$$U(0) = 0 \text{ and } U(x) = \frac{1}{2}kx^2$$

This expression for potential energy is same as that for a spring and has been used so far in this chapter.

$\omega = \sqrt{\frac{k}{m}}$, $k = m\omega^2$

so we can write $U(x) = \frac{1}{2}m\omega^2 x^2$ ...(i)

The displacement and the velocity of a particle executing a simple harmonic motion are given by $x = A\sin(\omega t)$ and $v = A\omega\cos(\omega t)$

The potential energy at time t is, therefore,

$$U = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t)$$

$$= \frac{m\omega^2 A^2}{4}(1-\cos 2\omega t)$$

and the kinetic energy at time t is

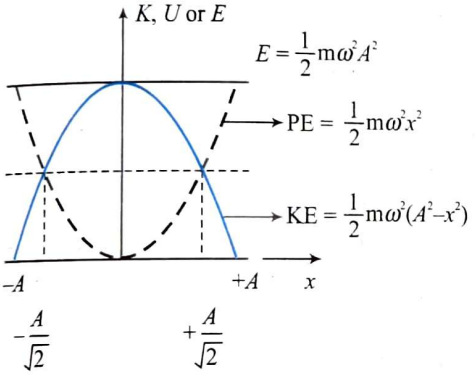
$$K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t) = \frac{m\omega^2 A^2}{4}(1+\sin 2\omega t)$$

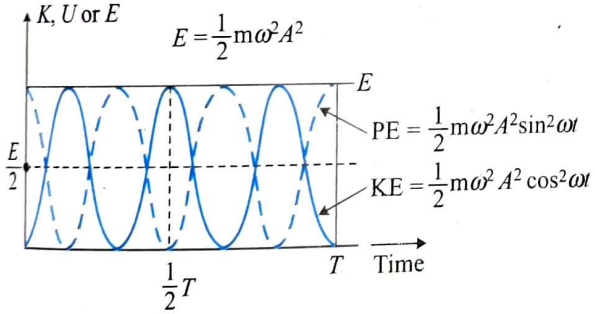
The total mechanical energy at time t is

$$E = U + K = \frac{1}{2}m\omega^2 A^2\left[\sin^2(\omega t)+\cos^2(\omega t)\right]$$

$$= \frac{1}{2}m\omega^2 A^2 \quad \text{...(ii)}$$

We see that the total mechanical energy at time t is independent of t. Thus, the mechanical energy remains constant as expected.





Note:

- In SHM U–x graph must be parabolic.
- Kinetic energy versus time equation can also be written as

$$K = \frac{m\omega^2 A^2}{4}(1+\sin 2\omega t)$$

This function is also periodic with angular frequency 2ω. Thus, kinetic energy in SHM is also periodic with double the frequency than that of x, v and a. But these oscillations are not simple harmonic in nature, because $\frac{d^2K}{dt^2}$ is not proportional to K.

Same is the case with potential energy function. U also oscillate with angular frequency 2ω but the oscillations are not simple harmonic in nature. Total energy does not oscillate. It is constant.

Thus,

$x \rightarrow$ oscillates simple harmonically with angular frequency ω

$v \rightarrow$ oscillates simple harmonically with angular frequency ω

$a \rightarrow$ oscillates simple harmonically with angular frequency ω

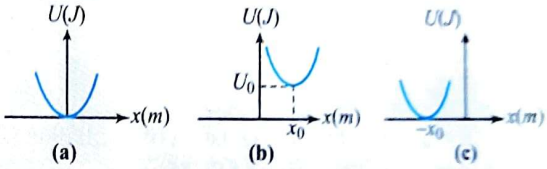
$K \rightarrow$ oscillates with angular frequency 2ω but not simple harmonically

$U \rightarrow$ oscillates with angular frequency 2ω but not simple harmonically

$E \rightarrow$ does not oscillate.

- In the above discussion, we have read that potential energy is zero at mean position and maximum at extreme positions and kinetic energy is maximum at mean position and zero at extreme positions. But at mean position, kinetic energy is maximum and potential energy is minimum (it may be zero also, but it is not necessarily zero).

At extreme positions, kinetic energy is zero and potential energy is maximum.



Thus, in Fig. (a), oscillations will take place about the mean position $x = 0$ and minimum potential energy at mean position is zero.

In Fig. (b), mean position is at $x = x_0$ and the minimum potential energy in this position is U_0.

In Fig. (c), mean position is at $x = -x_0$ and the minimum potential energy in this position is again zero.

- The energy of oscillation and total mechanical energy of the system both are different terms. We can write

$$E_{total} = E_{oscillation} + U_0$$

where $E_{oscillation}$ is the maximum kinetic energy when the particle passes through the mean position.

AVERAGE VALUE OF PE AND KE

By Eq. (i), PE at distance x is given by

$$U = \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t+\phi)$$

{since at time t, $x = A\sin(\omega t+\phi)$}

The average value of PE of complete vibration is given by

$$U_{average} = \frac{1}{T}\int_0^T U dt = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 A^2\sin^2(\omega t+\phi)$$

$$= \frac{m\omega^2 A^2}{4T}\int_0^T 2\sin^2(\omega t+\phi)dt = \frac{1}{4}m\omega^2 A^2$$

because the average value of sine or of cosine function for the complete cycle is equal to zero.

Now KE at x is given by

$$KE = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\left[\frac{d}{dt}\{A\sin(\omega t+\phi)\}\right]^2$$

$$= \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)$$

The average value of KE for complete cycle,

$$KE_{average} = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)dt$$

$$= \frac{m\omega^2A^2}{4T}\int_0^T\{1+\cos 2(\omega t+\phi)\}dt$$

$$= \frac{m\omega^2A^2}{4T}T = \frac{1}{4}m\omega^2A^2$$

Thus average values of KE and PE of harmonic oscillator are equal to half of the total energy.

Note: We can write total mechanical energy of a particle/body performing SHM as

$$KE + PE = \text{constant}$$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

or $$v^2 + \frac{k}{m}x^2 = \text{constant} \quad ...(i)$$

Here k is the equivalent force constant of oscillation ($k = m\omega^2$)

We have differential equation of SHM as

$$\frac{d^2x}{dt^2} = -\omega^2x \quad \text{or} \quad a + \omega^2x = 0 \quad ...(ii)$$

If we compare Eqs. (i) and (ii), both equations are same.

ILLUSTRATION 5.16

A particle of mass 0.50 kg executes a simple harmonic motion under a force $F = -(50\ \text{N/m})x$. If it crosses the centre of oscillation with a speed of 10 m/s, find the amplitude of motion.

Sol. The kinetic energy of the particle when it is mean position, it is also its total energy as the potential energy is zero here.

$$E = \frac{1}{2}mv^2 = \frac{1}{2}(0.50\ \text{kg})(10\ \text{m/s})^2 = 25\ \text{J}$$

At the maximum displacement $x = A$, the speed is zero and hence the kinetic energy is zero. The potential energy here is $1/2\ kA^2$. At this position it will also its total energy. Hence

$$\frac{1}{2}kA^2 = 25\ \text{J} \quad ...(i)$$

The force on the particle is given by $F = -(50\ \text{N/m})x$

Thus, the spring constant is $k = 50$ N/m.

Equation (i) gives

$$\frac{1}{2}(50\ \text{N/m})A^2 = 25\ \text{J} \Rightarrow A = 1\ \text{m}$$

ILLUSTRATION 5.17

A 20 g particle is oscillating simple harmonically with a period of 2 second and maximum kinetic energy 2 J. The total mechanical energy of the particle is zero. Find

(a) Amplitude of oscillation

(b) Potential energy as a function of displacement x relative to mean position.

Sol.

(a) Given, $m = 20\ \text{g} = 20 \times 10^{-3}$ kg

Time period, $T = 2s$ and $K_{max} = 2$ J

We have maximum kinetic energy; $K_{max} = \frac{1}{2}m\omega^2A^2$

Hence, $2 = \frac{1}{2}\times 20\times 10^{-3}\times\left(\frac{2\pi}{2}\right)^2\cdot A^2$

$$\Rightarrow \quad A = \frac{10\sqrt{2}}{\pi}\ \text{m}$$

(b) We know total energy of a particle is summation of oscillation energy and minimum potential energy.

$$E_{total} = E_{oscillation} + U_0$$

But oscillation energy of particle; $E_{oscillation} = K_{max} = 2$ J

As we are given $E_{total} = 0$

Hence, $0 = 2 + U_0 \Rightarrow U_0 = -2$ J

The potential energy in relation with displacement x from mean position is

$$U(x) = \frac{1}{2}kx^2 + U_0 = \frac{1}{2}(m\omega^2)x^2 + U_0$$

$$= \frac{1}{2}m\omega^2x^2 - 2$$

$$= \frac{1}{2}\times\frac{20}{1000}\times\left(\frac{2\pi}{2}\right)^2x^2 - 2$$

$$= \frac{\pi^2}{100}x^2 - 2 \quad \text{(SI units)}$$

ILLUSTRATION 5.18

A particle is executing SHM under the action of force whose maximum magnitude is 50 N. Find the magnitude of force acting on the particle at the time when its energy is half kinetic and half potential.

Sol. Total energy of oscillation, $E = 1/2\ mA^2\omega^2$

Given, $U = K \Rightarrow \frac{1}{2}(m\omega^2)x^2 = \frac{1}{2}mv^2$

$$\therefore \quad \frac{1}{2}mA^2\omega^2\sin^2\omega t = \frac{1}{2}mA^2\omega^2\cos^2\omega t$$

or $$\tan^2\omega t = 1 \quad \Rightarrow \quad \omega t = \frac{\pi}{4}$$

But, $a = -\omega^2x = -\omega^2A\sin\omega t$

Hence, force acting on the particle at this time is

$$|F| = ma = m\omega^2A\sin\frac{\pi}{4} \quad \text{(in the sense of magnitude)}$$

But $F_{max} = m\omega^2 A = 50$ N

$\therefore \quad F = \frac{50}{\sqrt{2}} = 25\sqrt{2}$ N

ILLUSTRATION 5.19

A point particle of mass 0.1 kg is executing SHM with amplitude of 0.1 m. When the particle passes through the mean position, its K.E. is 8×10^{-3} J. Obtain the equation of motion of this particle if the initial phase of oscillation is $\pi/4$.

Sol. The kinetic energy of a particle performing SHM is maximum when it passes through means position. Hence,

$$\frac{1}{2}mv_{max}^2 = 8 \times 10^{-3}$$

or $$\frac{1}{2}mA^2\omega^2 = 8 \times 10^{-3}$$

$\Rightarrow \quad \omega^2 = 16$ or $\omega = 4$ rad/s

Hence, equation of SHM

$$x = A\sin\left(\omega t + \frac{\pi}{4}\right) = 0.1\sin\left(4t + \frac{\pi}{4}\right)$$

ILLUSTRATION 5.20

A particle executes SHM with an amplitude of 10 cm and frequency 2 Hz. At $t = 0$, the particle is at a point where potential energy and kinetic energy are same. Find the equation of displacement of particle.

Sol. Let $x = A \sin(\omega t + \phi)$; then

$$v = \frac{dx}{dt} = A\omega\cos(\omega t + \phi)$$

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$$

$$(\text{KE})_{max} = \frac{1}{2}mA^2\omega^2 \qquad [\text{For } \cos^2(\omega t + \phi) = 1]$$

$$\text{PE} = \frac{1}{2}mA^2 - \text{KE} = \frac{1}{2}mA^2\omega^2 - \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$$

$$= \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$$

According to the problem, KE = PE; therefore,

$$\frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi) = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi)$$

$$\tan^2(\omega t + \phi) = 1 \quad \Rightarrow \tan^2(\omega t + \phi) = \tan^2\frac{\pi}{4}$$

$\Rightarrow \quad \omega t + \phi = \pi/4 \quad \Rightarrow \phi = \pi/4 \ (t = 0)$

$x = A \sin(\omega t + \phi)$

Here, $A = 10$ cm $= 0.1$ m

$\omega = 2\pi f = 2\pi \times 2 = 4\pi$ rad/s

$\phi = \pi/4$

Hence, the equation of SHM is $x = 0.1\sin\left(4\pi t + \frac{\pi}{4}\right)$

ILLUSTRATION 5.21

A particle of mass 0.2 kg undergoes SHM according to the equation $x(t) = 3 \sin(\pi t + \pi/4)$.

(a) What is the total energy of the particle if potential energy is zero at mean position?

(b) What are the kinetic and potential energies of the particle at time t = 1 s?

(c) At what time instants is the particle's energy purely kinetic?

Sol. Comparing the given equation with $x(t) = A \sin(\omega t + \phi_0)$, we get $A = 3$ m, $\omega = \pi$ rad/s, $\phi_0 = \pi/4$

$$T = \frac{2\pi}{\omega} = 2 \text{ s}$$

$x(0) = 3 \sin \pi/4 = 1.5\sqrt{2}$ m

$dx/dt = v(t) = 3\pi \cos(\pi t + \pi/4)$

$\Rightarrow \quad v(0) = 3\pi/\sqrt{2}$ m/s

(a) Total energy $= \frac{1}{2}KA^2 = \frac{1}{2}m\omega^2A^2 = 1/2\ (0.2)\ (\pi)^2\ (3)^2$

$= 0.9\ \pi^2$

(b) At $t = 1$, $x(t) = 3 \sin(\pi + \pi/4) = -\frac{3}{\sqrt{2}}$ m

$$v(t) = 3\pi \cos(\pi + \pi/4) = -\frac{3\pi}{\sqrt{2}} \text{ m/s}$$

$$K = \frac{1}{2}mv^2 = \frac{1}{2}(0.2)\left(\frac{9\pi^2}{2}\right) = \frac{9\pi^2}{20}\text{J}$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}(0.2)(\pi)^2\left(\frac{9}{2}\right) = \frac{9}{20}\pi^2 \text{ J}$$

(c) Energy is purely kinetic at mean position, i.e., when $x = 0$

Using $x(t) = 3 \sin(\pi t + \pi/4)$, we have

$0 = 3 \sin(\pi t + \pi/4)$

$\pi t + \pi/4 = 0, \pi, 2\pi, 3\pi, \ldots$

$\Rightarrow \quad t = \frac{3}{4}\text{s}, \frac{7}{4}\text{s}, \frac{11}{4}\text{s}\ldots$

At these time instants, particle crosses origin and hence its energy is purely kinetic.

ILLUSTRATION 5.22

A particle of mass 0.2 kg executes simple harmonic motion along a path of length 0.2 m at the rate of 600 oscillations per minute. Assume at $t = 0$ the particle starts SHM in positive direction. Find the kinetic and potential energies in joules when the displacement is $x = A/2$, where A stands for the amplitude.

Sol. Given that $f = 600$ oscillations/min, $m = 0.2$ kg

A = Amplitude = 1/2 × Length of path = 0.1 m $= \frac{1}{10}$ m

As $\quad v = \frac{1}{T} = \frac{600}{60} = 10$ Hz $\Rightarrow \quad \omega = 2\pi T = 20\pi$ rad/s

The magnitude of velocity of a particle performing SHM is given as $v = \omega\sqrt{A^2 - x^2}$.

At $x = A/2 = \frac{1}{20}$ m, the velocity at this position

$$v_{x=A/2} = 20\pi\sqrt{\left(\frac{1}{10}\right)^2 - \left(\frac{1}{20}\right)^2} = \sqrt{3}\pi \text{ m/s}$$

Hence, kinetic energy at this position

$$k_{x=A/2} = \frac{1}{2}mv^2_{x=A/2} = \frac{1}{2}(0.2)(\sqrt{3}\pi)^2 = \frac{3\pi^2}{10}\text{J}$$

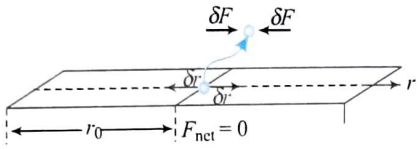
We can write potential energy at any position x as

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(m\omega^2)x^2$$

Hence, $U_{x=A/2} = \frac{1}{2}(0.2)(20\pi)^2\left(\frac{1}{20}\right)^2 = \frac{\pi^2}{10}\text{J}$

Finding Effective Force Constant (K_{eff}) of SHM from Given Potential Energy Relation [$U = f(r)$]

Let us assume that the potential energy of a system varies as $U = f(r)$. If the particle in the system oscillates simple harmonically about the mean position, the net force acting at equilibrium point, $(r = r_0)$ is zero.



$$F_{net} = \frac{dU}{dt} = 0$$

Let us now displace the particle from mean position by a small distance δr. Now, the particle experiences a net (restoring) force

$$F_{net} = \delta F = -k_{eff}\delta r \quad \text{at } r = r_0$$

Then, we have $k_{eff} = -\left(\frac{\delta F}{\delta r}\right)_{r=r_0}$ ...(i)

Since the force acting on the particle is equal to negative of maximum potential energy gradient (because, force acts in the direction of maximum decrease in potential energy), we can write

$$F = -\frac{dU}{dr} \quad \text{...(ii)}$$

Substituting $F = -\frac{dU}{dr}$ from Eq. (ii) in Eq. (i), we have

$$k_{eff} = -\frac{d}{dr}\left(-\frac{dU}{dr}\right)$$

This gives $k_{eff} = \left(\frac{d^2U}{dr^2}\right)_{r=r_0}$

Since $k_{eff} > 0$, we have $\frac{d^2U}{dr^2} > 0$

This tells us that U is minimum at the stable equilibrium point $r = r_0$. Any particle possessing stable equilibrium has minimum potential energy. Then, the particle can oscillate about the mean (stable equilibrium) position. If the particle is displaced slightly from mean position $(r = r_0)$, it oscillates simple harmonically with an effective spring constant given as

$$k_{eff} = \left(\frac{d^2U}{dr^2}\right)_{r=r_0}$$

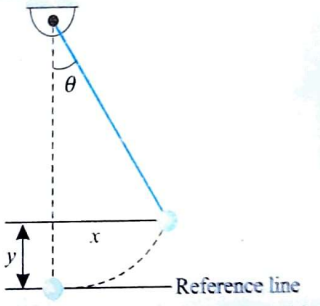
The physical significance of $\frac{d^2U}{dr^2}\left(=\frac{-dF}{dr}\right)$: It is the ratio of unbalanced (restoring) force F_{net} $(= \delta F)$ and the small (elementary) displacement x $(= \delta r)$ from the mean position. Since, the particle oscillates simple harmonically at $r = r_0$ the ratio $\frac{\delta F}{\delta r}\left(=\frac{F_{net}}{x}\right)$ equal to the effective spring constant of the system. Remember that $k \neq \frac{F}{r}$; rather it is $\frac{\delta F}{\delta r} = \frac{dF}{dr}$ at $r = r_{eq}$ $(= r_0)$.

ILLUSTRATION 5.23

Find

(a) variation of potential energy in the direction of small oscillation of a simple pendulum.

(b) effective spring constant for the simple pendulum. Assume m = mass of the bob and l = length of the string of the simple pendulum.



Sol.

(a) If you displace the bob by a small horizontal distance x, the string rotates by a small angle θ. In consequence, the bob moves up by a height y. Then the gravitational potential energy of the bob relative to its lowest position can be given as

$U = mgy$, where $y = l(l - \cos\theta)$

This gives $U = mgl(1 - \cos\theta)$

Since $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$,

We have $U = 2mgl\sin^2\frac{\theta}{2}$

As θ is small, $U = \frac{1}{2}mgl\theta^2$

Substituting $\theta = \frac{x}{l}$ (for small x), we have $U = \frac{mg}{2l}x^2$

(b) Since $F = \frac{dU}{dx} = 0$ at $x = 0$

We have $k_{eff} = \left(\frac{d^2U}{dx^2}\right)_{x=0} = \frac{mg}{l}$

ILLUSTRATION 5.24

A particle of mass m is located in a unidimensional potential field where potential energy of the particle depends on the coordinates x as: $U(x) = U_0(1 - \cos Ax)$; U_0 and A are constants.

Find the period of small oscillations that the particle performs about the equilibrium position.

Sol. The equation of potential energy of the particle as a function of position is given as

$$U_x = U_0(1 - \cos Ax) \quad ...(i)$$

Hence, force acting on the particle is given as

$$F_x = -\frac{dU_x}{dx} = U_0 A \sin Ax \quad ...(ii)$$

So, for the equilibrium of the body the force given by the above equation must be zero and d^2U/dx^2 should be negative. So, $F = 0$ situations are given by $0 = -U_0 A \sin Ax$

$$x = 0, \quad x = \frac{\pi}{A} \quad ...(iii)$$

Since $\frac{d^2U}{dx^2} = -U_0A^2\cos Ax$

So, at $x = 0$, $\frac{d^2U}{dx^2} = -U_0A^2\cos 0° = -U_0A^2$ i.e., negative.

So, $x = 0$ satisfies the condition for stable equilibrium situation.

Hence $K_{eff} = \left(\frac{d^2U}{dx^2}\right)_{r=0} = U_0A^2$ and $\omega^2 = \frac{K_{eff}}{m} = \frac{U_0A^2}{m}$

or $\omega = \sqrt{\frac{U_0A^2}{m}} = \frac{2\pi}{T}$ or $T = 2\pi\sqrt{\frac{m}{U_0A^2}}$

ILLUSTRATION 5.25

If a particle moves in a potential energy field $U = U_0 - ax + bx^2$, where a and b are positive constants, obtain an expression for the force acting on it as a function of position. At what point does the force vanish? Is this a point of stable equilibrium? Calculate the force constant and frequency of the particle.

Sol. $F = -\frac{dU}{dx} = a - 2bx$

$F = 0$, at $x = a/2b$

If stable equilibrium is present at this position, $\frac{d^2U}{dx^2} > 0$

Hence, $\frac{d^2U}{dx^2} = 2b > 0$

i.e., $x = \frac{a}{2b}$ is a point of minimum potential energy.

Hence, the equilibrium is stable. The particle will oscillate about $x = \frac{a}{2b}$. The effective force constant of oscillation

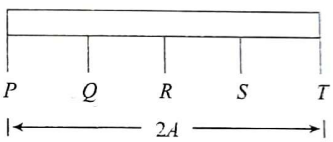
$$k_{eff} = \left(\frac{d^2U}{dx^2}\Big|_{x=\frac{a}{2b}}\right) = 2b$$

As $k_{eff} = m\omega^2 \Rightarrow 2b = m\omega^2$

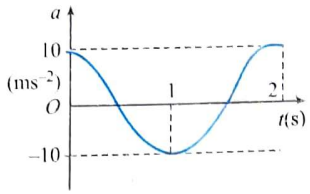
or $\omega^2 = \frac{2b}{m} \Rightarrow \omega = 2\pi f = \sqrt{\frac{2b}{m}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2b}{m}}$

CONCEPT APPLICATION EXERCISE 5.2

1. The motion of a particle is described by $x = 30\sin(\pi t + \pi/6)$, where x is in cm and t in sec. Find the position where potential energy of the particle is twice of kinetic energy for the first time after $t = 0$.
2. A particle starts oscillating simple harmonically with time period T from its equilibrium position. Find the ratio of kinetic energy and potential energy of the particle at the time $T/12$. (T = time period).
3. A body performs simple harmonic oscillations along the straight line $PQRST$ with R as the midpoint of PT. Its kinetic energies at Q and S are each one fourth of its maximum value. If $PT = 2A$, find the distance between Q and S.



4. The potential energy of a simple harmonic oscillator of mass 2 kg in its mean position is 5 J. If its total energy is 9 J and its amplitude is 0.01 m, find its time period.
5. An object of mass 0.2 kg executes SHM along the x-axis with frequency of $(25/\pi)$ Hz. At the point $x = 0.04$ m, the object has kinetic energy 0.5 J and potential energy 0.4 J. Find the amplitude of oscillation.
6. If you start measuring the time from the mean position, after what minimum time the energy of a horizontal spring mass system oscillating with time period T is half kinetic and half potential?
7. The potential energy between two atoms in a diatomic molecule varies with x as $U = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are positive constants. Find the equivalent spring constant for the oscillation of one atom if the other atom is kept stationary.
8. Potential energy (U) of a body of unit mass moving in a one-dimension conservative force field is given by, $U = (x^2 - 4x + 3)$ J. All units are in SI.
   (a) Find the equilibrium position of the body.
   (b) Show that oscillations of the body about this equilibrium position is simple harmonic motion and find its time period.
   (c) Find the amplitude of oscillations if speed of the body at equilibrium position is $2\sqrt{6}$ m/s.
9. **(i)** The acceleration versus time graph of a particle executing SHM is shown in the following figure. Plot the displacement versus time graph.



(ii) The frequency of oscillation is _____.
(iii) The displacement amplitude is _____.
(iv) At $t = 0$, the velocity of the particle is _____.
(v) The kinetic energy of the particle is maximum at $t =$ _____ and $t =$ _____.
(vi) The potential energy is maximum at $t =$ _____; $t =$ _____ and $t =$ _____.

10. A linear harmonic oscillator has a total mechanical energy of 200 J. Potential energy of it at mean position is 50 J. Find
(i) the maximum kinetic energy
(ii) the minimum potential energy
(iii) the potential energy at extreme positions.

11. The potential energy of a particle oscillating on x-axis is given as $U = 20 + (x - 2)^2$.
Here U is in joules and x in metres. Total mechanical energy of the particle is 36 J.
(a) State whether the motion of the particle is simple harmonic or not.
(b) Find the mean position.
(c) Find the maximum kinetic energy of the particle.

ANSWERS

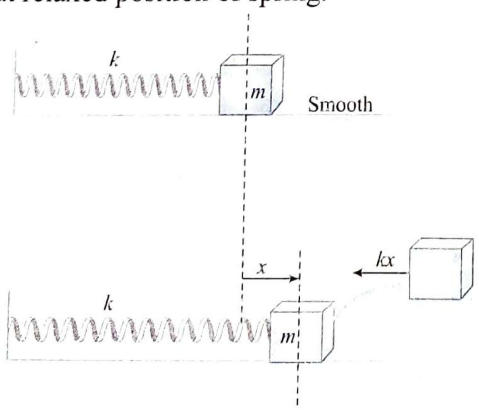
1. $x = \frac{\sqrt{6}}{10}$ m **2.** 3:1 **3.** $\sqrt{3}A$ **4.** $\frac{\pi}{100}$ s **5.** 0.06 m
6. $\frac{T}{8}$ **7.** $\frac{18b^2}{a}\left(\frac{b}{2a}\right)^{1/3}$ **8.** (a) $x = 2$ m (b) $\sqrt{2}\,\pi$s (c) $2\sqrt{3}$ m
9. (ii) 0.5 s^{-1} (iii) $\frac{10}{\pi^2}$ m (iv) zero (v) 0.5 s, 1.5 s (vi) 0, 1 s and 2 s
10. (i) 150 J (ii) 50 J (iii) 200 J **11.** (a) SHM (b) $x = 2$ m (c) 16 J

SPRING–MASS SYSTEM

CASE 1: SPRING–MASS SYSTEM OSCILLATING ON A SMOOTH HORIZONTAL SURFACE

Method 1: Force method

Let us find out the time period of a spring–mass system oscillating on a smooth horizontal surface as shown in the figure. The SHM will be about relaxed position of spring.



At the equilibrium position the spring is relaxed. When the block is displaced through a distance x towards right, it experiences a net restoring force $F = -kx$ towards left.

The negative sign shows that the restoring force is always opposite to the displacement. That is, when x is positive, F is negative, the force is directed to the left. When x is negative, F is positive, the force always tends to restore the block to its equilibrium position $x = 0$.

$$F = -kx$$

Applying Newton's second law,

$$F = m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$

Comparing the above equation with $a = d^2x/dt^2 = -\omega^2 x$, we get

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Note: Time period is independent of the amplitude. For a given spring constant, the period increases with the mass of the block that means more massive block oscillates more slowly. For a given block, the period decreases as k increases. A stiffer spring produces quicker oscillations.

Method 2: Energy method

The mechanical energy of the system will always remain conserved.

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Differentiating Eq. (i) w.r.t. time

$$\frac{1}{2}m2va + \frac{1}{2}k2xv = 0$$

$$a = -\left(\frac{k}{m}\right)x \quad \Rightarrow \omega^2 = \frac{k}{m}$$

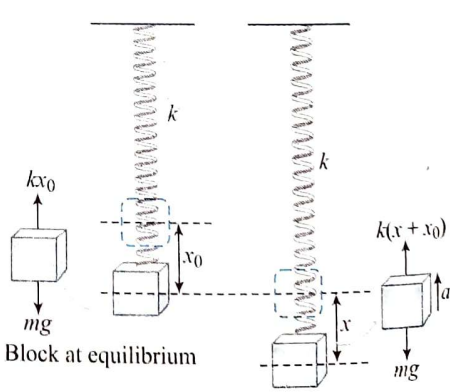
and $\omega = \sqrt{\frac{k}{m}} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$ H

CASE 2: SPRING–MASS SYSTEM OSCILLATING IN A VERTICAL PLANE

A block of mass m attached to a spring of stiffness k oscillates in a vertical plane. Let us find the period of oscillation of a vertical spring–mass system.

Method 1: Force method

Let x_0 be the deformation in the spring in equilibrium. Then $kx_0 = mg$. When the block is further displaced by x, the net restoring force is given by $F = -[k(x + x_0) - mg]$



Using second law of motion, $m\frac{d^2x}{dt^2} = -kx \quad \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$

Thus, $\omega^2 = \frac{k}{m} \quad \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$

Note: Gravity does not influence the time period of the spring–mass system; it merely changes the equilibrium position.

Method 2: Energy method

The mechanical energy of the system will always remain conserved.

$$\frac{1}{2}mv^2+\frac{1}{2}k\left(x_0+x\right)^2-mgx=\text{constant} \qquad ...(iii)$$

(considering equilibrium position as datum)

Differentiating Eq. (iii) w.r.t. time

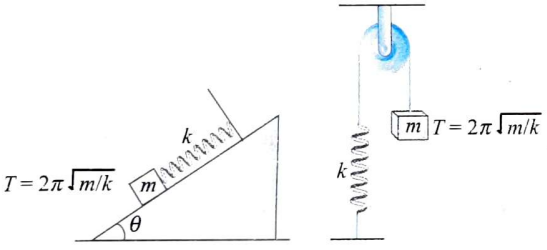
$$\frac{1}{2}m2va+\frac{1}{2}k2\left(x_0+x\right)v-mgv=0 \qquad ...(iv)$$

From Eqs. (i) and (iv), we get

$$a=-\left(\frac{k}{m}\right)x \quad \Rightarrow \omega=\sqrt{\frac{k}{m}} \quad \Rightarrow T=2\pi\sqrt{\frac{m}{k}}$$

Note: As we have seen that the time period of block–spring system depends only on the mass of the block and force constant of the spring, so the time period in the following cases is

$$T=2\pi\sqrt{\frac{m}{k}}$$



(a) (b)

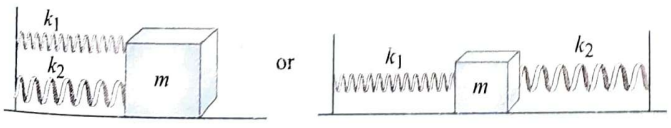
SERIES AND PARALLEL COMBINATION OF SPRINGS

(a) **Series combination of springs:** When two springs are joined in series, the equivalent stiffness of the combination may be obtained as

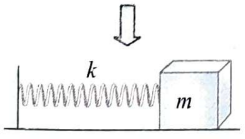
$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$$

Series combination of springs Equivalent spring

(b) **Parallel combination of springs:** When two springs are joined in parallel, the equivalent stiffness of the combination is given by $k=k_1+k_2$



Parallel combination of springs



Equivalent spring

ILLUSTRATION 5.26

A spring of stiffness constant k and natural length l is cut into two parts of length $3l/4$ and $l/4$, respectively, and an arrangement is made as shown in Figs. (a) and (b). If the mass is slightly displaced, find the time period of oscillation.

(a)

(b)

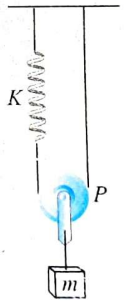
Sol. The stiffness of a spring is inversely proportional to its length. Therefore the stiffness of each part is

$$k_1=\frac{4}{3}k \quad \text{and} \quad k_2=4k$$

Time period, $T=2\pi\sqrt{\frac{m}{k_1+k_2}}=2\pi\sqrt{\frac{3m}{16k}}=\frac{\pi}{2}\sqrt{\frac{3m}{k}}$

ILLUSTRATION 5.27

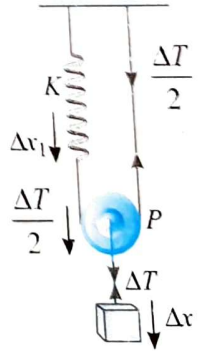
A block of mass m hangs from a smooth light pulley and by an inextensible string fitted a spring of stiffness k as shown in case (a) and case (b). Find the period of oscillation of the block.



Case (a)

Case (b)

Sol. **Case (a)**: In case of spring–particle system, gravity does not influence the time period. In such cases the force acting on the particle beyond equilibrium state of the particle is responsible for providing restoring force on it. This restoring force is responsible for acceleration of the particle. If we displace the block by a distance Δx and release, the tension in the string connecting the block and pulley will increase. This increased tension (say ΔT) will provide restoring force to the block. This increased tension (ΔT) will further increase the tension in the string connecting spring and the pulley.



Due to this increased tension in this string, the spring further elongate by a distance Δx_1.

By using constraint relation, we can write,

$$\Delta x_1=2\Delta x \qquad ...(i)$$

Also increase in spring force beyond equilibrium position,

$$\frac{\Delta T}{2} = k\Delta x_1 \quad \text{or} \quad \Delta x_1 = \frac{\Delta T}{2k} \qquad ...(ii)$$

From (i) and (ii), $\frac{\Delta T}{2k} = 2\Delta x$

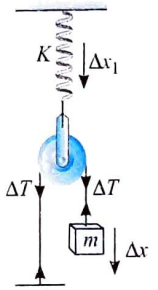
or $\Delta T = 4k\Delta x$

It means $ma = 4k\Delta x \Rightarrow a = \frac{4k\Delta x}{m}$

In vector form we can write $\vec{a} = -\frac{4k}{m}\Delta\vec{x}$ ...(iii)

Hence $\omega^2 = \frac{4k}{m} \Rightarrow T = 2\pi\sqrt{\frac{m}{4k}} = \pi\sqrt{\frac{m}{k}}$

Case (b) Here in this case also, we will consider additional force acting on the particle beyond its equilibrium condition. If we further displace the block Δx and release, the tension in the string connecting the block and pulley will increase. This increased tension (say ΔT) will provide restoring force to the block. This increased tension (ΔT) will further increase the tension in the string connecting spring and the pulley. Due to this increased tension in this string, the spring further elongate by a distance Δx_1.



By using constraint relation, we can write, $\Delta x_1 = \frac{\Delta x}{2}$ ...(i)

Also increase in spring force $k\Delta x_1 = 2\Delta T$ ...(ii)

From (i) and (ii), we have, $\Delta T = \frac{k\Delta x}{4}$

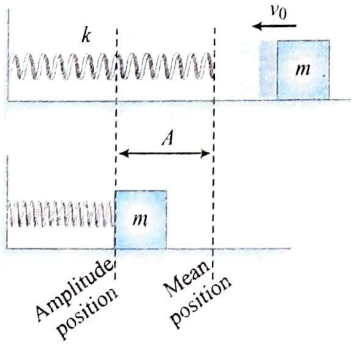
It means $ma = \frac{k}{4}\Delta x \Rightarrow a = \frac{k}{4m}\Delta x$

In vector form we can write $\vec{a} = -\frac{k}{4m}\Delta\vec{x}$ ...(iii)

Hence $\omega^2 = \frac{k}{4m} \Rightarrow T = 2\pi\sqrt{\frac{4m}{k}} = 4\pi\sqrt{\frac{m}{k}}$

Oscillation of a Block Not Connected with a Spring

Let us take a block of mass m is projected with a velocity v_0 so that it strikes a light spring of stiffness k. The block comes in contact with spring, compresses the spring to its maximum value then return back and leaves spring with same speed but in opposite direction. Here the time of oscillation of the block will not be same as we have calculated earlier.



As long as the block is in contact with the spring, it moves in SHM obeying the relation,

$$y = A\sin\omega t \qquad ...(i)$$

here $\omega = \sqrt{\frac{k}{m}}$ as described earlier.

For finding amplitude we can use conservation of mechanical energy. Using $\Delta K + \Delta U = 0$.

At maximum compression the speed of the block will be zero.

$$\left(0 - \frac{1}{2}mv_0^2\right) + \frac{1}{2}kA^2 = 0$$

Which gives $A = \left(\sqrt{\frac{m}{k}}\right)v_0$

If the block takes a time t_1 to reach its extreme (maximum compressed position of the spring) position ($y = A$), substituting $t = t_1$ and $y = A$ in the equation $y = A\sin\omega t$, we have

$$t_1 = \frac{1}{\omega}\sin^{-1}\left(\frac{A}{A}\right) = \frac{1}{\omega}\frac{\pi}{2} = \frac{\pi}{2\omega},$$

where $\omega = \sqrt{\frac{k}{m}}$

Hence, $t_1 = \frac{\pi}{2}\sqrt{\frac{m}{k}}$

Similarly we can prove that the time of the return journey of the block while in contact with the spring is equal to $\frac{\pi}{2}\sqrt{\frac{m}{k}}$.

Hence the time of contact of the block with the spring is

$$t_{\text{contact}} = 2t_1 = \pi\sqrt{\frac{m}{k}}$$

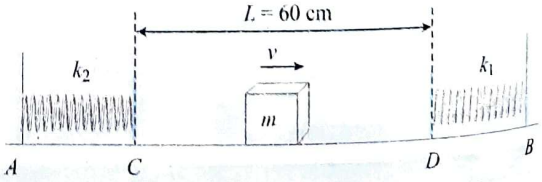
Another Approach

The time taken by block to move from mean position to amplitude position is one fourth of time period of SHM. At mean position the spring is relaxed so when the block comes in contact with spring is its mean position and position of maximum compression is the amplitude position. Hence time of contact of the block with spring should be

$$t_{\text{contact}} = 2\times\frac{T}{4} = \frac{T}{2} = \frac{1}{2}\left(2\pi\sqrt{\frac{m}{k}}\right) = \pi\sqrt{\frac{m}{k}}$$

Illustration 5.28

Two light springs of force constants k_1 and k_2 and a block of mass m are in the line AB on a smooth horizontal table such that one end of each spring is fixed on rigid supports and the other end is free as shown in the figure.



The distance CD between the free ends of the springs is 60 cm. If the block moves along AB with a velocity 120 cm/s, in between the springs, calculate the period of oscillation of the block (k_1 = 1.8 N/m, k_2 = 3.2 N/m and m = 200 g). Is the motion simple harmonic?

Sol. When the block touches D, it will compress the spring and its KE will be converted into elastic energy of the spring. The compressed spring will push the block to D with same speed; so time taken by the block to move from D towards B and back to D will be

$$t_1 = \frac{T_1}{2} = \pi\sqrt{\frac{m}{k_1}} = \pi\sqrt{\frac{0.2}{1.8}} = \frac{\pi}{3}\text{ s}$$

Similarly, time t_2 taken by block in contact with spring between A and C.

$$t_2 = \frac{T_2}{2} = \pi\sqrt{\frac{m}{k_2}} = \pi\sqrt{\frac{0.2}{3.2}} = \frac{\pi}{4}\text{ s}$$

Moreover during complete oscillation between A and B, the block moves the distance CD twice with uniform velocity v, once from C to D and again from D to C. So

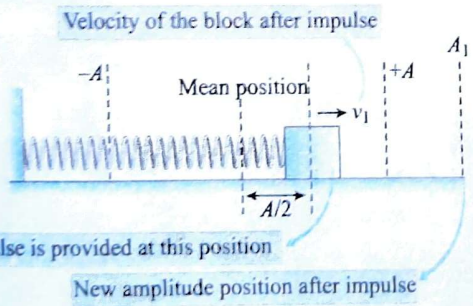
$$T_3 = \frac{2L}{v} = \frac{2\times 0.6}{1.2} = 1\text{ s}$$

$$T = t_1 + t_2 + t_3 = \pi\left(\frac{1}{3}+\frac{1}{4}\right)+1 = 2.82\text{ s}$$

Now a motion is simple harmonic only and only if throughout the motion $F = -kx$. Here between C and D, $F = 0$ (as v = constant); the motion is not simple harmonic but oscillatory.

ILLUSTRATION 5.29

A particle of mass m connected with a spring of force constant k is executing simple harmonic motion with amplitude A. When it is at $+\frac{A}{2}$ from equilibrium position moving towards right, it receives an impulse which doubles its velocity without changing direction.



Find

(a) velocity of the particle after the impulse,
(b) amplitude after the impulse, and
(c) time taken by the block to reach the right extreme position.

Sol. Time period of oscillation of the particle

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ or } \omega = \sqrt{\frac{k}{m}} \qquad \ldots(i)$$

We define velocity of the particle in SHM by relation

$$v = \omega\sqrt{A^2 - y^2} = \frac{2\pi}{T}\sqrt{A^2 - y^2} \qquad \ldots(ii)$$

Here $y = \frac{A}{2}$, hence the velocity of the particle at this position,

$$v = \frac{2\pi}{T}\sqrt{A^2 - \left(\frac{A}{2}\right)^2} = \frac{\pi\sqrt{3}A}{T}$$

After impulse the velocity of the particle doubles, hence velocity of the particle after impulse

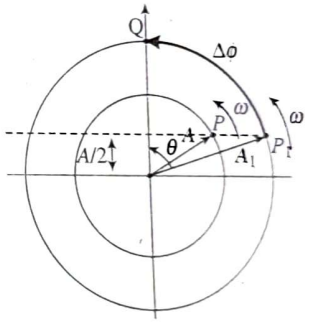
$$v' = 2v = \frac{2\pi\sqrt{3}A}{T} \qquad \ldots(iii)$$

After impulse the time period of the particle and angular frequency remains unchanged as neither mass nor spring constant is changing but amplitude of oscillation will change. Let new amplitude is A_1, then again using Eq. (ii), we get

$$v' = \frac{2\pi\sqrt{3}}{T} = \frac{2\pi}{T}\sqrt{A^2 - y^2}$$

$$\sqrt{3}A = \sqrt{A_1^2 - \left(\frac{A}{2}\right)^2} \Rightarrow A_1 = \frac{\sqrt{13}}{2}A$$

For calculating time taken by the block to reach the right extreme position, let us draw phasor circle correspond to this situation.



The circle corresponding to radius A represents the situation just before impulse and the circle of radius A represents phasor circle after the impulse. As the direction of motion of the particle remains unchanged, hence we need to find the time in reaching from position P_1 to Q or time corresponding to change in phase angle θ on the phasor circle corresponding to radius A_1.

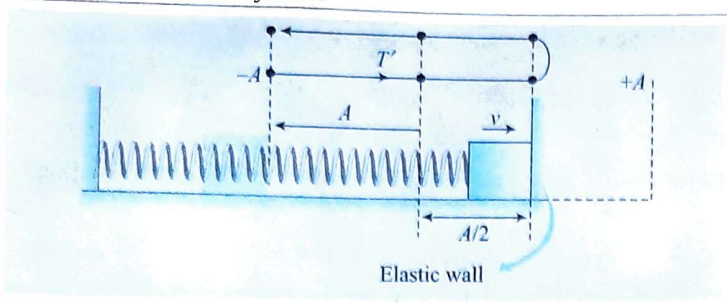
Hence $\cos\theta = \frac{A/2}{A_1} = \frac{A\cdot 2}{2\sqrt{13}A} = \frac{1}{\sqrt{13}}$

But $\theta = \omega t \Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{13}}\right) = \omega t$

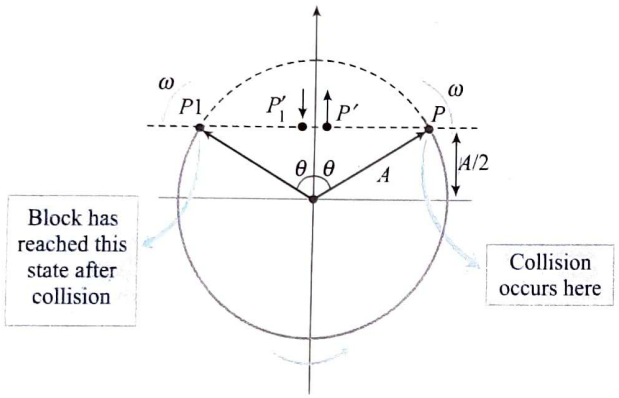
$$\cos^{-1}\left(\frac{1}{\sqrt{13}}\right) = \left(\sqrt{\frac{k}{m}}\right)t \quad \Rightarrow \quad t = \sqrt{\frac{m}{k}}\cos^{-1}\left(\frac{1}{\sqrt{13}}\right)$$

ILLUSTRATION 5.30

A block of mass m connected to a spring of force constant k is performing simple harmonic motion. Initially, the block is compressed by a distance A and released. An elastic wall is located in front of block at a distance of $A/2$. Find the time period of oscillation of this block.



Sol. In given situation, the block will not oscillate with time period T, as it is not free to move upto amplitude position on right side of its mean position. The block collides elastically with the wall at a distance $A/2$ from mean position, returns back and completes one oscillation. We can draw a phasor circle corresponding to this situation as shown in the figure.



Here $\cos\theta = \dfrac{A/2}{A} = \dfrac{1}{2}$ or $\theta = \dfrac{\pi}{3}$

The phase of the block has changed by $\dfrac{2\pi}{3}$ during collision.

The time interval (Δt) corresponding to phase change $\Delta\phi = \dfrac{2\pi}{3}$,

$$\Delta t = \frac{T}{2\pi}(\Delta\phi) = \frac{T}{2\pi}\times\frac{2\pi}{3} = \frac{T}{3}$$

Hence, the time period corresponding to this situation,

$$T' = T - \frac{T}{3} = \frac{2T}{3}$$

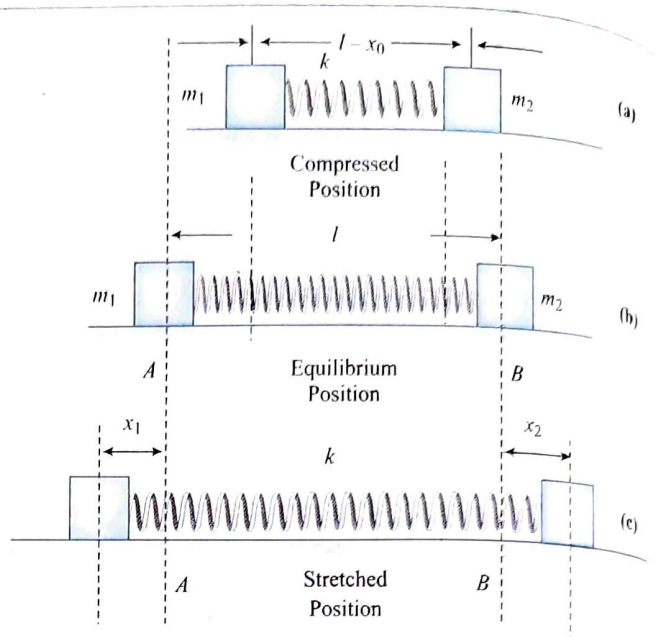
The time period of the block connected with spring is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Hence, $T' = \dfrac{2}{3}\left(2\pi\sqrt{\dfrac{m}{k}}\right) = \dfrac{4\pi}{3}\sqrt{\dfrac{m}{k}}$

OSCILLATION OF A TWO-PARTICLE SYSTEM TWO-BODY PROBLEM

Two blocks of masses m_1 and m_2 are connected with a spring of natural length l and spring constant k. The system is lying on a frictionless horizontal surface. Initially spring is compressed by a distance x_0 as shown in the figure.



The two blocks will perform SHM about their equilibrium position. We will discuss in this section about the oscillation of the system. If we release the blocks.

(a) **Time period of the blocks:** Here both the blocks will be in equilibrium at the same time when spring is in its natural length. Let A and B be equilibrium positions of blocks m_1 and m_2 as shown in the figure.

Let at any time during oscillations, blocks are at a distance of x_1 and x_2 from their equilibrium positions (A and B) [Fig. (c)].

As no external force is acting on the spring–block system, the displacement of centre of mass of the system, $\Delta x_{CM} = 0$

$$\therefore \quad (m_1 + m_2)\Delta x_{CM} = m_1 x_1 - m_2 x_2$$

$$\Rightarrow \quad m_1 x_1 = m_2 x_2$$

For first particle, force equation can be written as

$$k(x_1 + x_2) = m_1\frac{d^2x_1}{dt^2}$$

or, $$k\left(x_1\frac{m_1}{m_2} + x_1\right) = m_1 a_1$$

or, $$a_1 = \frac{k(m_1 + m_2)}{m_1 m_2}x_1$$

$$\because \quad \omega^2 = \frac{k(m_1 + m_2)}{m_1 m_2}$$

Using vector we can write $\vec{a}_1 = -\dfrac{k(m_1 + m_2)}{m_1 m_2}\vec{x}_1$

'–' sign is included as $\vec{a}_1$ and $\vec{x}_1$ are opposite to each other.

Hence, $T = 2\pi\sqrt{\dfrac{m_1 m_2}{k(m_1 + m_2)}} = 2\pi\sqrt{\dfrac{\mu}{k}}$

where $\mu = m_1 m_2/(m_1 + m_2)$ which is known as reduced mass.

Similarly time period of 2nd particle can be found. Both will have the same time period.

(b) Amplitude of the particles: Let the amplitude of blocks be A_1 and A_2.

$m_1A_1 = m_2A_2$

By energy conservation, $\frac{1}{2}k(A_1 + A_2)^2 = \frac{1}{2}kx_0^2$

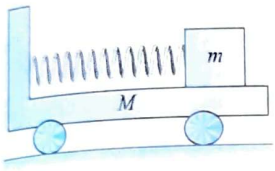
$A_1 + A_2 = x_0$ or $A_1 + \frac{m_1}{m_2}aA_1 = x_0$

or $A_1 = \frac{m_2x_0}{m_1 + m_2}$

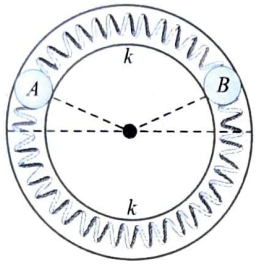
Similarly, $A_2 = \frac{m_1x_0}{m_1 + m_2}$

ILLUSTRATION 5.31

Figures (a) and (b) represent spring–block system. If m is displaced slightly, find the time period of oscillation of the system.



(a)



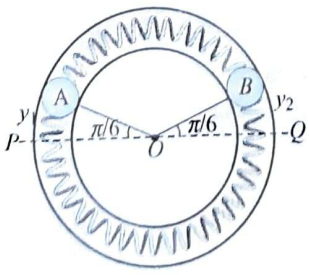
(b)

Sol. Both the cases are as follows:

Reduced mass of the system	Reduced mass of the system
$\mu = \left(\frac{mM}{m+m}\right)$	$\mu = \frac{mm}{m+m} = \frac{m}{2}$
	and $k_e = k + k = 2k$
$\therefore \quad T = 2\pi\sqrt{\frac{\mu}{k}}$	$\therefore \quad T = 2\pi\sqrt{\frac{\mu}{k_e}}$

ILLUSTRATION 5.32

Two identical balls A and B, each of mass 0.1 kg, are attached to two identical massless springs. The spring–mass system is constrained to move inside a rigid smooth pipe bent in the form of a circle as shown in figure. The pipe is fixed in a horizontal plane. The centres of the balls can move in a circle of radius 0.06 m. Each spring has a natural length of 0.06 π m and force constant 0.1 N/m. Initially, both the balls are displaced by an angle $\theta = \pi/6$ radian with respect to diameter PQ of the circle and released from rest.



(a) Calculate the frequency of oscillation of the ball B.

(b) What is the total energy of the system?

(c) Find the speed of the ball A when A and B are at the two ends of the diameter PQ.

Sol.

(a) As here two masses A and B are connected by two springs. this problem is equivalent to the oscillation of a reduced mass m by a spring of effective force constant k_{eff} given by

$$m = \frac{m_1m_2}{m_1 + m_2} = \frac{0.1 \times 0.1}{0.1 + 0.1} = 0.05 \text{ kg}$$

and $k_{eff} = k_1 + k_2 = 0.1 + 0.1 = 0.2$ N/m

So $f = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m}} = \frac{1}{2\pi}\sqrt{\frac{0.20}{0.05}} = \frac{1}{\pi}$ Hz

(b) As here one spring is compressed while the other is stretched by same amount (say y) and balls are at rest at A and B, so

$$E = \frac{1}{2}k_1y^2 + \frac{1}{2}k_2y^2 = ky^2 \qquad [\text{as } k_1 = k_2]$$

But from the above figure

$y = y_1 + y_2 = R\theta_1 + R\theta_2 = 2R\theta$

$y = 2 \times 0.06 \times (\pi/6) = 0.02\,\pi$ m

So $E = (0.1)(0.02\pi)^2 = 4\pi^2 \times 10^{-5}$ J

(c) As at P and Q springs are unstretched so the whole energy becomes kinetic of the balls A and B, i.e.

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = E = 4\pi^2 \times 10^{-5} \text{ J}$$

Here $m_1 = m_2 = 0.1$ kg and $v_1 = v_2 = v$

So $0.1\,v^2 = 4\pi^2 \times 10^{-5}$, i.e., $v = 2\pi \times 10^{-2}$ m/s

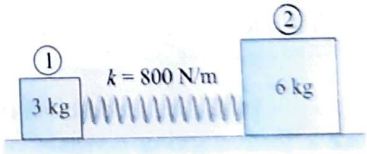
ILLUSTRATION 5.33

The system shown in the figure can move on a smooth surface. The spring is initially compressed by 6 cm and then released. Find

(a) time period

(b) amplitude of 3 kg block

(c) maximum momentum of 6 kg block.



Sol.

(a) $T = 2\pi\sqrt{\frac{\mu}{k}}$

Here, $\mu = \frac{m_1m_2}{m_1 + m_2} = \frac{3 \times 6}{9} = 2$ kg

$\therefore \quad = 2\pi\sqrt{\frac{2}{800}} = 2\pi \times \frac{1}{20}$

$T = \frac{\pi}{10}$ sec

(b) Here compression of spring, $x_0 = 6$ cm

Let displacements of block (1) and (2) w.r.t. center of mass be x_1 and x_2

But $x_1 + x_2 = x_0$ and $m_1x_1 = m_2x_2$

or $3x_1 = 6x_2$ or $x_1 = 2x_2$

$\therefore \quad x_1 + x_2 = x_0 \quad \text{or} \quad x_1 + \frac{x_1}{2} = x_0$

or $\quad \frac{3}{2}x_1 = x_0 \Rightarrow x_1 = \frac{2}{3}x_0 = \frac{2}{3} \times 6 = 4 \text{ cm}$

(c) Applying conservation principle of momentum

$3 \times v_1 - 6 v_2 = 0$

$\therefore \quad 3v_1 = 6v_2$

$\therefore \quad v_1 = 2v_2$

Applying mechanical energy conservation, we get

$$\frac{1}{2} \times 3 \times v_1^2 + \frac{1}{2} \times 6 \times v_2^2 = \frac{1}{2} k x_0^2$$

or $$\frac{3}{2} v_1^2 + 3v_2^2 = \frac{1}{2} \times 800 \times (0.06)^2$$

or $$\frac{3}{2}(2v_2)^2 + 3v_2^2 = 400 \times 36 \times 10^{-4}$$

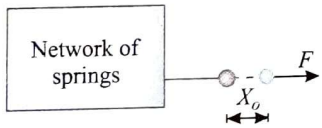
or $$v_2 = \frac{1.2}{3} = 0.4 \text{ m/s}$$

$\therefore \quad P_{2\max} = 6 \times 0.4 = 2.4 \text{ kg m/s}$

CALCULATION OF K_{EFF} FOR A COMPLICATED SPRING–MASS SYSTEM

If we analyze the oscillation of a particle connected with a complicated network of springs, it may take a longer time to find the frequency of oscillation of the block by the usual procedure. Hence, there is a need to design a procedure so as to quickly solve the problem of oscillation of a particle connected with such network.

We have a formula of angular velocity of a particle connected with a single spring, $\omega = \sqrt{\frac{k}{m}}$



This formula is also valid for a combination of springs if we take 'k' as the effective spring constant of the system of springs. Instead of k, let us write k_{eff} and recast the formula as:

$$\omega = \sqrt{\frac{k_{eff}}{m}}$$

For finding k_{eff}, we pull the particle by applying an external force F. If it remains in equilibrium after moving through a distance x_0, the effective spring constant can be given as:

$$k_{eff} = \frac{F}{x_0}$$

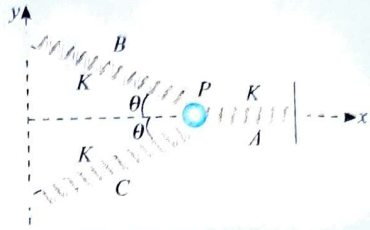
Once we have k_{eff}, we can find ω by using the formula,

$$\omega = \sqrt{\frac{k_{eff}}{m}}, \text{ where } k_{eff} = \frac{F}{x_0}$$

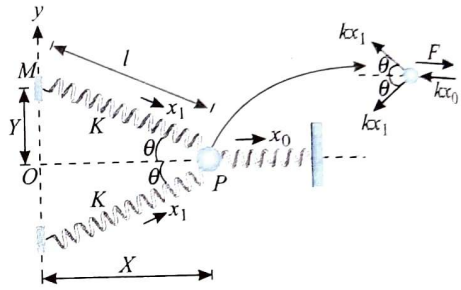
This gives $\omega = \sqrt{\frac{F}{mx_0}}$, where x_0 is called "static deformation" of the system when the force F is applied.

ILLUSTRATION 5.34

A particle 'P' of mass m is placed on a smooth horizontal surface is attached to three identical springs A, B and C each of force constant K as shown in figure. If the particle is pushed slightly against spring A and released. Find the time period of oscillation of the particle.



Sol. Let us apply a force F at 'P' along positive x-direction so that the system remains in equilibrium. This will compress the spring A by 'x_0' and stretch the springs B and C by 'x_1'.



In ΔMOP: $X^2 + Y^2 = l^2$

Differentiating above equation both sides, we get

$$2X\,dX + 2Y\,dY = 2l \cdot dl$$

As y is constant hence $dY = 0$

Now Eq. (i) becomes $2X\,dX = 2l\,dl$

or $$\frac{dl}{dX} = \frac{x_1}{x_0} = \frac{X}{l} \Rightarrow x_1 = x_0 \cos\theta$$

As the block is at equilibrium, then $F = Kx_0 + 2Kx_1 \cos\theta$ ....

From (ii) and (iii) we get, $F = Kx_0 + 2Kx_0 \cos^2\theta$

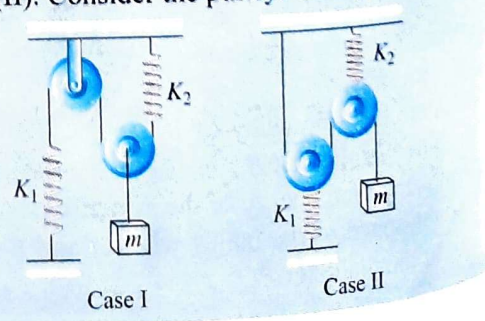
or $$\frac{F}{x_0} = K(1 + 2\cos^2\theta) = K_{eff}$$

It means, $$\omega^2 = \frac{K_{eff}}{m} = \frac{K(1 + 2\cos^2\theta)}{m}$$

$$\Rightarrow \quad T = 2\pi \sqrt{\frac{m}{K(1 + 2\cos^2\theta)}}$$

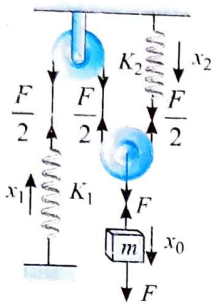
ILLUSTRATION 5.35

Find the time period of oscillation of the block of mass m in case (I) and (II). Consider the pulleys are light and small.



Sol. **Case (I):** Let the block is displaced by a distance 'x_0' beyond its initial equilibrium position by applying a force F so that the system remains in equilibrium. At this position the tension in the string connecting the block the pulley will increase and this increased tension is equal to the force F.

This increased tension (F), further increase the tension in the string connecting the springs. Because of this increased tension, both the springs will further extend. Let the further extensions of the springs are x_1 and x_2.



Hence $\frac{F}{2} = K_1x_1 = K_2x_2$ ...(i)

Using constraint relation, we can write, $x_1 + x_2 = 2x_0$ ...(ii)

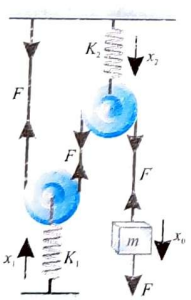
From (i) and (ii), we can write $\frac{F}{2K_1} + \frac{F}{2K_2} = 2x_0$

or $F = \frac{4K_1K_2}{(K_1 + K_2)}x_0$ ... (iii)

or $\frac{F}{x_0} = \frac{4K_1K_2}{(K_1 + K_2)} = K_{\text{eff}}$

Hence $\omega^2 = \frac{K_{\text{eff}}}{m} = \frac{4K_1K_2}{m(K_1 + K_2)}$ or $T = 2\pi\sqrt{\frac{m(K_1 + K_2)}{(4K_1K_2)}}$

Case (II): Adopting the same procedure as we have done in case (I). Let the block is displaced by a distance 'x_0' beyond its initial equilibrium position by applying a force F so that the system remains in equilibrium. Let the springs K_1 and K_2 will further elongate by x_1 and x_2 respectively.



The constraint relation should be: $x_0 = 2(x_1 + x_2)$ ...(i)

If both the pulleys are light, we can write $2F = K_1x_1 = K_2x_2$

$x_1 = \frac{2F}{K_1}$ and $x_2 = \frac{2F}{K_2}$ ...(ii)

From (i) and (ii) we can write, $x_0 = 2\left[\frac{2F}{K_1} + \frac{2F}{K_2}\right]$

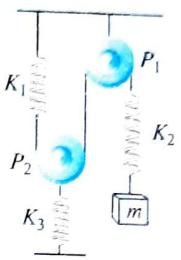
$F = \frac{K_1K_2}{4(K_1 + K_2)}x_0$ or $\frac{F}{x_0} = \frac{K_1K_2}{4(K_1 + K_2)} = K_{\text{eff}}$

Hence $\omega = \frac{K_{\text{eff}}}{m} = \frac{K_1K_2}{4m(K_1 + K_2)}$

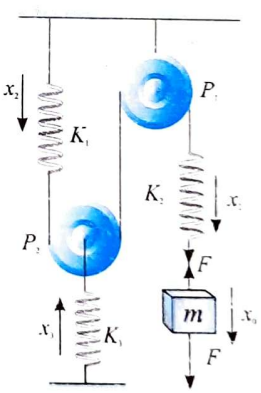
Or, $T = 2\pi\sqrt{\frac{4m(K_1 + K_2)}{K_1K_2}} = 4\pi\sqrt{\frac{m(K_1 + K_2)}{K_1K_2}}$

ILLUSTRATION 5.36

A block of mass m hangs from a smooth fixed pulley P_1 by an inextensible string filled with two springs of stiffness k_1 and k_2. The string passes over a movable smooth light pulley P_2 which is connected with another ideal spring of stiffness k_3. Find the period of oscillation of the block.



Sol. Let the block is displaced by a distance 'x_0' beyond its initial equilibrium position by applying a force F so that the system remains in equilibrium. This will increase in the tension in the string connecting block and spring and this increased tension is equal to the force F. Due to this increase in tension in the springs k_1, k_2 and k_3 further elongate by x_1, x_2 and x_3 respectively.



The constraint relation should be: $x_1 + x_2 + 2x_3 = x_0$ ...(i)

The spring k_1 and k_2 are connected with string, hence

$K_1x_1 = K_2x_2 = F$ ...(ii)

It gives $x_1 = \frac{F}{K_1}$ and $x_2 = \frac{F}{K_2}$ as pulleys are light, $2F = K_3x_3$

or $x_3 = \frac{2F}{K_3}$

Substituting the values of x_1, x_2 and x_3 in Eq. (i)

$\left(\frac{F}{K_1} + \frac{F}{K_2} + 4\frac{F}{K_3}\right) = x_0$ or $F = \frac{1}{\left(\frac{1}{K_1} + \frac{1}{K_2} + \frac{4}{K_3}\right)} \cdot x_0$

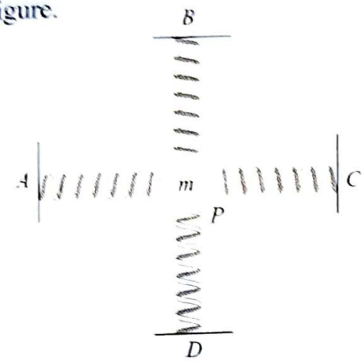
Hence $\frac{F}{x_0} = \frac{1}{\left(\frac{1}{K_1} + \frac{1}{K_2} + \frac{4}{K_3}\right)} = K_{\text{eff}}$

$$\Rightarrow \quad \omega^2 = \frac{K_{eff}}{m} = \frac{1}{m\left(\frac{1}{K_1}+\frac{1}{K_2}+\frac{4}{K_3}\right)}$$

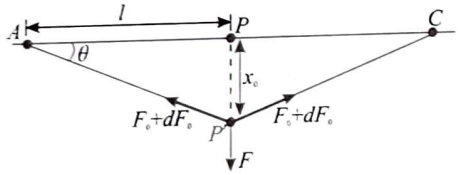
or $$T = 2\pi\sqrt{m\left(\frac{1}{K_1}+\frac{1}{K_2}+\frac{4}{K_3}\right)}$$

ILLUSTRATION 5.37

The figure shows a particle of mass 'm', attached with four identical springs, each of length l. Initial tension in each spring is F_0. Neglecting gravity, calculate period of small oscillations of the particle along a line perpendicular to the plane of the figure.



Sol. Let the particle be displaced slightly through a displacement dx along a line normal to the plane of the figure by applying a dF so that the particle is still in equilibrium. Since, springs are identical, therefore increase in tension of each spring will be the same. Let this increase be dF_0. Let at this time, the spring force in each spring be $F_0 + dF_0$. The horizontal components of this force will get cancelled and we will be left with vertical upward components of four springs.



For the equilibrium of the particle, $dF = 4(F_0 + dF_0)\sin\theta$ ...(i)

As θ is small angle we can write $\sin\theta \simeq \tan\theta$. Now Eq. (i) can be written as

$$dF = 4(F_0 + dF_0)\tan\theta = 4(F_0 + dF_0)\frac{dx}{l} = 4F_0\frac{dx}{l} + 4dF_0\frac{dx}{l}$$

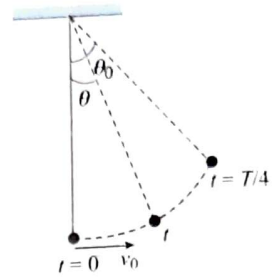
Neglecting product of small quantities above equation can be written as $dF = \frac{4F_0}{l}\cdot dx$. Hence $\frac{dF}{dx} = \frac{4F_0}{l} = K_{eff}$.

It means $\omega^2 = \frac{K_{eff}}{m} = \frac{4F_0}{ml}$ or $T = 2\pi\sqrt{\frac{ml}{4F_0}} = \pi\sqrt{\frac{ml}{F_0}}$

ANALYSIS OF ANGULAR SHM

In case of a simple pendulum a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support.

When the particle is at its extreme position, its angular displacement θ_0 can be regarded as the angular amplitude of oscillations. As the displacement of bob from its mean position is given as



$$x = A\sin(\omega t + \alpha) \quad \text{[general equation of SHM]} \qquad ...(i)$$

If θ and θ_0 are angular displacement and angular amplitude of bob, respectively, we have $\theta = \frac{x}{l}$ and $\theta_0 = \frac{A}{l}$.

Thus, general equation of SHM of bob in angular form can be given by substituting the values of x and in Eq. (i) as

$$\theta = \theta_0\sin(\omega t + \phi_0) \qquad ...(ii)$$

Using the above equation we can find the angular velocity of the body which in angular SHM is

$$\dot{\theta} = \frac{d\theta}{dt} = \theta_0\omega\cos(\omega t + \phi_0) \qquad ...(iii)$$

$$= \omega\sqrt{\theta_0^2 - \theta^2} \qquad ...(iv)$$

Note: Here we represent angular velocity $\frac{d\theta}{dt}$ by $\dot{\theta}$, not ω, as the notation ω is already being used for angular frequency of body in SHM.

Similarly angular acceleration of body is given as

$$\alpha = \frac{d^2\theta}{dt^2} = -\theta_0\omega^2\sin(\omega t + \phi_0) \qquad ...(v)$$

$$\tau_R = -I\alpha = -I\omega^2\theta \qquad ...(vi)$$

Thus restoring torque on body is given as

$$\tau_R = -I\ddot{\alpha} = -I\omega^2\theta \qquad ...(vii)$$

Thus we can state that in angular SHM, angular acceleration of the body and the restoring torque on the body are directly proportional to the angular displacement of body from its mean position and are directed toward mean position. Similarly basic differential equation for angular SHM can be written as

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0 \qquad ...(viii)$$

In terms of angular velocity and angular displacement we can write above equation as $(\dot{\theta})^2 + \omega^2(\theta)^2 = \text{constant}$. where $\dot{\theta}$ = angular velocity and θ = angular displacement.

OSCILLATION OF SIMPLE PENDULUM

As Angular SHM

Length of the simple pendulum is the distance between the point of suspension and the centre of mass of the suspended mass.

Consider the bob when string deflects through a small angle θ.

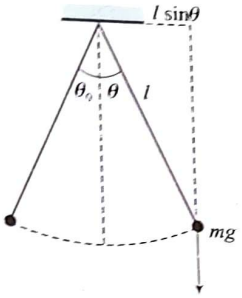
Forces acting on the bob are tension T in the string and weight mg of the bob.

Torque on the bob about point O is $\tau = \tau_{mg} + \tau_T = mgl\sin\theta + 0$

$$\tau = mgl\theta \quad \text{(as } \theta \text{ is very small)} \qquad ...(i)$$

Since MI of the bob about point O is $I = ml^2$,

$$\tau = ml^2\frac{d^2\theta}{dt^2} \qquad ...(ii)$$



As torque τ and θ are oppositely directed, from Eqs. (i) and (ii), we get

$$ml^2\frac{d^2\theta}{dt^2} = -mgl\theta$$

$$\frac{d^2\theta}{dt^2} = -(g/l)\theta$$

Comparing with the equation $d^2x/dt^2 = -\omega^2 x$, we get

$$\omega = \sqrt{\frac{g}{l}}$$

Since $T = 2\pi/\omega \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{l}{g}}$

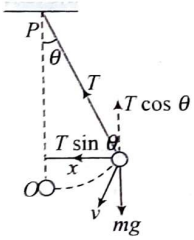
As Linear SHM

At equilibrium position the bob of pendulum is at lowest position, i.e., the string is in vertical position.

Let us pull the bob from its mean (lowest) position O, through a small distance x.

The pendulum bob oscillates simple harmonically experiencing a restoring force $mg \sin\theta$ with small linear and angular displacements.

Referring to the FBD (see figure), the net horizontal force acting on the bob is $F_{net} = T\sin\theta$.



Since θ is very small, $\sin\theta = \theta$; $\theta \simeq \frac{x}{l}$.

Then, we have $F_{net} = \frac{T}{l}x$.

As the net force, that is, restoring force acts (pushes the bob) towards the mean position, it opposes the displacement x from the mean position. Then, we can write vectorially as

$$\vec{F}_{net} = -\left(\frac{T}{l}\right)\vec{x}$$

Since, $T = mv^2/l + mg\cos\theta$ neglecting v^2/l for small speed of the bob (as it falls from a very small angle) and substituting $\cos\theta = 1$, we have

Equivalent force constant (k_{eq}) for simple pendulum: Finally substituting $T = mg$ from Eq. (ii) in Eq. (i), we have

$$\vec{F}_{net} = -\left(\frac{T}{l}\right)x = \frac{-mg}{l}\vec{x}$$

Frequency and time period: Comparing the above equation with $F_{net} = -kx$, the effective spring constant is $k_{eff} = mg/l$.

As we know $k_{eq} = m\omega^2$

This gives $\omega = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{g}{l}}$

Then the time period of the simple pendulum is $T = 2\pi/\omega$, where $\omega = \sqrt{g/l}$

This gives $T = 2\pi\sqrt{\frac{l}{g}}$

Note:

- In simple pendulum we can find the value of equivalent force constant of oscillation $k_{eq} = T/l$; where T is the tension in the string at equilibrium position of the bob and l is the length of the string.
- Even in case of simple pendulum in accelerated frame we can use the same approach for calculating the equivalent force of oscillation. In this case T is the tension in the string at equilibrium position with respect to accelerated frame.

TIME PERIOD OF SIMPLE PENDULUM IN ACCELERATING REFERENCE FRAME

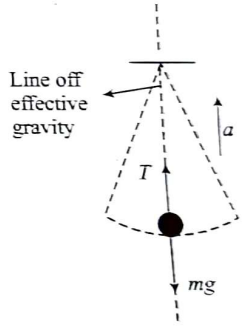
Time Period of Simple Pendulum When Point of Suspension is Accelerating

1. **Acceleration up or retardation down:**

From free body diagram of bob

$$T - mg = ma, \text{ i.e., } T = m(g + a)$$

$$k_{eq} = \frac{T}{l} = \frac{m(g+a)}{l}$$



Hence $\omega = \sqrt{\frac{k_{eq}}{m}}$

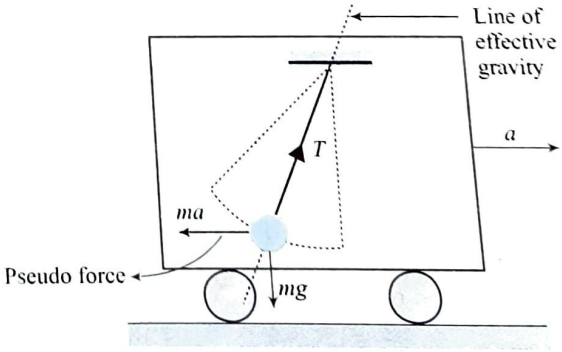
$$\Rightarrow \quad T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{(g+a)}}$$

2. **Acceleration down or retardation up:**

$$K_{eq} = \frac{T}{l} = \frac{m(g-a)}{l}$$

$$\omega = \sqrt{\frac{k_{eq}}{m}} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{(g-a)}}$$

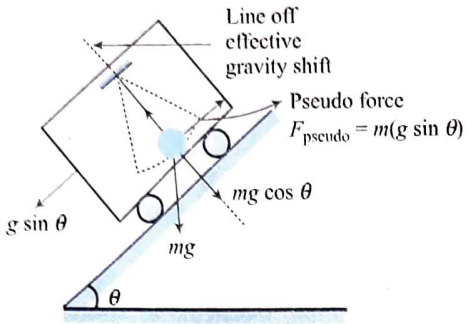
3. **Pendulum is accelerating horizontally:**



$$k_{eq} = \frac{T}{l} = \frac{m\sqrt{a^2+g^2}}{l}$$

$$\omega = \sqrt{\frac{k_{eq}}{m}} \Rightarrow T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g^2+a^2}}$$

4. **Pendulum accelerating down an incline:**



$$T = m\sqrt{g^2+(g\sin\theta)^2+2g(g\sin\theta)\cos(90°+\theta)}$$
$$= mg\cos\theta$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

$$\therefore \quad k_{eq} = \frac{T}{l} = \frac{mg\cos\theta}{l}$$

ILLUSTRATION 5.38

A simple pendulum of length 40 cm oscillates with an angular amplitude of 0.04 rad. Find **(a)** the time period, **(b)** the linear amplitude of the bob, **(c)** the speed of the bob when the string makes 0.02 rad with the vertical and **(d)** the angular acceleration when the bob is in momentary rest. Take $g = 10\ m/s^2$.

Sol.

(a) The angular frequency is $\omega = \sqrt{\frac{g}{l}} = \sqrt{\frac{10\ m/s^2}{0.4\ m}} = 5\ s^{-1}$

The time period is $\frac{2\pi}{\omega} = \frac{2\pi}{5\ s^{-1}} = 1.26\ s$

(b) Linear amplitude = 40 cm × 0.04 = 1.6 cm

(c) Angular speed $\dot{\theta} = \omega\sqrt{\theta_0^2 - \theta^2}$

$$\dot{\theta} = (5\ s^{-1})\sqrt{(0.04)^2-(0.02)^2}\ rad = 0.17\ rad/s$$

where speed of the bob at this instant

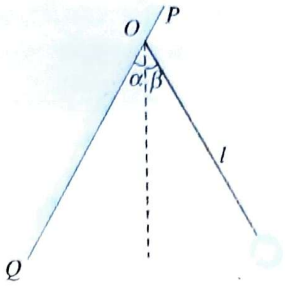
$$V = \dot{\theta}A = (40\ cm)\times(0.17\ s) = 6.8\ cm/s$$

(d) At momentary rest, the bob is in extreme position. Thus, the angular acceleration $|\alpha| = \omega^2\theta$.

$$|\alpha| = (0.04\ rad)\ (25\ s^{-2}) = 1\ rad/s^2$$

ILLUSTRATION 5.39

A ball is suspended by a thread of length l at the point O on an incline wall as shown. The inclination of the wall with the vertical is α. The thread is displaced through a small angle β away from the vertical and the ball is released. Find the period of oscillation of pendulum. Consider both cases.



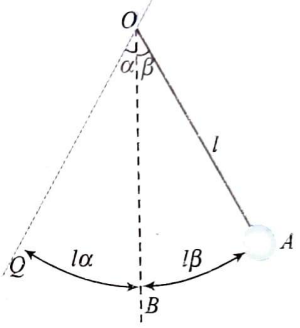
(a) $\alpha > \beta$ **(b)** $\alpha < \beta$

Assume that any impact between the wall and the ball is elastic.

Sol.

(a) If $\alpha > \beta$, the ball does not collide with the wall and it performs full oscillations like simple pendulum. Thus,

$$\text{Period} = 2\pi\sqrt{\frac{l}{g}}$$



(b) If $\alpha < \beta$, the ball collides with the wall and rebounds with same speed; the motion of ball from A to Q is one part of a simple pendulum time period of ball = 2 (t_{AQ}).

Consider A as the starting point ($t = 0$), equation of motion is $x(t) = A\cos\omega t$.

$x(t) = l\beta\cos\omega t$, because amplitude = $A = l\beta$.

Time from A to Q is the time t when x becomes $-l\alpha$.

$$\Rightarrow \quad -l\alpha = l\beta\cos\omega t \quad \Rightarrow t = t_{AQ} = 1/\omega\cos^{-1}\left(\frac{-\alpha}{\beta}\right)$$

The return path from Q to A will involve the same time interval.

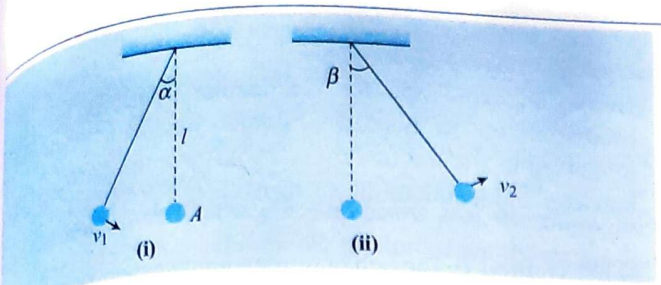
Hence time period of ball $T = 2t_{AQ}$

$$\Rightarrow \quad T = \frac{2}{\omega}\cos^{-1}\left(-\frac{\alpha}{\beta}\right) = 2\sqrt{\frac{l}{g}}\cos^{-1}\left(\frac{-\alpha}{\beta}\right)$$

$$= 2\pi\sqrt{\frac{l}{g}} - 2\sqrt{\frac{l}{g}}\cos^{-1}\left(\frac{\alpha}{\beta}\right)$$

ILLUSTRATION 5.40

Figure shows two identical simple pendulums of length l. One is tilled at an angle α and imparted an initial velocity v_1 toward mean position and at the same time other one is projected away from mean position at a velocity v_2 at an initial angular displacement β. Find the phase difference in oscillations of these two pendulums.



Sol. It is given that first pendulum bob is given a velocity v_1 at a displacement $l\alpha$ from mean position, using the formula for velocity we can find the amplitude of its oscillations as

$$v_1 = \omega\sqrt{A_1^2 - (l\alpha)^2}$$ [If A_1 is the amplitude of SHM of this bob]

As for simple pendulum $\omega = \sqrt{\frac{g}{l}}$ we have

$$v_1^2 = \frac{g}{l}[(A_1^2) - (l\alpha)^2]$$

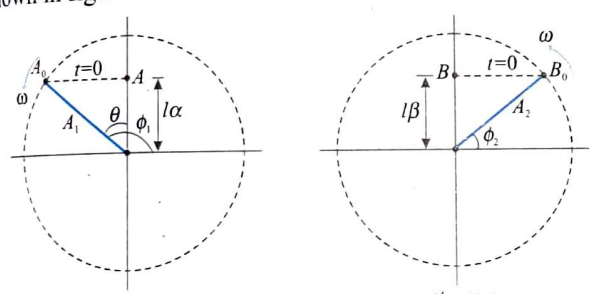
$$A_1 = \sqrt{l^2\alpha^2 + \frac{v_1^2 l}{g}} \quad ...(i)$$

Similarly if A_2 is the amplitude of SHM of second pendulum bob we have

$$v_2 = \omega\sqrt{A_2^2 - (l\beta)^2}$$

or $$A_2 = \sqrt{l^2\beta^2 + \frac{v_2^2 l}{g}} \quad ...(ii)$$

Now we represent the two SHMs by circular motion representation as shown in figure.



In first pendulum at $t = 0$ the bob is thrown from a displacement $l\alpha$ from mean position with a velocity v_1 towards mean position. As it is moving towards mean position, in Fig. (a), we consider the corresponding circular motion particle A_0 of the bob A is second quadrant, as the reference direction of ω, we consider anticlockwise. As shown in figure initial phase of bob A is given as

$$\phi_1 = \pi - \theta$$

here $$\sin\theta = \frac{l\alpha}{A_1}$$

$$\phi_1 = \pi - \sin^{-1}\left(\frac{l\alpha}{A_1}\right) \quad ...(iii)$$

Similarly for second pendulum bob B, its corresponding circular motion particle B_0 at $t = 0$ is considered as shown in Fig. (b) its initial base is given as

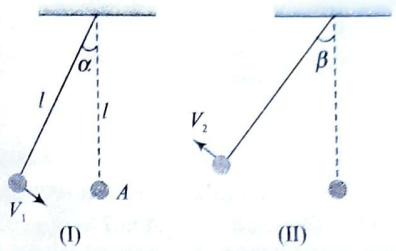
$$\phi_2 = \sin^{-1}\left(\frac{l\beta}{A_2}\right) \quad ...(iv)$$

As both the pendulums are identical, their angular frequency for SHM must be same, so their phase difference will not change with time, hence their phase difference can be given as

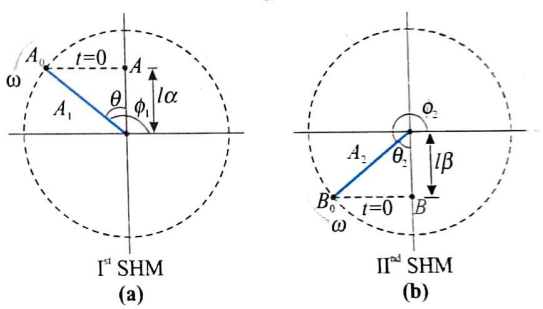
$$\Delta\phi = \phi_2 - \phi_1 = \sin^{-1}\left(\frac{l\beta}{A_2}\right) + \sin^{-1}\left(\frac{l\alpha}{A_1}\right) - \pi$$

ILLUSTRATION 5.41

In previous question, if the second pendulum bob is thrown at velocity v_2 at an angle β from mean position, but on other side of mean position, find the phase difference in the two SHMs now as shown in the figure.



Sol. In this case, amplitudes of the two SHMs will still remain the same and are given by Eqs. (i) and (ii) but when we represent the two SHMs on their corresponding circular motions, the position of the particle in circular motion in second pendulum is now different as shown in the figure.



As shown in Figs. (a) and (b) the initial phases of the two pendulum bobs are

$$\phi_1 = \pi - \sin^{-1}\left(\frac{l\alpha}{A_1}\right) \text{ and } \phi_2 = \pi + \sin^{-1}\left(\frac{l\beta}{A_2}\right)$$

As ω for both SHM are same, their phase difference remains constant so it is given as

$$\Delta\phi = \phi_2 - \phi_1 = \sin^{-1}\left(\frac{l\beta}{A_2}\right) + \sin^{-1}\left(\frac{l\alpha}{A_1}\right)$$

PENDULUM OF LARGE LENGTH BUT SMALL AMPLITUDE

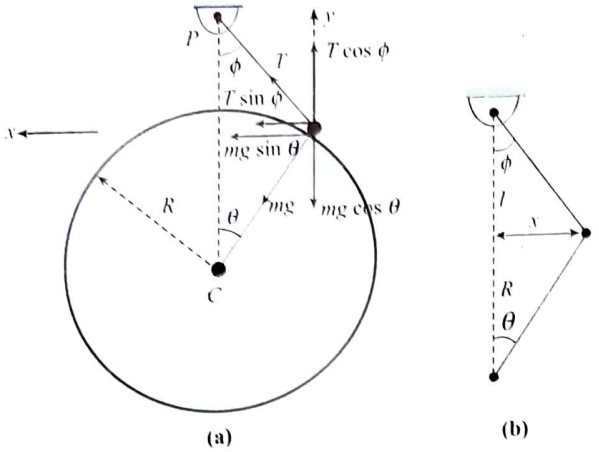
Consider a pendulum of length of the string l which is not negligible compared with the radius R of earth and assume that the bob is oscillating simple harmonically very close to earth's surface. Let us derive an expression for the angular frequency of oscillation.

At any angular position ϕ of the string of the simple pendulum, relative to the point of suspension P, the net force acting on the bob in x-direction is

$$F_{net} = T\sin\phi + mg\sin\theta \quad ...(i)$$

The net force acting on the bob in y-direction is

$$F_y = T\cos\phi - mg\cos\theta \quad ...(ii)$$



(a) (b)

Since, the bob has a small speed and θ and ϕ are very small, we can neglect its acceleration in y-direction. That gives $T\cos\phi - mg\cos\theta = 0$. Hence, we have $T = mg$.

Substituting $T = mg$ in Eq. (i) and writing $\sin\phi = \phi$ and $\sin\theta = \theta$ for small angle ϕ and θ, we have

$$F_{net} = mg(\phi + \theta) \text{ where } \phi = \frac{x}{l} \quad \text{and} \quad \theta = \frac{R}{l}$$

This gives $F_{net} = mg\frac{x}{l} + mg\frac{x}{R} = mg\left(\frac{1}{l} + \frac{1}{R}\right)x$

The direction of F_{net} opposes the displacement x. Then,

$$k_{eff} = \frac{F_{net}}{x} = mg\left(\frac{1}{l} + \frac{1}{R}\right)$$

Substituting $k_{eff} = mg(1/l + 1/R)$ in the formula $\omega = \sqrt{k_{eff}/m}$, we have $\omega = \sqrt{g[(1/l)) + (1/R)]}$

Hence $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1}{g\left[\frac{1}{l} + \frac{1}{R}\right]}}$

Special Cases

(i) For $l = R$, $T = 2\pi\sqrt{R/2g}$

On substituting the value of R and g, we get $T = 59.8$ min

(ii) For $l \to \infty$ $\frac{R}{l} \to 0$. Therefore, $T = 2\pi\sqrt{\frac{R}{g}}$ = 84.6 min

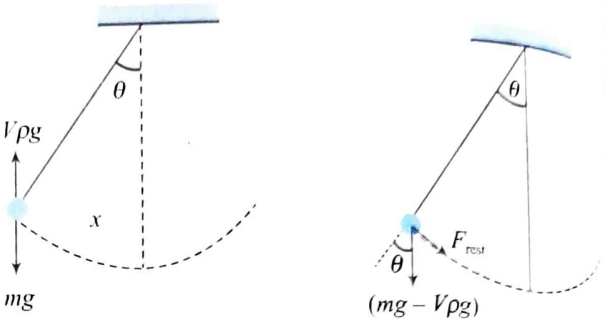
SIMPLE PENDULUM IN A LIQUID

If we immerse a simple pendulum in a liquid, the bob of the pendulum will experience buoyant force in upward direction in addition to the other forces such as gravity and tension. To find the k_{eff}, we need to find the net force on the bob near (or at) the equilibrium position. Dividing F_{net} by the small displacement x of the bob from the equilibrium position, we can find $k_{eff}\left(= \frac{F_{net}}{x} = \frac{T}{l}\right)$. After finding k_{eff}, substitute k_{eff} in the formula $\omega = \sqrt{k_{eff}/m}$ to find frequency (or time period). We will explain the above idea through the following illustration.

ILLUSTRATION 5.42

Derive an expression for the angular frequency of small oscillation of the bob of a simple pendulum when it is immersed in a liquid of density σ. Assume the density of the bob as ρ and length of the string as l.

Sol. **Method 1:** The bob of the pendulum experiences buoyant force due to liquid = $V\rho g$, in addition to gravitational force. The net force on the bob = $(mg - V\rho g)$.



(a) (b)

For small displacement x of the bob, restoring force

$$F_{rest} = (mg - V\rho g)\sin\theta = -(mg - V\rho g)\frac{x}{l}$$

and acceleration $= -\left(g - \frac{V\rho g}{m}\right)\frac{x}{l}$ m

On comparing with standard equation of SHM, $a = -\omega^2 x$ we get

$$\omega = \sqrt{\frac{\left(g - \frac{V\rho g}{m}\right)}{l}} = \sqrt{\frac{g}{l}\left(1 - \frac{\rho}{\sigma}\right)}$$

and $T = 2\pi\sqrt{\frac{l}{g\left(1 - \frac{\rho}{\sigma}\right)}}$

Method 2: At equilibrium position the pendulum should be vertical. Free body diagram of the bob at equilibrium position

$$T + B = mg$$

$$T = mg - B = V\sigma g - V\rho g = Vg(\sigma - \rho)$$

Equivalent force constant of oscillation $k_{eq} = \frac{T}{l} = \frac{Vg(\sigma - \rho)}{l}$

But we know $k_{eq} = m\omega^2$

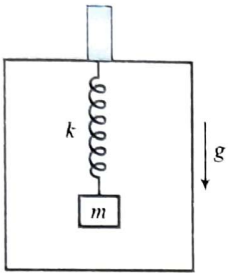
$$\omega = \sqrt{\frac{k_{eq}}{m}}$$

i.e., $\omega = \sqrt{\frac{Vg(\sigma - \rho)}{V\sigma \cdot l}} = \sqrt{\frac{g}{l}\left(1 - \frac{\rho}{\sigma}\right)}$

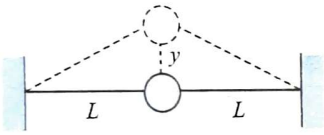
Hence $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g\left(1 - \frac{\rho}{\sigma}\right)}}$

CONCEPT APPLICATION EXERCISE 5.3

1. A block of mass m is suspended from the ceiling of a stationary elevator through a spring of spring constant k; it is in equilibrium. Suddenly, the cable breaks and the elevator starts falling freely. Show that the block now executes a simple harmonic motion of amplitude mg/k in the elevator.

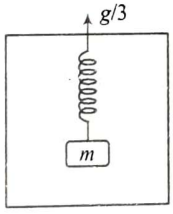


2. A ball of mass m is connected to two rubber bands of length L, each under tension T as shown in the figure. The ball is displaced by a small distance y perpendicular to the length of the rubber bands. Assuming the tension does not change, show that **(a)** the restoring force is $-(2T/L)y$ and **(b)** the system exhibits simple harmonic motion with an angular frequency $\omega = \sqrt{2T/mL}$.



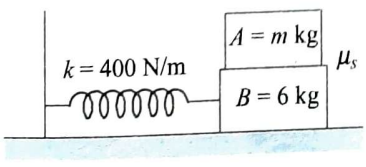
3. A mass M attached to a spring oscillates with a period of 2 s. If the mass is increased by 2 kg, the period increases by 1 s, find the initial mass M assuming that Hooke's law is obeyed.

4. A horizontal string–mass system of mass M executes oscillatory motion of amplitude a_0 and time period T_0. When mass M is passing through its equilibrium position another mass m is placed on it such that both move together. Find the new amplitude and time period.

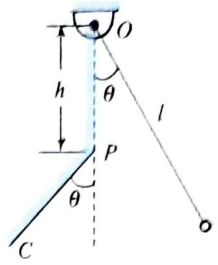


5. A spring of spring constant 200 N/m has a block of mass 1 kg hanging at its one end and from the other end the spring is attached to a ceiling of an elevator. The elevator rises upwards with an acceleration of $g/3$. When acceleration is suddenly ceased, then what should be the angular frequency and elongation during the time when the elevator is accelerating?

6. Two blocks lie on each other and are connected to a spring as shown in the figure. What should be the mass of block A placed on block B of mass 6 kg so that the system is period is 1 s. Assuming no slipping, what should be the minimum value of coefficient of static friction μ_s for which block A will not slip relative to block B, if block B is displaced 50 mm from equilibrium position and released?
(Given: $g = 10$ m/s^{-2} and $\pi^2 = 10$)



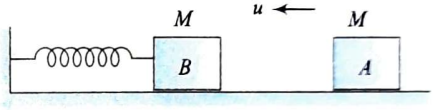
7. A simple pendulum of length l swings from a small angle θ. Its swinging is constrained by the smooth inclined planes OP and PC. Assuming elastic collision of the bob with the plane PC, find
(a) angular amplitude for the motion of the bob in the left hand side of its mean position.
(b) time period for a complete cycle of motion of the bob.



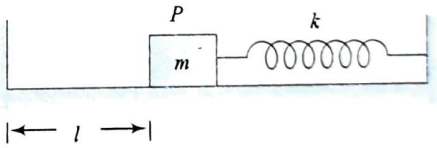
8. The period of oscillation of a spring pendulum is T. If the spring is cut into four equal parts, then find the time period corresponding to each part.

9. A body is in SHM with period T when oscillated from a freely suspended spring. If this spring is cut in two parts of length ratio 1 : 3 and again oscillated from the two parts separately, then the periods are T_1 and T_2. Find T_1/T_2.

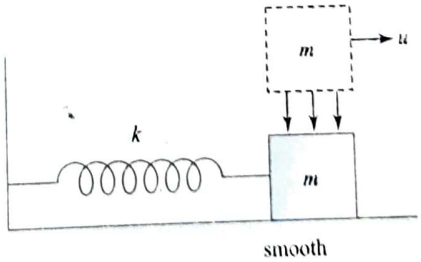
10. In the figure shown, the block A of mass M collides with the identical block B and after collision they stick together. Calculate the amplitude of resultant vibration.



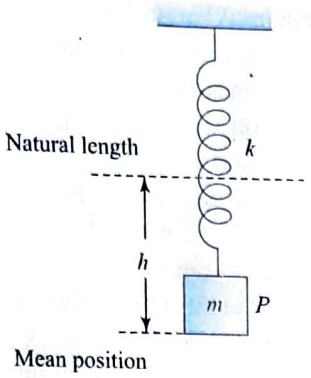
11. Figure shows a block P of mass m resting on a horizontal smooth floor at a distance l from a rigid wall. Block is pushed towards right by a distance $3l/2$ and released. When block passes from its mean position another block of mass m_1 is placed on it which sticks to it due to friction. Find the value of m_1 so that the combined block just collides with the left wall.



12. Figure shows a block of mass m resting on a smooth horizontal ground attached to one end of a spring of force constant k in natural length. If another block of same mass and moving with a velocity u towards right is placed on the block which stick to it due to friction, find the time it will take to reach its extreme position. Also find the amplitude of oscillations of the combined mass $2m$.



13. Figure shows a spring block system hanging in equilibrium. If a velocity v_0 is imparted to the block in downward direction, find the amplitude of SHM of the block and the time after which it will reach a point at half of the amplitude of block.



ANSWERS

3. 1.6 **4.** $a_0\sqrt{\frac{M}{M+m}}, \sqrt{\frac{M+m}{M}}T_0$ **5.** 14 rad/s, 0.07 m

6. 4 kg, 0.20 **7.** (a) $\left(\sqrt{\frac{l}{l-h}}\right)\theta$ (b) $\pi\sqrt{\frac{l}{g}}+\sqrt{\frac{l-h}{l}}\sin^{-1}\sqrt{\frac{l-h}{l}}$

8. $4k$ **9.** $1:\sqrt{3}$ **10.** $\sqrt{\frac{M}{2K}}u$ **11.** $\frac{5}{4}m$

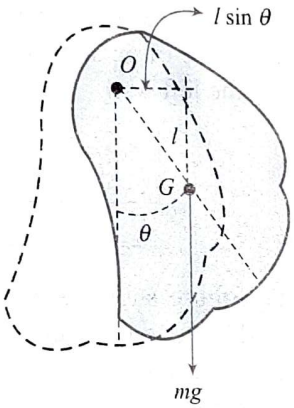
12. $\frac{\pi}{2}\sqrt{\frac{k}{2m}}, u\sqrt{\frac{m}{2k}}$ **13.** $v_0\sqrt{\frac{m}{k}}, \frac{\pi}{6}\sqrt{\frac{m}{k}}$

PHYSICAL PENDULUM (COMPOUND PENDULUM)

Any rigid body suspended from a fixed support constitutes a physical pendulum. Consider the situation when a body is displaced through a small angle θ. Torque on the body about O is given by

$$\tau = mgl\sin\theta \quad ...(i)$$

where l is the distance between point of suspension and centre of mass of the body.



If I is the MI of the body about O, then $\tau = I\alpha$ ...(ii)

From Eqs. (i) and (ii), we get $I\frac{d^2\theta}{dt^2} = -mgl\sin\theta$

as θ and $d^2\theta/dt^2$ are oppositely directed.

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = \frac{-mgl}{I}\theta, \text{ since } \theta \text{ is very small.}$$

Comparing with the equation $d^2\theta/dt^2 = -\omega^2\theta$, we get

$$\omega = \sqrt{\frac{mgl}{I}} \quad \Rightarrow T = 2\pi\sqrt{\frac{I}{mgl}}$$

Note:

Time period, $T = 2\pi\sqrt{\frac{I}{mgl}}$; $I = I_{CM} + ml^2$

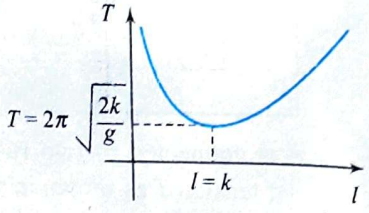
where I_{CM} is moment of inertia relative to the axis which passes from the center of mass and parallel to the axis of oscillation.

$$T = 2\pi\sqrt{\frac{I_{CM}+ml^2}{mgl}}, \quad \text{where} \quad I_{CM} = mk^2$$

k is gyration radius (about axis passing from centre of mass)

$$T = 2\pi\sqrt{\frac{mk^2+ml^2}{mgl}} = 2\pi\sqrt{\frac{k^2+l^2}{lg}} = 2\pi\sqrt{\frac{L_{eq}}{g}}$$

$L_{eq} = \frac{k^2}{l} + l$ = equivalent length of simple pendulum;



Graph of T vs l

T is minimum when $l = k$ = radius of gyration

$$T_{min} = 2\pi\sqrt{\frac{2k}{g}}$$

ILLUSTRATION 5.43

What is the period of a pendulum formed by pivoting a metre stick so that it is free to rotate about a horizontal axis passing through the 75 cm mark?

Sol. Let m be the mass and l be the length of the stick.

$l = 100$ cm.

The distance of the point of suspension from centre of gravity is $d = 25$ cm

Moment of inertia about a horizontal axis through O is

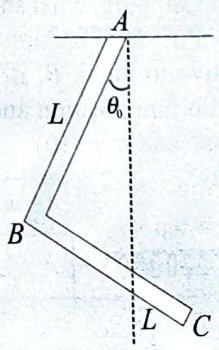
$$I = I_C + md^2 \Rightarrow I = \frac{ml^2}{12} + md^2$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{\frac{ml^2}{12}+md^2}{mgd}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{I^2+12d^2}{12gd}} = 2\pi\sqrt{\frac{l^2+12(0.25)^2}{12\times9.8\times0.25}} = 1.53\text{s}$$

ILLUSTRATION 5.44

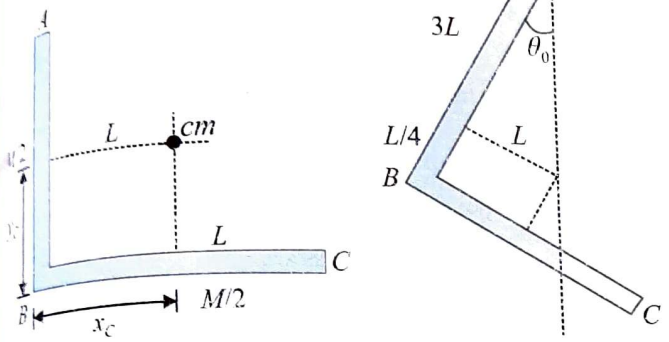
An L-shaped bar of mass M is pivoted at one of its end so that it can freely rotate in a vertical plane, as shown in the figure.

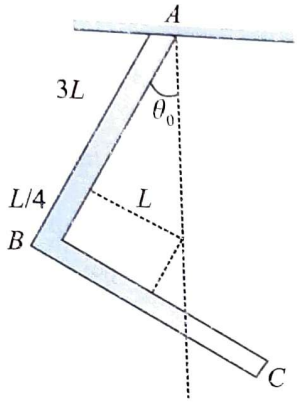


(a) Find the value of θ_0 at equilibrium.
(b) If it is slightly displaced from its equilibrium position, find the frequency of oscillation.

Sol. (a) Taking B as the origin, the coordinates of its centre of mass are

$$x_C = \frac{\frac{M}{2}\frac{L}{2}}{\frac{M}{2}+\frac{M}{2}} = \frac{L}{4}$$





and $y_C = \frac{\frac{M}{2}\frac{L}{2}}{M} = \frac{L}{4} \Rightarrow \tan\theta_0 = \frac{L/4}{3L/4} = \frac{1}{3}$

$$\theta_0 = \tan^{-1}\left(\frac{1}{3}\right)$$

(b) The frequency of oscillation for a compound pendulum is

$$f = \frac{1}{2\pi}\sqrt{\frac{mgl}{I}}$$

where d = distance of the cm from the point of suspension.
and I = moment of inertia about the point of suspension.

$$d = \sqrt{\left(\frac{3L}{4}\right)^2+\left(\frac{L}{4}\right)^2} = \frac{L}{4}\sqrt{10}$$

$$I = \left(\frac{M}{2}\right)\frac{L^2}{3}+\left(\frac{M}{2}\right)\frac{L^2}{12}+\left(\frac{M}{2}\right)\left[L^2+\left(\frac{L}{2}\right)^2\right]$$

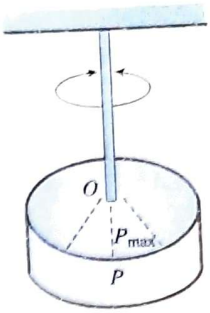
$$= \frac{ML^2}{3}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{Mg\frac{L}{4}\sqrt{10}}{\frac{ML^2}{3}}} \quad \text{or} \quad f = \frac{1}{2\pi}\sqrt{(2.37)\frac{g}{L}}$$

TORSIONAL PENDULUM

Figure shows a rigid object suspended by a wire attached at the top to a fixed support. When the object is twisted through some angle θ, the twisted wire exerts on the object a restoring torque that is proportional to the angular position. That is, $\tau = -k\theta$, where k (Greek letter kappa) is called the torsion constant of the support wire. The value of k can be obtained by applying a known torque to twist the wire through a measurable angle θ. Applying Newton's second law for rotational motion, we find that

$$\tau = -k\theta = I\frac{d^2\theta}{dt^2} \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta$$



Again, this result is the equation of motion for a simple harmonic oscillator with $\omega = \sqrt{k/I}$ and a period $T = 2\pi\sqrt{I/k}$.

This system is called a torsional pendulum. There is no small angle restriction in this situation as long as the elastic limit of the wire is not exceeded.

ILLUSTRATION 5.45

A uniform disc of radius 5.0 cm and mass 200 g is fixed at its centre to a metal wire, the other end of which is fixed to a ceiling. The hanging disc is rotated about the wire through an angle and is released. If the disc makes torsional oscillations with time period 0.20 s, find the torsional constant of the wire.

Sol. The situation is shown in the figure. The moment of inertia of the disc about the wire is

$$I = \frac{mr^2}{2} = \frac{(0.200\text{ kg})(5.0\times10^{-2}\text{ m})^2}{2} = 2.5\times10^{-4}\text{ kg-m}^2$$

The time period is given by $T = 2\pi\sqrt{\frac{I}{k}}$

or $$k = \frac{4\pi^2 I}{T^2} = \frac{4\pi^2(2.5\times10^{-4}\text{ kg-m}^2)}{(0.20\text{ s})^2} = 0.25\text{ kg-m}^2/\text{s}^2$$

SHM IN CASE OF ROTATION AND ROLLING

In case of linear SHM the linear acceleration of the particle is proportional to the displacement of the particle from its mean position and direction of both vectors are opposite to each other. For analyzing the cases of SHM in case of rotation and rolling, we need to analyze restoring torque and angular displacement. In such cases, the angular acceleration of the body is directly proportional to its angular displacement from its mean position. Here the direction of the angular acceleration and angular displacement both are opposite to each other.

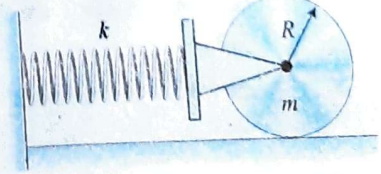
Here we can write the restoring torque on the body: $\vec{\tau} = -I\vec{\alpha}$

Here I is the moment of inertia about rotational axis or about instantaneous axis of rotation.

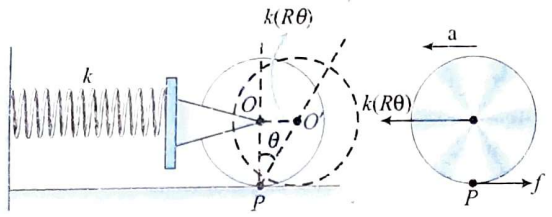
Now we will take few selected illustration to understand the application of the concept.

ILLUSTRATION 4.46

A solid cylinder is attached to a massless spring so that it can roll without slipping along a horizontal surface. Calculate the period of oscillation made by the cylinder if m = mass of cylinder and k = spring constant.



Sol. In the mean position of the cylinder, spring will be in its original length. Let us rotate the cylinder clockwise through an angle θ. This causes a linear displacement $R\theta$ of the centre of mass of the cylinder towards right as shown. Hence the spring is elongated by $R\theta$.



Cylinder is displaced slightly towards right

The pull of the spring towards left creates a static friction 'f' at the point of contact towards right. The net force on the cylinder is $(kR\theta - f)$ towards the mean position.

Applying $\tau = I_{CM}\,\alpha$ and $F = ma$, we have

$fR = I\alpha$ and $kR\theta - f = ma$

where a is the linear acceleration of the centre of mass and α is the angular acceleration of the cylinder.

For rolling without slipping, we have $a = R\alpha$

Eliminating f and a, we have $(kR\theta - mR\alpha)\,R = I_{CM}\,\alpha$

$$\alpha = \frac{-kR^2\theta}{I_{CM} + mR^2} \quad \text{[with proper vector sign]}$$

The magnitude of the angular acceleration is proportional to θ. Hence the cylinder performs angular SHM.

Comparing with $\alpha = -\omega^2\,\theta$, we have $T = 2\pi/\omega$

$$T = 2\pi\sqrt{\frac{I_{CM} + mR^2}{kR^2}} = \sqrt{\frac{\frac{3}{2}mR^2}{kR^2}} = 2\pi\sqrt{\frac{3m}{2k}}$$

Alternative method 1: Taking torques about the point P which is the instantaneous centre of rotation ($v_p = 0$) for pure rolling,

$\tau = (kR\theta)\,R = I_p\alpha$

$kR^2\theta = (I_{CM} + MR^2)\,\alpha$

$$\alpha = \frac{-kR^2\theta}{I_{CM} + mR^2} \quad \text{[with proper vector sign]}$$

and α (clockwise) is opposite to θ (anticlockwise)

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_{CM} + mR^2}{kR^2}} = 2\pi\sqrt{\frac{3m}{2k}}$$

Alternative method 2: The total energy in the cylinder–spring system at any instant when v is the speed of centre of mass, ω is the angular velocity and x is the spring elongation:

$$\text{Total energy} = \frac{1}{2}mv^2 + \frac{1}{2}I_{CM}\,\omega^2 + \frac{1}{2}kx^2$$

as this remains constant with time and $\omega = v/R$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}I_{CM}\frac{v^2}{R^2} + \frac{1}{2}kx^2\right) = 0$$

$$\frac{1}{2}m(2v)\frac{dv}{dt} + \frac{1}{2R^2}I_{CM}\left(2v\frac{dv}{dt}\right) + \frac{1}{2}k(2x)\frac{dx}{dt} = 0$$

Substituting $\frac{dv}{dt} = a$; $\frac{dx}{dt} = v$; $I_{CM} = \frac{1}{2}mR^2$,

$$mva + \frac{m}{2}va + kxv = 0 \quad \Rightarrow a = \frac{-2k}{3m}x$$

Comparing with $a = -\omega^2 x$; $\omega = \sqrt{\frac{2k}{3m}}$ $\Rightarrow T = 2\pi\sqrt{\frac{3m}{2k}}$

Alternative method 3: As the cylinder is rolling, we can take motion of cylinder as pure rotation about point of contact. The total energy associated with oscillation

$$\frac{1}{2}I_P\omega^2 + \frac{1}{2}kx^2 = \text{constant}$$

But $I_P = \frac{3}{2}mR^2$ and $\omega = \frac{v}{R}$

Substituting I_p and ω in Eq. (i) we get

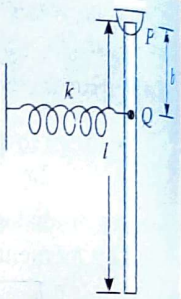
$$\frac{1}{2}\left(\frac{3}{2}mR^2\right)\left(\frac{v}{R}\right)^2 + \frac{1}{2}kx^2 = \text{constant}$$

$$\frac{3}{4}mv^2 + \frac{1}{2}kx^2 = \text{constant} \Rightarrow v^2 + \left(\frac{2k}{3m}\right)x^2 = \text{constant}$$

Comparing with equation $v^2 + \omega^2 x = \text{constant}$ $\Rightarrow$ $\omega = \sqrt{\frac{2k}{3m}}$

ILLUSTRATION 4.47

A uniform rod of mass m and length l hangs from a rigid support (smooth pivot) P. A light spring of stiffness k is rigidly attached with the rod at a point Q at a distance b below the pivot. Find the angular frequency of oscillation of the rod.



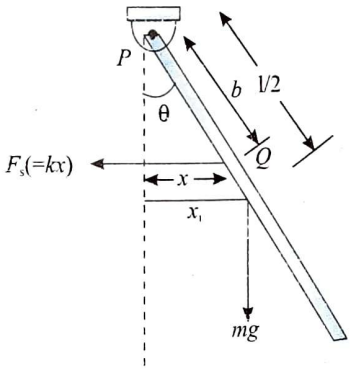
Sol. If rod is displaced by a small angle θ as shown in figure, the spring will be stretched by a distance $x = b\theta$. The net (restoring) torque acting on the rod about P is $\tau_R = \tau_{spring} + \tau_{gravity}$

$$\tau_R = -\left[kx \times b\cos\theta + mg \times x_1\right] = -\left[kb\theta \times b\cos\theta + mg \times \frac{l}{2}\sin\theta\right]$$

[negative sign for restoring nature]

$$= -\left[kb^2\theta + mg\frac{l}{2}\theta\right] = -\left[kb^2 + mg\frac{l}{2}\right]\theta$$

[As for small θ, $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$]



Substituting $\tau = I_P\alpha$ (Torque equation), we have:

$$\alpha = -\left(\frac{mg\frac{l}{2} + kb^2}{I_P}\right)\theta$$

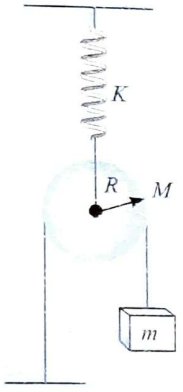
Substituting $I_P = \frac{ml^2}{3}$. We have $\alpha = -\left(\frac{3g}{2l} + \frac{3kb^2}{ml^2}\right)\theta$

Comparing this equation with $\alpha = -\omega^2\theta$

Hence $\omega = \sqrt{\frac{3g}{2l} + \frac{3kb^2}{ml^2}}$

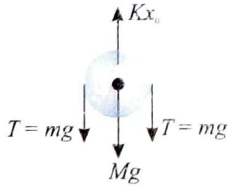
ILLUSTRATION 4.48

A block of mass m hangs by means of a string which goes over a pulley of mass M and moment of inertia I, as shown in the diagram. The string does not slip relative to the pulley. Find the frequency of small oscillations.

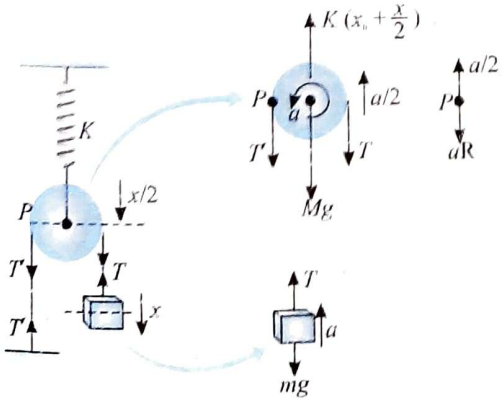


Sol. Method 1:

At the time of equilibrium: $Kx_0 = 2mg + Mg$ ...(i)



If the block is depressed by x, the pulley owing to the constraint is depressed by $x/2$. If the acceleration of the block is a, then the acceleration of the pulley should be $a/2$.



Writing equation of motion

For block: $T - mg = ma$ ...(ii)

For pulley: $K\left(x_0 + \frac{x}{2}\right) - T - T' - Mg = \frac{Ma}{2}$ ...(iii)

The acceleration of point 'P' should be zero as it is directly connected to a fixed point.

Hence, $\alpha R = \frac{a}{2} \Rightarrow \alpha = \frac{a}{2R}$ ...(iv)

Now writing torque equation for pulley: $T'R - TR = I\alpha = I\cdot\frac{a}{2R}$

or $T' - T = \frac{Ia}{2R^2}$ ...(v)

Using above equations, we get $a = \frac{Kx}{\left[4m + M + \frac{I}{R^2}\right]}$

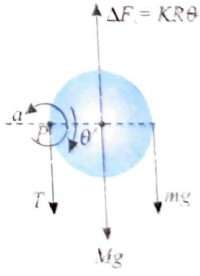
As the acceleration vector and displacement vectors both are opposite to each other, hence above equation can be written in vector form as, $\vec{a} = \frac{K}{\left[4m + M + \frac{I}{R^2}\right]}\cdot\vec{x}$

Comparing with equation: $\vec{a} = -\omega^2\vec{x}$; we get

$$\omega = \frac{K}{\left(4m + M + \frac{I}{R^2}\right)} \quad \text{or} \quad \omega = \sqrt{\frac{K}{\left(4m + M - \frac{I}{R^2}\right)}}$$

$$\Rightarrow \quad T = 2\pi\sqrt{\frac{\left(4m + M + \frac{I}{R^2}\right)}{K}}$$

Method 2: Point 'P' is the instantaneous center of rotation. Let the pulley be rotated by a small angle θ about an axis passing through P.



Due to this rotation, the spring will be elongated by a length $R\theta$. This increased elongation of spring will provide a restoring torque about point P. Now writing torque equation about P.

$$(\Delta F_S)R = I_P\alpha$$

$$(KR\theta)\cdot R = \left(I + MR^2 + m(2R)^2\right)\alpha$$

or $$\alpha = \frac{KR^2\theta}{(I + MR^2 + 4MR^2)}$$ ...(i)

Here we are not writing the torques due to weight of the pulley and block, as these torques are not responsible for providing restoring torques.

Equation (i) can be written in vector form, $\vec{\alpha} = \frac{K}{\left(4m + M + \frac{I}{R^2}\right)}\vec{\theta}$

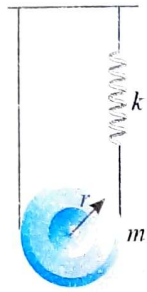
Comparing with standard equation of angular SHM, $\vec{\alpha} = -\omega^2 \vec{\theta}$

We get $\omega^2 = \dfrac{K}{\left(4m + M + \dfrac{I}{R^2}\right)}$ or $T = 2\pi\sqrt{\dfrac{\left(4m + M + \dfrac{I}{R^2}\right)}{K}}$

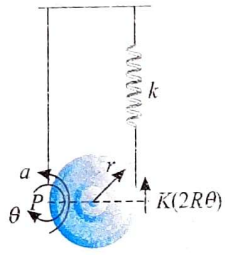
The time period of oscillation of the pulley and block will be same.

ILLUSTRATION 5.49

The pulley shown in figure has a radius r, moment of inertia I about its axis and mass m. Find the time period of vertical oscillations of its center of mass. The spring constant of spring is K and the spring does not slip over the pulley.



Sol. Here point P is the instantaneous center of rotation. Let the pulley is rotated in clockwise sense by a small angle θ beyond its equilibrium position. Because of this rotation the spring will be further elongated by a length $2R\theta$. This increased length will develop an additional force $K(2R\theta)$, which will provide a restoring torque about an axis passing through P. Now writing torque equation about P.



Now writing torque equation about P, $\tau_P = I_P\alpha$

$$I_P\alpha = \left[K(2R\theta)^2\right] \times 2R \Rightarrow \left[I + MR^2\right]\alpha = 4KR^2\theta$$

Hence, $\alpha = \dfrac{4K}{\left[M + \dfrac{I}{R^2}\right]}\theta$ ...(i)

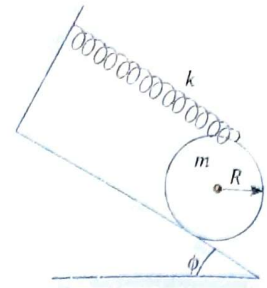
As angular acceleration (α) and angular displacement (θ), both are directed in opposite direction. Hence we can write Eq. (i) in vertical form as

$$\vec{\alpha} = -\frac{4K}{\left(M + \dfrac{I}{R^2}\right)}\vec{\theta} \text{ hence } \omega^2 = \frac{4K}{\left(M + \dfrac{I}{R^2}\right)}$$

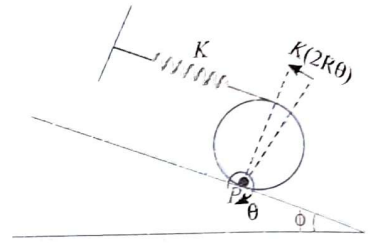
or Time period $T = 2\pi\sqrt{\dfrac{\left(M + \dfrac{I}{R^2}\right)}{4K}} = \pi\sqrt{\dfrac{\left(M + \dfrac{I}{R^2}\right)}{K}}$

ILLUSTRATION 5.50

A uniform cylinder of mass m and radius R is in equilibrium on an inclined plane by the action of a light spring of stiffness k, gravity and reaction force acting on it. If the angle of inclination of the plane is ϕ, find the angular frequency of small oscillation of the cylinder.



Sol. Here point of contact P, will be instantaneous center of rotation. Let the cylinder rotates a small angle θ beyond its equilibrium position about an axis passing through P. Here the increase in length of spring will be $2R\theta$. Thus the increased spring force provides a restoring torque about P.



Now writing torque equation about P as: $\Delta\tau_P = I_P\alpha$

$$(K \times 2R\theta) \times 2R = \left[\frac{mR^2}{2} + mR^2\right]\alpha \Rightarrow 4KR^2\alpha = \frac{3}{2}mR^2\alpha$$

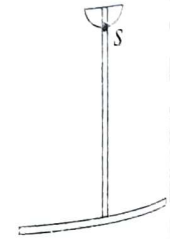
or $\alpha = \left(\dfrac{8K}{3m}\right)\theta \Rightarrow \vec{\alpha} = -\left(\dfrac{8K}{3m}\right)\vec{\theta}$

Hence $\omega^2 = \dfrac{8K}{3m}$ or $\omega = 2\sqrt{\dfrac{2K}{3m}}$

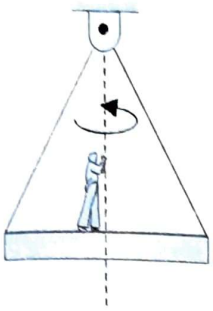
CONCEPT APPLICATION EXERCISE 5.4

1. A uniform rod of length l is pivoted at a distance x from the top of the rod. Neglecting friction, find the
   (a) value of x for minimum period of oscillation,
   (b) minimum period of oscillation of the rod.

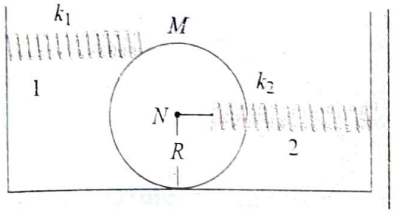
2. A ring of diameter 2 m oscillates as a compound pendulum about a horizontal axis passing through a point at its rim. It oscillates such that its centre moves in a plane which is perpendicular to the plane of the ring. Find the equivalent length of the simple pendulum.
3. Two identical rods each of mass m and length L, are rigidly joined and then suspended in a vertical plane so as to oscillate freely about an axis normal to the plane of paper passing through S (point of suspension). Find the time period of such small oscillations.



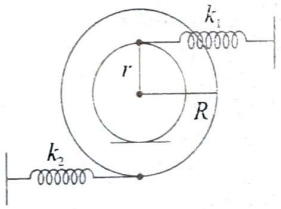
4. To determine the MI of an object, we can use torsional pendulum method. If the torsional constant of the apparatus (a disc suspended by the thread) is $C = 16$ Nm/rad. If the empty torsional pendulum (without the man) has a time period $T_0 = 2$ s and that with the man is $T = 2.5$ s, find the MI of the man. (Take $\pi^2 = 10$)



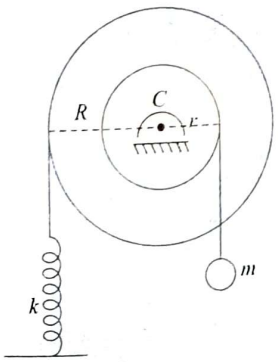
5. A uniform disc of mass m and radius R is connected with two light springs 1 and 2. The springs are connected at the highest point M and the CM 'N' of the disc. The other ends of the springs are rigidly attached with vertical walls. If we shift the CM in horizontal by a small distance, the disc oscillates simple harmonically. Assuming a perfect rolling of the disc on the horizontal surface, find the angular frequency of oscillation.



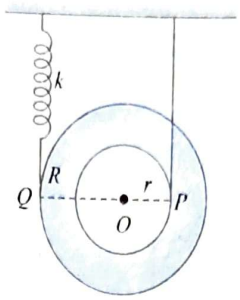
6. A stepped pulley having mass m, radius R and radius of gyration k is connected with two ideal springs of stiffness k_1 and k_2 as shown in the figure. If the pulley rolls without sliding, find the angular frequency of its oscillation.



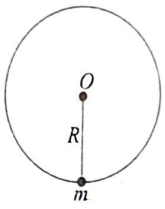
7. A stepped disc of mass M and radius R is pivoted at its center C smoothly. An inextensible string connected with a light spring of stiffness k passes over the pulley. One end of the string is rigidly connected with the ground and the other end is attached to a body of mass m. If the string does not slide on the pulley, find the angular frequency of oscillation of the system.



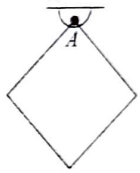
8. A disc of mass m hanged by a string is attached at P and a spring of stiffness k is attached at O. Find the frequency of small angular oscillation of the disc if the string does not slide over the pulley. Assume $I_0 = MI$ of the disc about O.



9. A disc of mass $M = 2m$ and radius R is pivoted at its centre. The disc is free to rotate in the vertical plane about its horizontal axis through its centre O. A particle of mass m is stuck on the periphery of the disc. Find the frequency of small oscillations of the system about its equilibrium position.



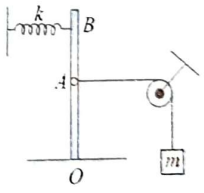
10. A square plate of mass M and side length L is hinged at one of its vertex (A) and is free to rotate about it. Find the time period of small oscillations if



(a) The plate performs oscillations in the vertical plane of the figure. (Axis is perpendicular to figure.)

(b) The plate performs oscillations about a horizontal axis passing through A lying in the plane of the figure.

11. A massless rod rigidly fixed at O. A string carrying a mass m at one end is attached to point A on the rod so that $OA = a$. At another point B ($OB = b$) of the rod, a horizontal spring of force constant k is attached as shown. Find the period of small vertical oscillations of mass m around its equilibrium position



ANSWERS

1. (a) $\frac{l}{2}\left(1-\frac{1}{\sqrt{3}}\right)$ (b) $2\pi\sqrt{\frac{l}{\sqrt{3}g}}$ **2.** 1.5 m **3.** $2\pi\sqrt{\frac{17l}{18g}}$

4. 0.9 kg m^2 **5.** $\sqrt{\frac{2(4k_1+k_2)}{3m}}$ **6.** $\sqrt{\frac{4k_1r^2+k_2(R-r)^2}{m(k^2+r^2)}}$

7. $\sqrt{\frac{2kR^2}{MR^2+2mr^2}}$ **8.** $\frac{1}{2\pi}\sqrt{\frac{k(R+r)^2}{I_0+mr^2}}$ **9.** $\frac{1}{2\pi}\sqrt{\frac{g}{2R}}$

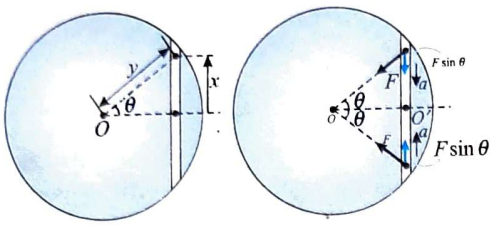
10. (a) $2\pi\sqrt{\frac{2\sqrt{2}}{3}\frac{a}{g}}$ (b) $2\pi\sqrt{\frac{7}{6\sqrt{2}}\frac{a}{g}}$ **11.** $2\pi\frac{a}{b}\sqrt{\frac{m}{k}}$

MOTION OF A BALL IN A TUNNEL THROUGH THE EARTH

Case I: If the tunnel is along a diameter and the ball is released from the surface. If the ball at any time is at a distance y from the centre of the earth, the restoring force will act on the ball due to gravitation between ball and earth.

Acceleration of the particle at the distance y from the centre of earth.

$$a = -GMy/R^3 = -(GM)y/R^3$$



(a) (b)

Furthermore as $g = GM/R^2$

$$a = \frac{-(gR^2)y}{R^3} = -\frac{g}{R}y$$

Comparing with $a = -\omega^2 y$

$$\omega^2 = \frac{g}{R} \quad \text{or} \quad \omega = \sqrt{\frac{g}{R}} \quad \Rightarrow T = 2\pi\sqrt{\left(\frac{R}{g}\right)} = 84.6 \text{ min}$$

Case II: If the tunnel is along a chord and ball is released from the surface. If the ball at any time is at a distance x from the centre of tunnel, acceleration of the particle at the distance y from the centre of earth

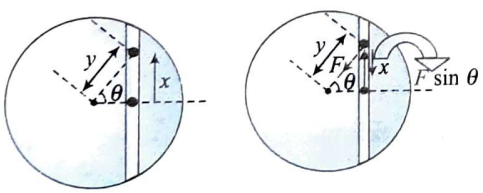
$$a = -GMy/R^3 = -(GM)y/R^3$$

Furthermore, as $g = GM/R^2$

$$a = -(gR^2)y/R^3$$

This acceleration will be towards the centre of earth. The component of acceleration towards the centre of earth

$$a' = a\sin\theta = \left(-\frac{g}{R}y\right)\left(\frac{x}{y}\right) = -\frac{g}{R}x$$



(a) (b)

Comparing with $a = -\omega^2 y$

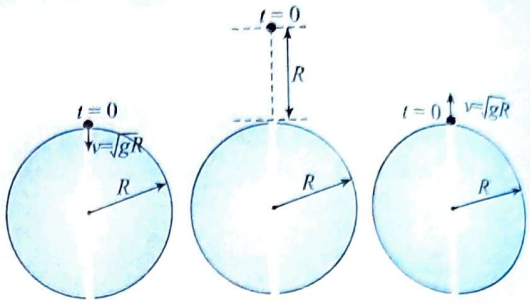
$$\omega^2 = \frac{g}{R} \quad \Rightarrow \omega = \sqrt{\frac{g}{R}} \quad \Rightarrow T = 2\pi\sqrt{\frac{R}{g}} = 84.6 \text{ min}$$

Note: Time period of oscillation is same in both the cases wheather the tunnel is along a distance or along the chord.

ILLUSTRATION 5.51

Assume that a tunnel is dug across the earth (radius = R) passing through its centre. Find the time a particle takes to cover the length of the tunnel if

(a) it is projected into the tunnel with a speed of $\sqrt{gR}$
(b) it is released from a height R above the tunnel
(c) it is thrown vertically upward along the length of tunnel with a speed of $\sqrt{gR}$



(a) (b) (c)

Sol. Let $y = A\sin(\omega t + \phi)$

At $t = 0$,

$R = A\sin\phi$ ...(i)

and $\dfrac{dy}{dt} = A\omega\cos(\omega t + \phi)$

At $t = 0$, the velocity of the particle is downward direction, i.e. $v = -\sqrt{gR}$.

$$\frac{dy}{dt} = -\sqrt{gR} = A\sqrt{\frac{g}{R}}\cos\phi \qquad \left[\text{as } \omega = \sqrt{\frac{g}{R}}\right]$$

$A\cos\phi = -R$ ...(ii)

$$\tan\phi = -1 \quad \Rightarrow \phi = \frac{3\pi}{4}$$

Squaring and adding Eqs. (i) and (ii), $A = \sqrt{2}R$

$$y = \sqrt{2}R\sin\left(\sqrt{\frac{g}{R}}t + \frac{3\pi}{4}\right)$$

$$\text{At } y = 0,\ \sin\left(\sqrt{\frac{g}{R}}t + \frac{3\pi}{4}\right) = \pi \ \Rightarrow\ t = \frac{\pi}{4}\sqrt{\frac{R}{g}}$$

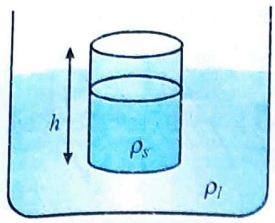
Hence time to cross the tunnel $= \dfrac{\pi}{2}\sqrt{\dfrac{R}{g}}$

OSCILLATION OF A FLOATING BODY IN A LIQUID

A floating body is in a stable equilibrium. When it is displaced up and released, it accelerates down and when it is pushed down and released, it accelerates up. It means, a floating body experiences a net (resting) force towards its stable equilibrium position. Hence, a floating body oscillates when displaced up or down from its mean position.

ILLUSTRATION 5.52

Consider a solid cylinder of density ρ_s, cross-sectional area A and height h floating in a liquid of density ρ_l, as shown in the given figure ($\rho_l > \rho_s$). It is depressed slightly and allowed to oscillate vertically. Find the frequency of small oscillations.



Sol. At equilibrium the net force on the cylinder is zero in the vertical direction.

$F_{net} = B - W = 0$

where, B = the buoyancy and W = the weight of the cylinder.

When the cylinder is depressed slightly by x, the buoyancy increases from B to $B + \delta B$, where $\delta B = |x| \rho_l A g$

The weight W remains the same.

Therefore the net force, $F'_{net} = B + \delta B - W = \delta B = |x| \rho_l A g$

The equation of motion is, therefore, $\rho_s A h \dfrac{d^2x}{dt^2} = -x \rho_l A g$

The minus sign takes into account the fact that x and restoring force are in opposite directions. Therefore

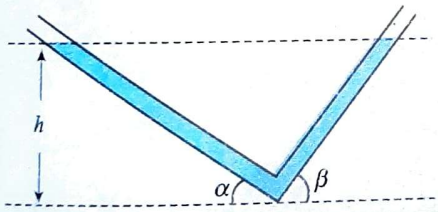
$\dfrac{d^2x}{dt^2} = -x \dfrac{\rho_l g}{\rho_s h}$ and the angular frequency, ω, is $\omega = \sqrt{\dfrac{g\rho_l}{h\rho_s}}$

OSCILLATION OF A LIQUID COLUMN IN A TWO ARM TUBE

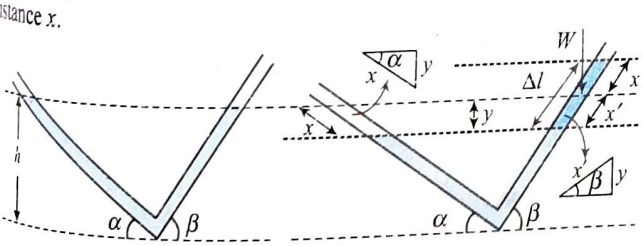
Let us take liquid in a glass tube with two arm and keep it in a vertical plane. In equilibrium, the level of liquid in both arms of the tube is equal. If we pass the liquid of any arm and release, the liquid in the tube starts oscillating. We can learn this concept through the following illustration.

ILLUSTRATION 5.53

A V-shaped glass tube of uniform cross section is kept in a vertical plane as shown. A liquid is poured in the tube. In equilibrium the level of liquid in both limbs of tube are equal. Find the angular frequency of small oscillations of liquid.



Sol. Let us displace the liquid of left limb by a distance x, the liquid of right limb will rise relative to equilibrium line by same distance x.



Excess length of the liquid on the right limb, $\Delta l = x + x'$ ...(i)

From the figure, $y = x \sin\alpha = x' \sin\beta$ ...(ii)

From Eq. (i), $x' = \dfrac{x \sin\alpha}{\sin\beta}$ ...(iii)

from Eqs. (i) and (iii), $\Delta l = \left(\dfrac{\sin\alpha + \sin\beta}{\sin\beta}\right) x$

This excess length Δl in right limb will provide restoring force to the liquid in tube.

Weight of the excess liquid column

$$W = \rho A \cdot \Delta l g = \left(\frac{\sin\alpha + \sin\beta}{\sin\beta}\right) \rho A x g$$

ρ is the density of liquid and A is the area of cross section of tube. The component of W along the tube will be equal to restoring force

$$F = W \sin\beta = \rho(\sin\alpha + \sin\beta) A x g$$

This force helps the entire liquid to restore its original position by moving it with an acceleration $a = F/m$; m = mass of total liquids

$$a = \frac{\rho(\sin\alpha + \sin\beta) A x g}{m}$$

Here m = mass of liquid in the tube = $A\rho l$

l = Length of total liquid column

$$= \frac{h}{\sin\alpha} + \frac{h}{\sin\beta} = h\left[\frac{\sin\alpha + \sin\beta}{\sin\alpha \cdot \sin\beta}\right]$$

Hence $a = \dfrac{\rho(\sin\alpha + \sin\beta) A x g}{A\rho\left[h\dfrac{(\sin\alpha + \sin\beta}{\sin\alpha \cdot \sin\beta}\right]}$

As the direction of acceleration vector $(\vec{a})$ and displacement vector $(\vec{x})$ both are opposite to each other, hence we can write

$$\vec{a} = -\left(\frac{g \sin\alpha \cdot \sin\beta}{h}\right) \cdot \vec{x} \begin{bmatrix} \text{with proper} \\ \text{vector sign} \end{bmatrix}$$

Compare with $\vec{a} = -\omega^2 \vec{x}$ which gives

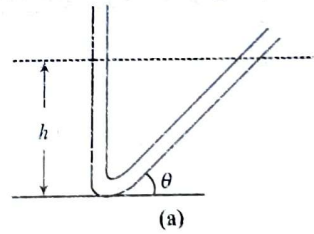
$$\omega = \sqrt{\frac{g \sin\alpha \cdot \sin\beta}{h}} \Rightarrow T = 2\pi \sqrt{\frac{h}{g \sin\alpha \cdot \sin\beta}}$$

Note: If we modify the tube as shown in the figure, the different time periods will be

Here $\sin\alpha = 1$,

$\sin\beta = \sin\theta$

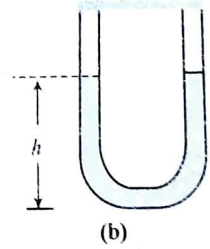
Hence $T = 2\pi \sqrt{\dfrac{h}{g \sin\theta}}$



(a)

Here $\alpha = \beta = 90°$

Hence $T = 2\pi \sqrt{\dfrac{h}{g}}$



(b)

SUPERPOSITION OF TWO SHM'S

- **In same direction and of same frequency.**

$x_1 = A_1 \sin\omega t$

$x_2 = A_2 \sin(\omega t + \theta)$, then resultant displacement

$x = x_1 + x_2 = A_1 \sin\omega t + A_2 \sin(\omega t + \theta) = A \sin(\omega t + \phi)$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\theta}$

and $\phi = \tan^{-1}\left[\dfrac{A_2 \sin\theta}{A_1 + A_2\cos\theta}\right]$

If $\theta = 0$, both SHMs are in phase and $A = A_1 + A_2$

If $\theta = \pi$, both SHMs are out of phase and $A = |A_1 - A_2|$

The resultant amplitude due to superposition of two or more than two SHM's of this case can also be found by phasor diagram.

- **In same direction but are of different frequencies**

$x_1 = A_1 \sin \omega_1 t$

$x_2 = A_2 \sin \omega_2 t$

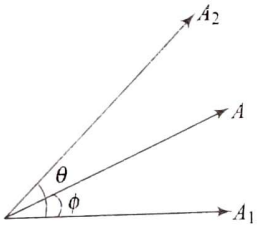
then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$

This resultant motion is not SHM.

- **Superposition of SHMs along the same direction (using phasor diagram)**

If two or more SHMs are along the same line, their resultant can be obtained by vector addition by making phasor diagram.

- Amplitude of SHM is taken as length (magnitude) of vector.
- Phase difference between the vectors is taken as the angle between these vectors. The magnitude of resultant of vectors give resultant amplitude of SHM and angle of resultant vector gives phase constant of resultant SHM.



Phasor diagram

For example, $x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin(\omega t + \theta)$

If equation of resultant SHM is taken as $x = A \sin(\omega t + \phi)$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$$

$$\tan\phi = \frac{A_2\sin\theta}{A_1 + A_2\cos\theta}$$

- **Superposition of two SHMs in perpendicular directions**

Let us take two SHMs of same angular frequency but one along x-axis and the other along y-axis.

$x = A_1 \sin \omega t$ ...(i)

$y = A_2 \sin(\omega t + \phi)$ ...(ii)

The amplitudes A_1 and A_2 may be different but phase difference ϕ and angular frequency ω are same.

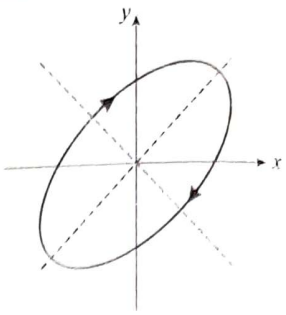
So, equation of the path may be obtained by eliminating t from (i) and (ii)

$$\sin\omega t = \frac{x}{A_1} \quad ...(iii)$$

$$\cos\omega t = \sqrt{1 - \frac{x^2}{A_1^2}} \quad ...(iv)$$

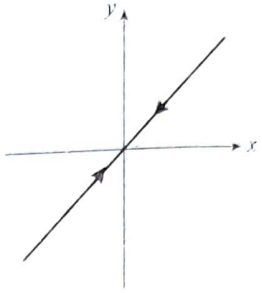
On rearranging, we get $\dfrac{x^2}{A_1^2} + \dfrac{y^2}{A_2^2} - \dfrac{2xy\cos\phi}{A_1A_2} = \sin^2\phi$ ...(v)

This is the general equation of ellipse.



Special Cases:

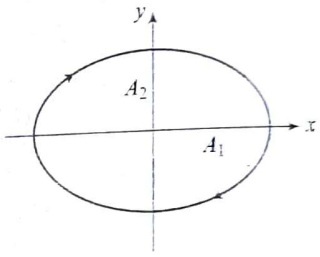
1. If $\phi = 0$



$$\therefore \quad \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1A_2} = 0$$

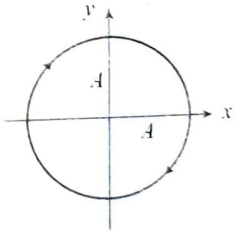
$$\therefore \quad y = \frac{A_2}{A_1}x \quad \text{(Equation of straight line)}$$

2. If $\phi = 90°$



$$\Rightarrow \quad \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \quad \text{(Equation of ellipse)}$$

3. If $\phi = 90°$ and $A_1 = A_2 = A$



Then $x^2 + y^2 = A^2$ (Equation of circle.)

The above figures are called Lissajous figures.

ILLUSTRATION 5.54

Find the amplitude and initial phase of a particle in SHM whose motion equation is given as

$y = A \sin \omega t + B \cos \omega t$

Sol. Here in the given equation we can write

$A = R \cos \phi$

and $B = R \sin \phi$

Thus the given equation transforms to $y = R \sin(\omega t + \phi)$

Equation (iii) is a general equation of SHM and here R is the amplitude of given SHM and ϕ is the initial phase of the oscillating particle at $t = 0$.

Here R is given by squaring and adding Eqs. (i) and (ii)

$$R = \sqrt{A^2 + B^2}$$

Initial phase ϕ can be given by dividing Eq. (ii) by (i)

$$\tan\phi = B/A$$

$$\phi = \tan^{-1}(B/A)$$

Note: Equations $y = Ae^{i(\omega t + \phi)}$ and $y = A\sin\omega t + B\cos\omega t$, can also represent the general equation of SHM.

ILLUSTRATION 5.55

Find the amplitude of the simple harmonic motion obtained by combining the motions

$x_1 = (2.0\text{ cm})\sin\omega t$ and $x_2 = (2.0\text{ cm})\sin(\omega t + \pi/3)$

Sol. The two equations given represent simple harmonic motions along the X-axis with amplitudes $A_1 = 2.0$ cm and $A_2 = 2.0$ cm. The phase difference between the two simple harmonic motions is $\pi/3$. The resultant simple harmonic motion will have an amplitude A given by

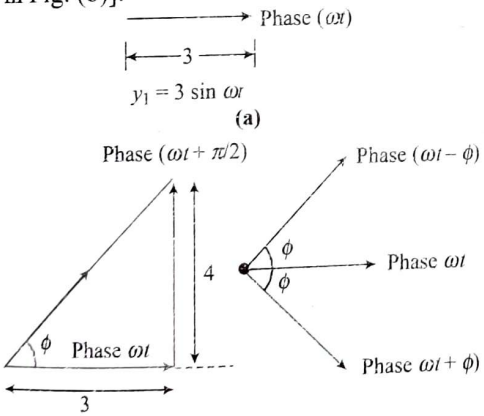
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\delta}$$

$$= \sqrt{(2.0\text{ cm})^2 + (2.0\text{ cm})^2 + 2(2.0\text{ cm})^2\cos\frac{\pi}{3}} = 3.5\text{ cm}$$

ILLUSTRATION 5.56

Two particles A and B execute simple harmonic motion according to the equations $y_1 = 3\sin\omega t$ and $y_2 = 4\sin[\omega t + (\pi/2)] + 3\sin\omega t$. Find the phase difference between them.

Sol. Representing a quantity with phase (ωt) along the X-axis, quantity with phase ($\omega t + \phi$) can be represented by a vector making an angle ϕ with the X-axis in the anticlockwise sense and any quantity with a phase ($\omega t - \phi$) can be represented by a vector making an angle an ϕ with the X-axis in the clockwise sense [as shown in Fig. (b)].



∴ Phase difference $= \phi = \tan^{-1}(4/3)$

Alternatively:

Given $y_1 = 3\sin\omega t$ ...(i)

and $y_2 = 4\sin(\omega t + \pi/2) + 3\sin\omega t = 4\cos\omega t + 3\sin\omega t$

$= 5[(4/5)\cos\omega t + (3/5)\sin\omega t]$

$= 5[\sin\phi\cos\omega t + \cos\phi\sin\omega t] = 5\sin(\omega t + \phi)$ ...(ii)

$\cos\phi = 3/5$; $\sin\phi = 4/5$ and $\tan\phi = 4/3$

From Eqs. (i) and (ii), the results follow.

ILLUSTRATION 5.57

Two simple harmonic motions are represented by the following equations

$$y_1 = 10\sin(\pi/4)(12t + 1)$$

$$y_2 = 5(\sin 3\pi t + \sqrt{3}\cos 3\pi t)$$

Here t is in seconds.

Find out the ratio of their amplitudes. What are the time periods of the two motions?

Sol. Given equations are

$$y_1 = 10\sin(\pi/4)(12t + 1)$$ ...(i)

$$y_2 = 5(\sin 3\pi t + \sqrt{3}\cos 3\pi t)$$ ...(ii)

We recast these in the form of standard equation of SHM which is

$$y = A\sin(\omega t + \phi)$$ ...(iii)

Equation (1) may be written as

$$y_1 = 10\sin[(12\pi t/4) + (\pi/4)] = 10\sin[(3\pi t + \pi/4)]$$ ...(iv)

Comparing Eq. (iv) with Eq. (iii), we have

Amplitude of first SHM $= A_1 = 10$ cm/s and $\omega_1 = 3\pi$

∴ Time period of first motion $T_1 = 2\pi/\omega_1 = 2\pi/3\pi = (2/3)$ s

Equation (ii) may also be written as

$$y_2 = 5\sin 3\pi t + 5\sqrt{3}\cos 3\pi t$$

Let us put $5 = A_2\cos\phi$ ...(v)

and $5\sqrt{3} = A_2\sin\phi$ ...(vi)

Then $y_2 = A_2\cos\phi\sin 3\pi t + A_2\sin\phi\cos 3\pi t$

$= A_2\sin(3\pi t + \phi)$ ...(vii)

Squaring (v) and (vi) and adding, we have

$$A_2 = \sqrt{[(5)^2 + (5\sqrt{3})^2]} = 10\text{ cm}$$

i.e., amplitude of second SHM is 10 cm and time period of second SHM is T_2

$$2\pi/\omega_2 = 2\pi/3\pi = 2/3\text{ s}$$

Thus, the ratio of amplitudes $A_1 : A_2 = 1 : 1$ and periodic times are $T_1 = T_2 = (2/3)$ s

CONCEPT APPLICATION EXERCISE 5.5

1. Given that

$x_1 = 5\sin\omega t$

$x_2 = 5\sin(\omega t + 53^\circ)$

$x_3 = -10\cos\omega t$

Find amplitude of resultant SHM.

2. $x_1 = 3\sin\omega t$; $x_2 = 4\cos\omega t$

Find **(a)** amplitude of resultant SHM, **(b)** equation of the resultant SHM.

3. A particle is subjected to two simple harmonic motions: $x_1 = A_1\sin\omega t$ and $x_2 = A_2\sin(\omega t + \pi/3)$.

Find **(a)** the displacement at $t = 0$, **(b)** the maximum speed of the particle and **(c)** the maximum acceleration of the particle.

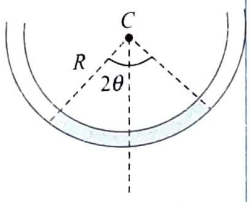
4. If the displacement of a moving point at any time is given by an equation of the form $y(t) = a\cos\omega t + b\sin\omega t$, show that the motion is simple harmonic. If $a = 3$ m, $b = 4$ m and $\omega = 2$; determine the period, amplitude, maximum velocity and maximum acceleration.

5. If two SHMs are represented by $y_1 = 10\sin(4\pi t + \pi/2)$ and $y_2 = 5[\sin 2\pi t + \sqrt{8}\cos 2\pi t]$, compare their amplitudes.

6. Find the amplitude and phase of the vibrations

$$x = A\cos\omega t + \frac{A}{2}\cos\left(\omega t + \frac{\pi}{2}\right) + \frac{A}{4}\cos(\omega t + \pi) + \frac{A}{8}\cos\left(\omega t + \frac{3\pi}{2}\right)$$

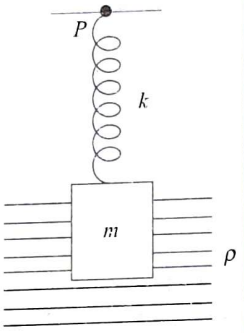
7. A smooth tube in the form of a semicircle of radius R is kept vertical. A non-viscous liquid is in equilibrium subtending an angle 2θ at the centre C of the circle. Find the angular frequency of small oscillations of the liquid.



8. A cylinder of mass m and cross-sectional area A floating in the liquid of density ρ hangs from a rigid support P by an ideal spring of stiffness k.

(i) Find the depth of immersion of the cylinder if compression of the spring is equal to x_0.

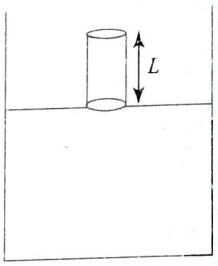
(ii) What is the frequency of small vertical oscillation of the cylinder?



9. A cylindrical container having area of cross-section equal to $4A$, contains a non-viscous liquid of density 2ρ. A cylindrical block of wood having cross sectional area A and length L has density ρ, held vertically with its lower surface touching the liquid. Now it is released from this position. Assume that the depth of the container is sufficient and the cylinder does not touch the bottom.

(a) Find amplitude of oscillation of the wooden cylinder.

(b) Find time period of its oscillation.



ANSWERS

1. 10 **2.** (a) 5 (b) $x = 5\sin(\omega t + 53°)$

3. (a) $A_2\frac{\sqrt{3}}{2}$ (b) $\omega\sqrt{A_1^2 + A_2^2 + A_1A_2}$ (c) $\omega^2\sqrt{A_1^2 + A_2^2 + A_1A_2}$

4. πs, 5 m, 10 m/s, 20 m/s^2 **5.** $\frac{2}{3}$ **6.** $\frac{3\sqrt{5}}{8}A, \frac{1}{2}$

7. $\sqrt{\frac{g\sin\theta}{R\theta}}$ **8.** (i) $\frac{mg - kx_0}{A\rho g}$ (ii) $\frac{1}{2\pi}\sqrt{\frac{k + A\rho g}{m}}$

9. (a) $\frac{3L}{8}$ (b) $2\pi\sqrt{\frac{3L}{8g}}$

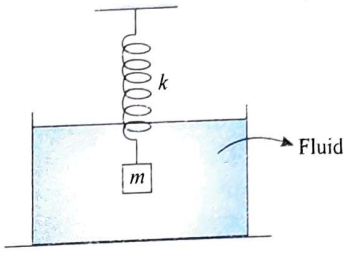
DAMPED AND FORCED OSCILLATIONS

DAMPED OSCILLATIONS

In previous section, in all of the oscillatory motion we have taken idealistic cases and neglected friction. Here, we can consider the frictional force proportional to the magnitude of the velocity, and the direction of this frictional force always opposes the motion. Here, one can observe a possible arrangement for oscillating system with a frictional force provided by a fluid.

Here in this case the frictional force is $F_f = -bv$, where b is called the damping constant. The net force on mass m with displacement x and velocity $v = dx/dt$ is given as

$$F_{net} = -kx - bv$$



Writing force equation, according to Newton's second law

$$-kx - bv = ma$$

Hence, $$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

or $$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The solution of differential equation depends on the relative sizes of the spring constant k and the damping constant b.

- Displacement of damped oscillator is given by

  $x = x_m e^{-bt/2m}\sin(\omega' t + \phi)$ where ω' = angular frequency of the damped oscillator = $\sqrt{\omega_0^2 - (b/2m)^2}$

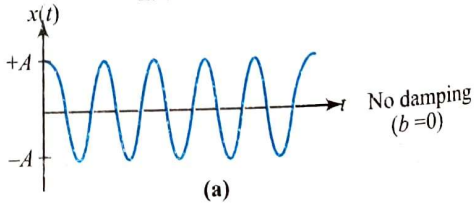
- The amplitude decreases continuously with time according to

  $x = x_m e^{-(b/2m)t}$

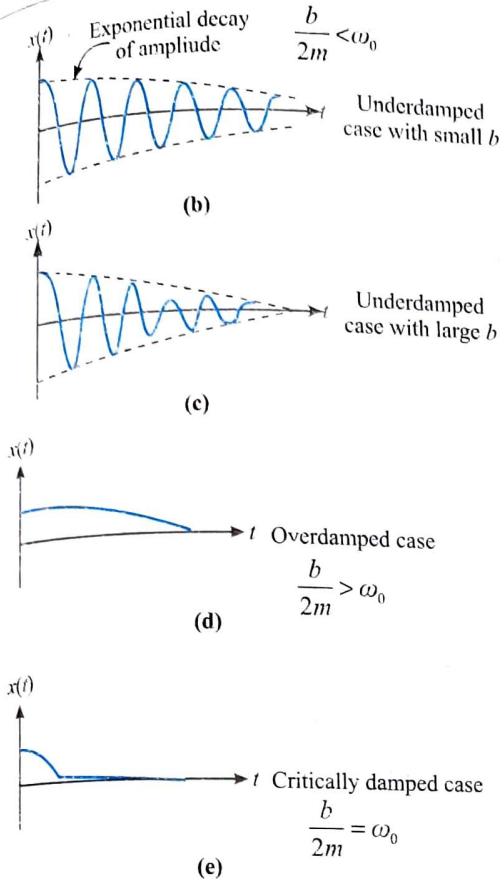
- For a damped oscillator if the damping is small then the mechanical energy decreases exponentially with time as

  $$E = \frac{1}{2}Kx_m^2 e^{-bt/m}$$

Graphs of $x(t)$ with no damping and $x(t)$ for the underdamped case are shown in the Figs. (a), (b) and (c). The motion is oscillatory, but the amplitude decreases exponentially. The SHM has an angular frequency ω and an amplitude $Ae^{-(b/2m)t}$ that decays exponentially in time. In Fig. (c) the damping constant b is larger than that in Fig. (a), yet still small enough so that $\frac{b}{2m} < \omega_0$.



(a)



FORCED OSCILLATIONS

The oscillation in which a body oscillates under the influence of an external periodic force is known as forced oscillation. In forced oscillation, frequency of damped oscillator is equal to the frequency of external force. When the frequency of external force is equal to the natural frequency of the oscillator. Then this state is known as the state of resonance. And this frequency is known as resonant frequency.

Suppose in addition to the spring and damping forces, the mass m is acted on by a periodic force

$$F(t) = F_0 \cos(\omega_d t)$$

where F_0 is a constant and ω_0 is the angular velocity of this driving force. The differential equation for this motion is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega_d t)$$

After a sufficiently long time this driving force is performing work on the mass at the same rate at which energy is dissipated by friction. In this case **steady-state condition** is reached and $x(t)$ settles down to a simple harmonic oscillation described by

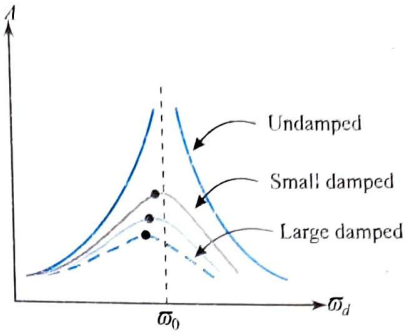
$$x(t) = A\cos(\omega_d t - \phi_0)$$

where $$A = \frac{F_0 / m}{\sqrt{(\omega_d^2 - \omega_0^2)^2 + (b\omega_d / m)^2}}$$

and $$\phi_0 = \arctan\left(\frac{d\omega_d / m}{\omega_0^2 - \omega_d^2}\right)$$

This angle specifies how far the displacement lags behind the driving force.

A graph of A versus driving frequency ω_d is shown in the figure. The steady-state oscillation amplitude depends on the driving frequency ω_d. As the driving frequency is changed the amplitude peaks dramatically at a particular value of ω_d known as the **resonance frequency**. For a large frictional force (large b) the peak is broad and low.



When the damping constant is small, the peak is high and centred closer to ω_0. If there were no friction at all, the amplitude would grow without bound as the driving angular frequency approaches ω_0. The angular frequency ω_0 is called the **natural frequency**, and the dramatic increase in amplitude near this frequency is called **resonance**.

Solved Examples

EXAMPLE 5.1

A force $F = -10x + 2$ acts on a particle of mass 0.1 kg, where k is in m and F in newton. If it is released from rest at $x = 0$, find:

(a) time period;

(b) equation of motion.

Sol.

(a) Given, $F = -10x + 2 \Rightarrow \Delta F = -10\,\Delta x$

But $\Delta F = ma$

$$ma = -10\,\Delta x$$

$$\therefore \quad a = -\left(\frac{10}{0.1}\right)\Delta x = -100\Delta x$$

or $\omega^2 = 100$ or $\omega = 10$ rad/s

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{10} = \frac{\pi}{5}\text{ sec}$$

(b) Let at $x = x_0$, mean position be situated. At mean position, $F = 0$.

$$\therefore \quad F = -10x + 2 \Rightarrow 0 = -10x_0 + 2$$

This gives $x_0 = \frac{2}{10} = 0.2$ m

Let us write equation of SHM about $x = x_0$. Here we are measuring displacement (Δx) from mean position.

Let $\Delta x = A\sin(\omega t + \phi) = A\sin(10t + \phi)$ ...(i)

At $t = 0$, the particle is released from rest, i.e., amplitude position.

Hence, amplitude of oscillation $|A| = (0.2 - 0) = 0.2$ m

Hence, from Eq (i), $-0.2 = 0.2\sin\phi$

$\Rightarrow$ $\sin\phi = -1$ which gives $\phi = -\dfrac{\pi}{2}$

Now Eq. (i) becomes

$$\therefore \quad \Delta x = 0.2 \sin\left(10t - \frac{\pi}{2}\right) = -0.2\cos 10t \ (m)$$

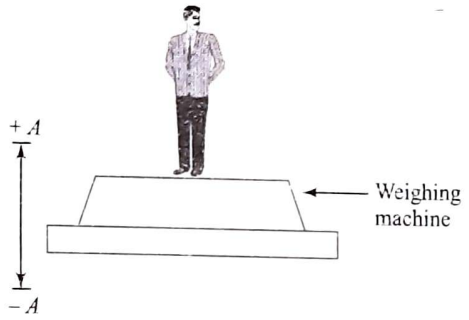
The displacement of the particle from origin will be

$$x = 0.2 + \Delta x = 0.2 - 0.2\cos 10t \ (m)$$

EXAMPLE 5.2

A person normally weighing 60 kg stands on a platform which oscillates up and down harmonically at a frequency 2.0 sec^{-1} and an amplitude 5.0 cm. If a machine on the platform gives the person's weight against time, deduce the maximum and minimum reading it will show, take $g = 10$ m/sec^2.

Sol. As shown in figure, platform is executing SHM with amplitude and angular frequency given as



$A = 5.0$ cm

and $\omega = 2\pi n = 4\pi$ rad/sec $\quad [\text{as } n = 2 \text{ sec}^{-1}]$

At Lower Amplitude Position: Here weighing machine will show weight more than that of man when it is below its equilibrium position when the acceleration of platform is in upward direction. In this situation the free body diagram of man relative to platform is shown in the figure. Here ma is a pseudo-force on man in downward direction relative to platform (or weighing machine). As weighing machine will read the normal reaction on it; thus, for equilibrium relative to platform, we have

$$N = mg + ma$$

or $\quad N = mg + m(\omega^2 y) \quad [\text{as } |a| = \omega^2 y]$

where y is the displacement of platform from its mean position. We wish to find the maximum weight shown by the weighing machine, which is possible when platform is at its lowest extreme position as shown in the figure; thus, maximum reading of weighing machine will be

$$N = mg + m\omega^2 A = 60 \times 10 + 60 \times (4\pi)^2 \times 0.05$$
$$= 600 + 480 = 1080 \text{ N} = 108 \text{ kg wt}$$

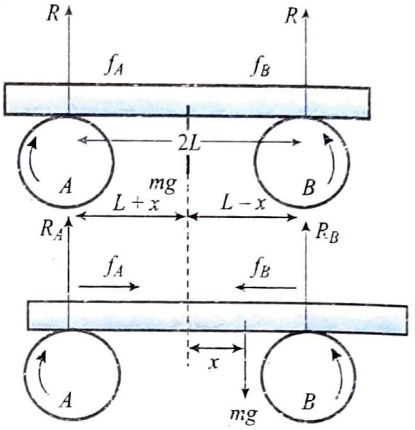
At Upper Amplitude Position: Similarly the machine will show minimum reading when it is at its upper extreme position when pseudo-force on man will be in upward direction; thus, minimum reading of weighing machine will be

$$N = mg - m\omega^2 A = 600 - 480 = 120 \text{ N} = 12 \text{ kg wt}$$

EXAMPLE 5.3

A uniform horizontal plank is resting symmetrically in a horizontal position on two cylindrical drums, which are spinning in opposite direction about their horizontal axes with equal angular velocity. The distance between the axes is $2L$ and the coefficient of friction between the plank and cylinder is μ. If the plank is displaced slightly from the equilibrium position along its length and released, show that it performs simple harmonic motion. Calculate also the time period of motion.

Sol. When the plank is situated symmetrically on the drums, the reactions on the plank from the drums will be equal and so the force of friction will be equal in magnitude but opposite in direction and hence, the plank will be in equilibrium along vertical as well as horizontal direction.



Now if the plank is displaced by x to the right, the reaction will not be equal. For vertical equilibrium of the plank,

$$R_A + R_B = mg \quad \ldots(i)$$

And for rotational of plank, taking moment about centre of mass we have

$$R_A(L + x) = R_B(L - x) \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get $R_A = mg\left(\dfrac{L - x}{2L}\right)$

and $R_B = mg\left(\dfrac{L + x}{2L}\right)$

Now as $f = \mu R$, so friction at B will be more than that at A and will bring the plank back, i.e., restoring force here is

$$F = -(f_B - f_A) = -\mu(R_B - R_A) = -\mu\frac{mg}{L}x$$

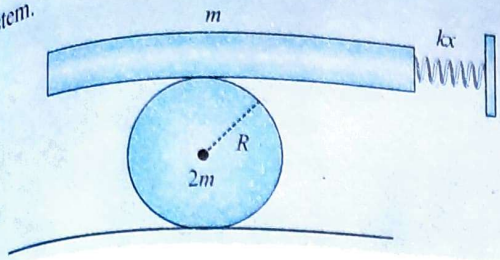
As the restoring force is linear, the motion will be simple harmonic motion with force constant $k = \dfrac{\mu mg}{L}$

So $\quad T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{L}{\mu g}}$

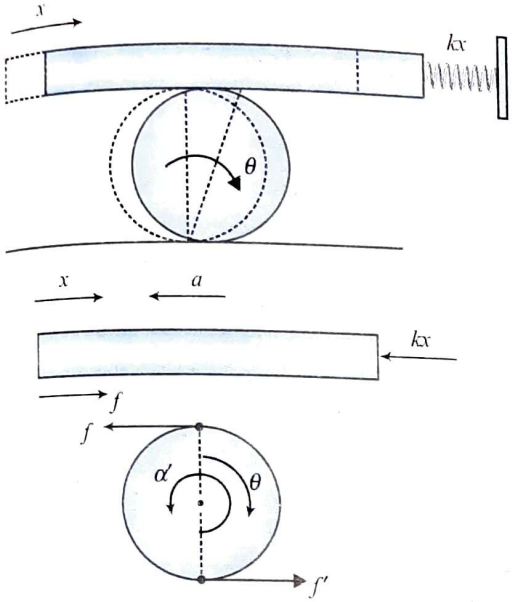
EXAMPLE 5.4

A uniform plank of mass m, free to move in the horizontal direction only, is placed at the top of a solid cylinder of mass $2m$ and radius R. The plank is attached to a fixed wall by means of a light spring of spring constant k. There is no slipping between the cylinder and the plank, and between the cylinder

and the ground. Find the time period of small oscillations of the system.



Sol. **Method 1:** When the plank is displaced slightly (say x) from equilibrium, as there is no sliding at any surfaces, the friction will be of static nature. As cylinder is rolling we can take the motion of the cylinder as pure rotation about point of contact. Free body diagrams of plank and cylinder



Equation of motion for plank

$$kx - f = m(a) \qquad \ldots(i)$$

Writing torque equation to cylinder about point of contact

$$f \cdot (2R) = I_P \cdot \alpha = \left[\frac{1}{2}(2m)R^2 + 2mR^2\right]\alpha = 3mR^2\alpha$$

or $$f = \frac{3}{2}mR\alpha \qquad \ldots(ii)$$

As cylinder is rolling we can write, the acceleration of topmost point of cylinder (or plank)

$$a = \alpha(2R) \Rightarrow \alpha R = \frac{a}{2} \qquad \ldots(iii)$$

Substituting the value of (iii) and (ii), we get

$$f = \frac{3}{4}ma \qquad \ldots(iv)$$

Now, from Eqs. (i) and (iv),

$$kx - \left(\frac{3}{4}ma\right) = ma$$

$$\Rightarrow \quad a = \frac{4k}{7m}x$$

Writing above equation with proper vector relation

$$\vec{a} = -\left(\frac{4k}{7m}\right)\vec{x} \qquad \ldots(v)$$

Comparing with standart equation $\vec{a} = -\omega^2\vec{x}$

which gives $\omega^2 = \frac{4k}{7m}$ or $T = \pi\sqrt{\frac{7m}{k}}$

Method 2: Suppose that the plank is displaced from its equilibrium position by x at time t, the centre of the cylinder is, therefore, displaced by $x/2$.

Therefore the mechanical energy of the system is given by

E = KE (plank) + PE (spring) + KE (cylinder)

$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \frac{1}{2}I_P\omega^2 \qquad \ldots(i)$$

$$I_P = I_G + (2m)R^2 = \frac{1}{2}(2m)R^2 + (2m)R^2 = 3mR^2$$

As cylinder is rolling velocity of centre of cylinder will be half of the velocity of the plank (top most point of the cylinder)

$$v_{\text{cylinder}} = \frac{v}{2} = \omega R \Rightarrow \omega = \frac{v}{2R}$$

Substituting the value of ω in Eq. (i), we get

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \frac{1}{2}(3mR^2)\left(\frac{v}{2R}\right)^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \frac{3}{8}mv^2 = \frac{7}{8}mv^2 + \frac{1}{2}kx^2 \qquad \ldots(ii)$$

Differentiating Eq. (ii) w.r.t. time

$$\frac{dE}{dt} = 0 = \frac{7}{8}m \cdot 2v\left(\frac{dv}{dt}\right) + \frac{1}{2}k \cdot 2x \cdot \frac{dx}{dt} = \frac{7}{4}mv \cdot a + kx \cdot v$$

Here $$\frac{7}{4}ma + kx = 0 \quad \text{or} \quad a = -\frac{4k}{7m}x$$

Comparing with $a = -\omega^2 x \Rightarrow T = 2\pi\sqrt{\frac{7m}{4k}} = \pi\sqrt{\frac{7m}{k}}$

Method 3: We can write Eq. (ii) as

$$E = v^2 + \frac{4k}{7m}x^2 = \text{constant}$$

as total energy of system performing SHM is constant. Comparing with standard equation $v^2 + \omega^2x^2 = \text{constant}$.

Which gives $\omega^2 = \frac{4k}{7m}$ or $T = \pi\sqrt{\frac{7m}{k}}$

EXAMPLE 5.5

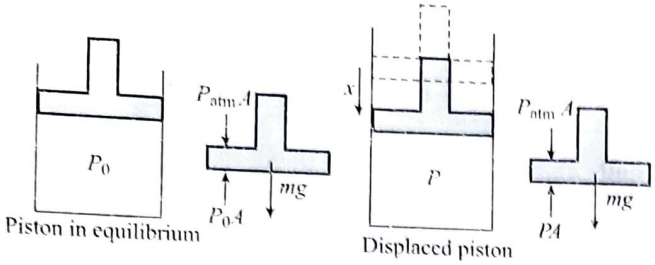
A certain mass of a perfect gas is enclosed in a cylinder of volume V_0 fitted with a smooth heavy piston of mass m and area a. The piston is displaced through a small distance downwards so as to compress the gas isothermally, and then it is let go. Show that the resultant motion is SHM and find its period. Take the atmospheric pressure as P_{atm}.

Sol. Let A be the area of cross section of the cylinder. When the piston is in equilibrium, force on it must balance.

$$P_0A = P_{atm}A + mg \qquad \ldots(i)$$

Let us now displace the piston through a small distance x in downward direction. As the compression of the gas is isothermal, the final pressure P is given by

$$P_0V_0 = P(V_0 - xA)$$



$$P = \frac{P_0V_0}{V_0 - xA} = P_0\left(1 - \frac{xA}{V_0}\right)^{-1}$$

$$P_0 = \left(1 + \frac{xA}{V_0}\right) \qquad [(1+t)^n = 1 + nt \text{ for } t \ll 1]$$

Considering the forces on the displaced piston, the thrust of the gas pressure now exceeds its weight, and hence, magnitude of net upward force is $F = PA - mg - P_{atm}A$

$$F = P_0\left(1 + \frac{xA}{V_0}\right)A - mg - P_{atm}A$$

$$= \frac{P_0A^2}{V_0}x \qquad \text{[using Eq. (i)]}$$

Hence the magnitude of force is proportional to x and its direction is towards equilibrium position.

Comparing with $F = kx$, we get $k = \dfrac{P_0A^2}{V_0}$

Time period $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{mV_0}{P_oA^2}}$

$$T = 2\pi\sqrt{\frac{mV_0}{\left(P_{atm} + \dfrac{mg}{A}\right)A^2}} \Rightarrow T = 2\pi\sqrt{\frac{mV_0}{P_{atm}A^2 + mgA}}$$

EXAMPLE 5.6

A sphere of mass m and radius r rolls without slipping on a rough concave surface of large radius R. It makes small oscillations about the lowest point. Find the time period.

Sol. We can imagine that when the CM of the body is shifted slightly by an angle θ relative to the centre of curvature of the surface, the magnitude of acceleration of the CM of the body can be given as

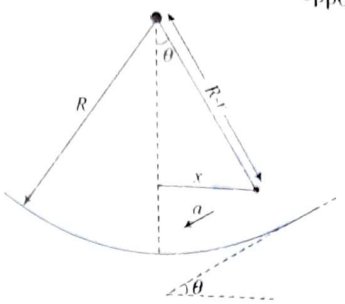
$$a = \frac{g\sin\theta}{1 + \dfrac{k^2}{r^2}}$$

(because the body is accelerating down the inclined plane of instantaneous angle θ)

Substituting $\sin\theta = \theta = \dfrac{x}{R-r}$, we have

$$a = \frac{-gx}{(R-r)\left(1 + \dfrac{k^2}{r^2}\right)}$$

(negative sign is given because $\vec{a}$ and $\vec{x}$ are oppositely directed)



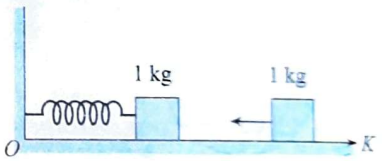
Comparing the above equation with $a = -\omega^2x$, we have

$$\omega = \sqrt{\frac{g}{(R-r)\left(1 + \dfrac{k^2}{r^2}\right)}}$$

For solid sphere $k^2 = 2/5\ r^2$. Hence $\omega = \sqrt{\dfrac{5}{7}\dfrac{g}{(R-r)}}$

EXAMPLE 5.7

One end of an ideal spring is fixed to a wall at origin O and axis of spring is parallel to the x-axis. A block of mass $m = 1$ kg is attached to the free end of the spring and it is performing SHM. Equation of position of the block in coordinate system shown in the figure is $x = 10 + 3\sin(10t)$, where t is in seconds and x in centimetres.



(a) Calculate force constant of the spring.
Another identical block, moving towards origin with velocity 0.6 m/s collides elastically with the block performing SHM at $t = 0$. Calculate
(b) new amplitude of oscillations
(c) equation of position of block performing SHM
(d) percentage increase in oscillation energy (neglect friction)

Sol. $x = 10 + 3\sin(10t)$

Since the block is oscillating in horizontal direction, its equilibrium position corresponds to natural length of the spring. Hence, natural length of spring is, $l_0 = 10$ cm.

Displacement of block from equilibrium position is

$s = x - l_0$

$s = 3\sin(10t)$ cm or $s = 0.03\sin(10t)$ m

But displacement of a block performing SHM is given by

$s = a\sin(\omega t)$.

Comparing it with Eq. (i), $A = 0.03$ m and $\omega = 10$ rad/s

But angular frequency of block, performing SHM and tied with an ideal spring is $\omega = \sqrt{\dfrac{K}{m}}$

Spring constant, $K = \omega^2 m = (10)^2 \times 1 = 100$ N/m

Velocity of the block (performing SHM) is given by

$$v = \frac{dx}{dt} = 30\cos(10t) \text{ cm}$$

Its velocity at time $t = 0$ is $v = 30$ cm/s or 0.30 m/s

Velocity of the block (just before collision) is 0.6 m/s (leftward); therefore it is to be taken (–0.6 m/s).

Since masses of two colliding blocks are equal and collision is elastic, therefore, velocity of oscillating block (just after collision) becomes equal to (–6.0 m/s).

Note: Whenever head-on elastic collision between two bodies having equal masses takes place, their velocities get interchanged.

It means now oscillating block starts to move leftwards with speed 0.6 m/s. But at instant $t = 0$, it is in equilibrium position; therefore, speed –0.6 m/s = $a'\omega$, where a' = new amplitude of oscillation and $\omega = 10$ rad/s (angular frequency), which remains unchanged.

$a' = 0.06$ m = 6 cm

Since at $t = 0$, now block is in equilibrium position and is moving towards negative x-direction, therefore just after collision, phase of the block is π.

Hence, equation of its position x is given by, $x = l_0 + \sin(\omega t + \pi)$ $= 10 + 6 \sin(10t + \pi)$ cm.

Initially oscillation energy was

$$U_1 = \frac{1}{2} ma^2\omega^2 = \frac{1}{2} \times 1 \times (0.03)^2 (10)^2 = 0.045 \text{ J}$$

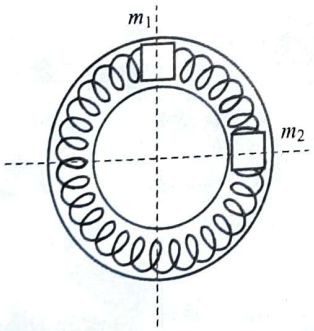
Oscillation energy just after collision $U_2 = \frac{1}{2} m(a')^2 \omega^2 = 0.18$ J

Percentage increase in oscillation energy is

$$\frac{U_2 - U_1}{U_1} \times 100\% = 300\%$$

EXAMPLE 5.8

A circular spring of natural length l_0 is cut and welded with two beads of masses m_1 and m_2 such that the ratio of the lengths of the springs between the beads is 4:1. If the stiffness of the original spring is k, find the frequency of oscillation of the beads in a smooth horizontal rigid tube. Assume $m_1 = m$ and $m_2 = 3m$.

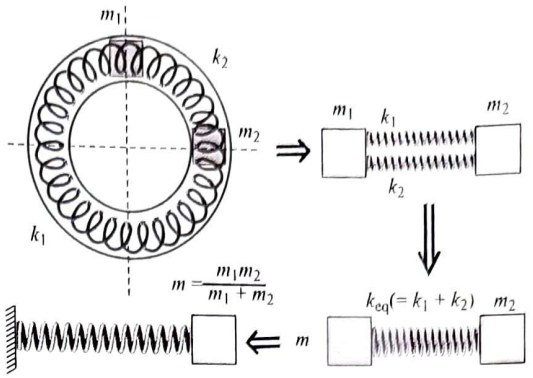


Sol. When m_1 is displaced relative to m_2 by a distance x, each spring will be deformed by same amount. Hence, the springs are connected in parallel. The equivalent spring constant is

$$k_{eq} = k_1 + k_2$$

If the spring is cut, the force constant of spring

$$k \propto \frac{1}{l} \Rightarrow k_1 l_1 = k_2 l_2 = kl$$



Substituting $l_1 = l/5$ and $l_2 = 4l/5$, we have

$$k_1 = 5k \quad \text{and} \quad k_2 = \frac{5}{4}k$$

Then $k_{eq} = \frac{25}{4}k$

Now we have two particles of masses m_1 and m_2 and one spring of stiffness

$$k_{eq} = \frac{25}{4}k$$

The reduced mass is $\mu = \frac{m_1 m_2}{m_1 + m_2}$

where $m_1 = m$ and $m_2 = 3m$

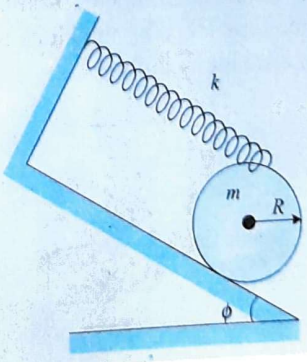
This gives $\mu = 3/4\ m$

Substituting $\mu = 3/4\ m$ and $k_{eq} = 25/4k$ in the formula

$$\omega = \sqrt{\frac{k_{eq}}{\mu}} \Rightarrow \omega = \sqrt{\frac{\frac{25}{4}k}{\frac{3}{4}m}} = \sqrt{\frac{25k}{3m}} = 5\sqrt{\frac{k}{3m}}$$

EXAMPLE 5.9

A uniform cylinder of mass m and radius R is in equilibrium on an inclined plane by the action of a light spring of stiffness k, gravity and reaction force acting on it. If the angle of inclination of the plane is ϕ, find the angular frequency of small oscillation of the cylinder.



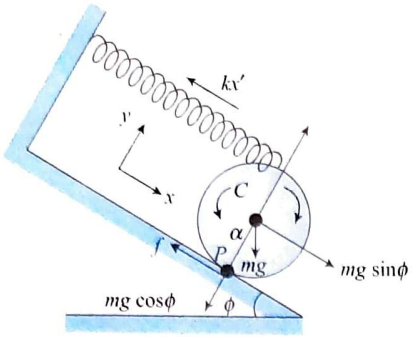
Sol. If the CM is pushed (shifted) down by the additional distance x, the additional elongation of the spring will be $2x$

Let x' be the equilibrium stretch of spring.

The total elongation of the spring $x'' = (x' + 2x)$

The spring force = $F_S' = kx' = k(x' + 2x)$

The torque acting on the cylinder about P by the spring force is $\tau_S = F_S'(2R) = 2k(x' + 2x)R$



The gravitational torque about P is $\tau_g = -mgR\sin\phi$

The net torque about P is

$$\tau_{net} = \tau_S + \tau_g = 2kx'R + 4kxR - mgR\sin\phi$$

Since, $2kx' = mg\sin\phi$, from Eq. (ii), we have $\tau_{net} = 4kxR$

Substituting $x = R\theta$, we have the vector equation,

$$\tau_{net} = 4kR^2\phi$$

Torque equation: The torque rotates the cylinder with angular acceleration α at the angular position θ. Applying Newton's second law of rotation, we have $\alpha = \dfrac{\tau_{net}}{I_P}$

where $\tau_{net} = -4kR^2\theta$

This gives $\alpha = \dfrac{-4kR^2}{I_P}\theta$

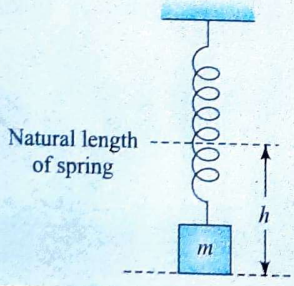
Comparing the above equation with $\alpha = -\omega^2\theta$,

we have $\omega = \sqrt{\dfrac{4kR^2}{I_P}}$, where $I_P = \dfrac{3mR^2}{2}$

Then, $\omega = 2\sqrt{\dfrac{2k}{3m}}$

EXAMPLE 5.10

A spring block pendulum is shown in the given figure. The system is hanging in equilibrium. A bullet of mass $m/2$ moving at a speed u hits the block from downward direction and gets embedded in it. Find the amplitude of oscillation of the block now. Also find the time taken by the block to reach its upper extreme position after hit by bullet.



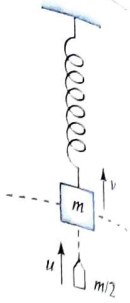
Sol. If block is in equilibrium, then spring must be at some stretch if it is h, we have $mg = kh$.

If a bullet of mass $m/2$ gets embedded in the block, due to this inelastic impact its new mass becomes $3m/2$ and now the new mean position of the block will be, say, at a depth h_1 from old mean position, then we must have

$$\frac{3}{2}mg = k(h + h_1)$$

$$\frac{3m}{2}g = k(h + h_1)$$

$$h_1 = \frac{mg}{2k} \qquad (\text{as } mg = kh)$$



Just after impact due to inelastic collision if the velocity of block becomes v, we have, according to momentum conservation,

$$\frac{m}{2}u = \frac{3m}{2}v \quad \text{or} \quad v = \frac{u}{3}$$

Now the block executes SHM and at $t = 0$ block is at a distance $h_1 = \dfrac{mg}{2k}$ above its mean position and having a velocity $u/3$. If amplitude of oscillation is A, we have

$$\frac{u}{3} = \omega\sqrt{A^2 - \left(\frac{mg}{2k}\right)^2}$$

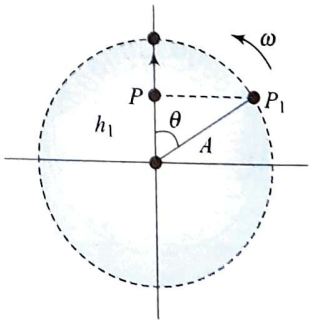
or $$\frac{u^2}{9} = \frac{2k}{3m}\left[A^2 - \left(\frac{mg}{2k}\right)^2\right]$$

[As for this spring block system $\omega = \sqrt{\dfrac{k}{(3m/2)}}$]

$$A = \sqrt{\frac{mu^2}{6k} + \left(\frac{mg}{2k}\right)^2}$$

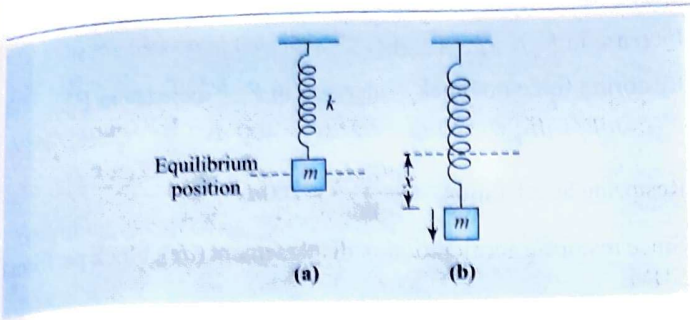
Now time taken by particle to reach the topmost point can be obtained by circular motion representation as shown in the figure. This figure shows the position of block P and its corresponding circular motion particle P_0 at the $t = 0$. Block P will reach its upper extreme position when particle P_0 will traverse the angle θ and reach the topmost point. As P_0 moves at constant angular velocity ω, it will take a time given as

$$t = \frac{\theta}{\omega} = \frac{\cos^{-1}(h_1/A)}{\sqrt{\dfrac{k}{3m/2}}} = \sqrt{\frac{3m}{2k}}\cos^{-1}\left(\frac{3mg}{2kA}\right)$$



EXAMPLE 5.11

Figure (a) shows a spring block system, hanging in equilibrium. The block of system is pulled down by a distance x and imparted a velocity v in downward direction as shown in Fig. (b). Find the time it will take to reach its mean position. Also find the maximum distance to which it will move before returning back towards mean position.



Sol. As shown in Fig. (b), when the block is pulled down by a distance x and thrown downward, it will start executing SHM. It will go further to a distance A (amplitude) from mean position before returning back which can be found by using the velocity of block v at a displacement x from its mean position as

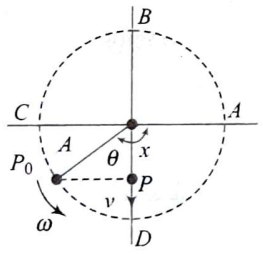
$$v = \omega\sqrt{A^2 - x^2}$$

or $$v^2 = \frac{k}{m}(A^2 - x^2)$$

$$\left[\text{as for spring block system } \omega = \sqrt{\frac{k}{m}}\right]$$

or $$A = \sqrt{\frac{mv^2}{k} + x^2}$$

Now time of motion of bob can be obtained by circular motion representation of the respective SHM. Corresponding circular motion representation for this SHM is shown in figure at $t = 0$. At $t = 0$, block P is at a distance x from its mean position in downward direction and it is moving downward so we consider its corresponding circular motion particle in III quadrant as shown in reference angular velocity. We consider P will reach its mean position when particle P_0 reaches position A by traversing an angle θ as shown in the figure.



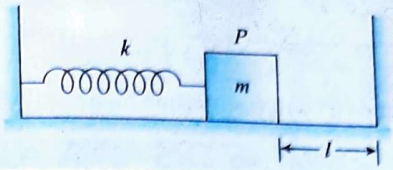
Thus it will take a time given as $t = \dfrac{\theta}{\omega} = \dfrac{\pi - \sin^{-1}(x/A)}{\sqrt{\dfrac{k}{m}}}$

or $$t = \sqrt{\frac{m}{k}}\left[\pi - \sin^{-1}\frac{x}{\sqrt{\dfrac{mv^2}{k} + x^2}}\right]$$

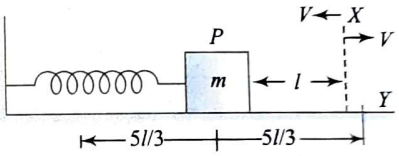
The maximum distance to which block will move from its initial position is $A - x$ as it gas upto its lower extreme at a distance equal to its amplitude A from to mean position.

EXAMPLE 5.12

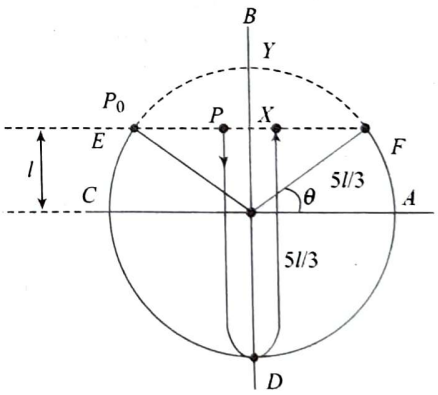
Figure shows a block P of mass m resting on a smooth horizontal surface, attached to a spring of force constant k which is rigidly fixed on the wall on left side, shown in the figure. At a distance l to the right of block there is a rigid wall. If block is pushed toward lift so that spring is compressed by a distance $5l/3$ and when released, it will start its oscillations. If collision of block with the wall is considered to be perfectly elastic. Find the time period of oscillations of the block.



Sol. As shown in the figure, as the block is released from rest at a distance $5l/3$ from its mean position, this will be the amplitude of oscillation. But on the other side of mean position block can move only up to a distance l from mean position and then it will return from this point with equal velocity due to elastic collision. Consider figure. If no right wall is present during oscillation, block P will execute complete SHM on right side of mean position also up to its amplitude $5l/3$. Thus, we can observe the block P at a point X at a distance l from mean position (where in our case wall is present), if block passes this position at speed v (which is the speed with which block hits the wall in our case), after reaching its extreme position Y, it will return and during return path it will cross the position X with the same speed v (as displacement from mean position is same).



Thus, we can state that in our case of given problem, that block P is executing SHM but it skips a part XYX on right half of its motion due to electric collision of the block with the wall. If T_0 is the time period of this oscillation we look at the following figure, which shows the corresponding circular motion representation.



Here during oscillation of point P, particle P_0 covers its circular motion along $ECDAF$, and from F it instantly jumps to E (due to elastic collision of P with wall at X) and again carry out $ECDAF$ and so on, thus we can find the time of this total motion as

$$t = \frac{\pi + 2\theta}{\omega} = \frac{\pi + 2\sin^{-1}(3/5)}{\sqrt{\dfrac{k}{m}}}$$

$$= \sqrt{\frac{m}{k}}\left[\pi + 2\sin^{-1}\left(\frac{3}{5}\right)\right] \quad \text{...(i)}$$

We can also find the time period of this motion by subtracting the time of FYE from the total time period as

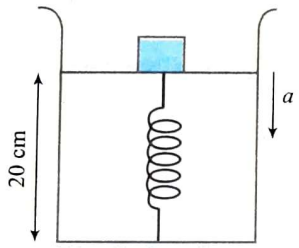
$$t = \frac{2\pi - 2\cos^{-1}\left(\frac{3}{5}\right)}{\omega} = \sqrt{\frac{m}{k}}\left[2\pi - 2\cos^{-1}\left(\frac{3}{5}\right)\right] \quad ...(ii)$$

Equations (i) and (ii) will result same numerical value.

EXAMPLE 5.13

A rectangular tank having base 15 cm × 20 cm is filled with water (density ρ = 1000 kg/m^3) up to 20 cm height. One end of an ideal spring of natural length h_0 = 20 cm and force constant K = 280 N/m is fixed to the bottom of the tank so that the spring remains vertical.

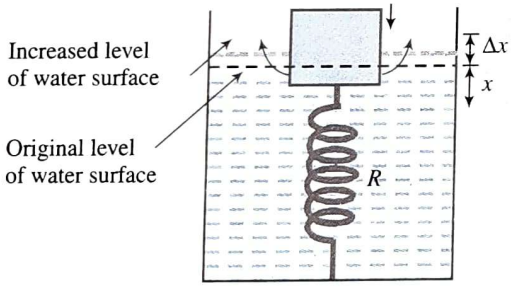
This system is in an elevator moving downwards with acceleration a = 2 m/s^2. A cubical block of side l = 10 cm and mass m = 2 kg is gently placed over the spring and released gradually, as shown in the figure.



(a) Calculate compression of the spring in equilibrium position.

(b) If the block is slightly pushed down from equilibrium position and released, calculate frequency of its vertical oscillations.

Sol. Let, in equilibrium position, compression of spring be x. Liquid of volume l^2x is displaced from its original position and level of liquid in tank rises as shown in the figure.



This rise in level, $\Delta x = \frac{l^2 x}{A - l^2}$ where A = 15 cm × 20 cm (base area of tank), $\Delta x = 0.5x$

Mass of water displaced by the block is $l^2(x + \Delta x)\rho = 15x$ kg

Upthrust exerted by water = apparent weight of water displaced.

Upthrust $F_1 = 1.5\,x\,(g - a) = 120\,x$ N

Upward force exerted by spring, $F_2 = Kx = 280x$

Considering free body diagram of the block, $mg - (F_1 + F_2) = ma$

Substituting values of F_1 and F_2, $x = 0.04$ m = 4 cm

If the block is slightly pushed downward by dx, both F_1 and F_2, increase.

Increase in F_1 is $dF_1 = 120\,dx$

Increase in F_2 is $dF_2 = 280\,dx$

Restoring force on block = increase in F_1 + increase in F_2

$$dF_1 + dF_2 = (120\,dx + 280\,dx) = 400\,dx$$

Restoring acceleration $= \frac{400dx}{m} = 200dx$

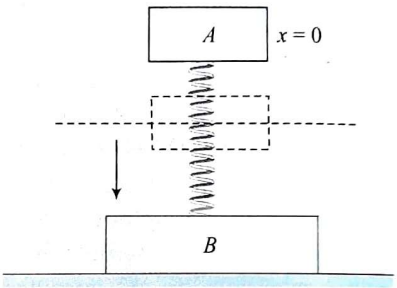
Since restoring acceleration ∝ displacement (dx), block performs SHM.

Hence, frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = \frac{1}{2\pi}\sqrt{200} = \frac{5\sqrt{2}}{\pi} \text{ per second}$$

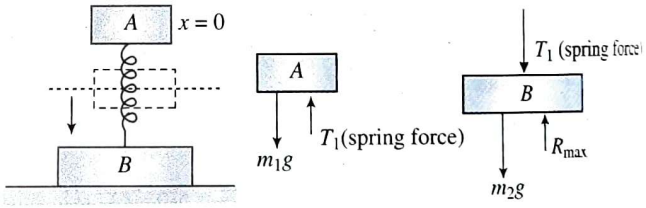
EXAMPLE 5.14

A body A of mass m_1 = 1 kg and body B of mass m_2 = 5.1 kg are interconnected by a spring as shown in figure. The body A performs free vertical harmonic oscillations with the amplitude 1.6 cm and frequency 25 Hz. Neglecting the mass of the spring, find the maximum and minimum value of force that the system exerts on the bearing surface.



Sol. As A oscillates up and down, the normal reaction between B and the surface is maximum. when A is at lowest position and it is minimum when A is at topmost position.

Case I: Lowest position of A



Let T_1 be the force in the compressed spring.

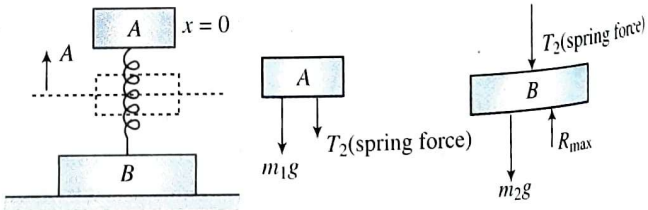
As B is at rest, $T_1 = m_2 g = R_{max}$

As acceleration of A is $\omega^2 A$ towards mean position (up),

$$T_1 - m_1 g = m_1 \omega^2$$

Combining, $R_{max} = (m_1 + m_2)g + m_2\omega^2 A$

Case II: Topmost position of A



Let T_2 be the tension in the elongated spring. As B is at rest,

$$T_2 + R_{min} = m_2 g$$

Acceleration of A is $\omega^2 A$ towards mean position (down)

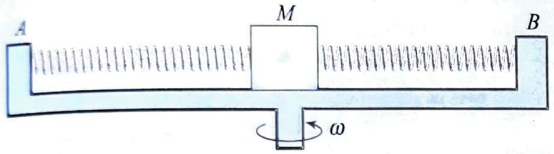
$$T_2 + m_1 g = m_1 \omega^2 A$$

Combining, we get $R_{min} = (m_1 + m_2)g - m_1\omega^2 A$

Note: B will leave the surface if $m_1\omega^2 A > (m_1 + m_2)g$.

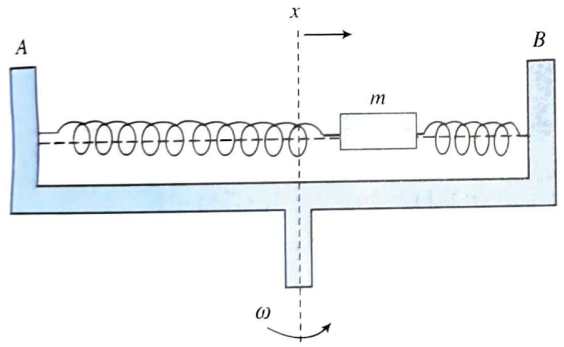
EXAMPLE 5.15

In the arrangement as shown in the figure, the sleeve of mass M is fixed between two identical spring whose combined force constant is k. The sleeve can slide without friction over a horizontal bar AB. The arrangement rotates with a constant angular velocity ω about a vertical axis passing through the middle of the bar. Find the period of small oscillations of the sleeve.



Sol. We will analyse the problem relative to the rotating bar AB. As the acceleration of bar is centripetal, a pseudo force will act on sleeve away from the centre and will be of magnitude $m(\omega^2 x)$.

If the sleeve is displaced by x, net force towards centre



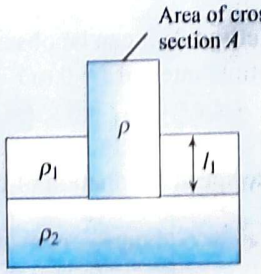
$$F = kx - m\omega^2 x = (m - m\omega^2)x$$

$$\Rightarrow \quad \text{Period} = 2\pi\sqrt{\frac{m}{k - m\omega^2}}$$

Note that if $k < m\omega^2$, there will be no oscillations of the sleeve. It will rush to the point B if it is displaced slightly (for $k < m\omega^2$) or will remain in the displaced position (for $k < m\omega^2$).

EXAMPLE 5.16

A vertical pole of length l, density ρ, area of cross section A floats in two immiscible liquids of densities ρ_1 and ρ_2. In equilibrium position the bottom end is at the interface of the liquids. When the cylinder is displaced vertically, find its time period of oscillation.



Sol. In equilibrium position, net force on pole is zero.

$$\rho A l g = \rho_1 A l_1 g \qquad \ldots(i)$$

In displaced position, net force,

$$F_1 = \rho A l g - \rho_1 A l_1 g - \rho_2 A y g = -\rho_2 A g y$$

Acceleration of the pole, $a_1 = -\dfrac{\rho_2 A g}{\rho A l} y$

Hence in the liquid of density r_2, $\omega_1 = \sqrt{\dfrac{\rho_2 g}{\rho l}} \Rightarrow T_1 = 2\pi\sqrt{\dfrac{\rho l}{\rho_2 g}}$

The pole system is in the liquid of density ρ_2 for half time period only. When it moves in the liquid of density ρ_1, the net force is $F_2 = -\rho A l g + (l_1 - y')A\rho_1 g = -\rho_1 A g y'$

$$a_2 = -\frac{\rho_1 A g}{\rho A l} y' \quad \Rightarrow \quad \omega_2 = \sqrt{\frac{\rho_1 g}{\rho l}}$$

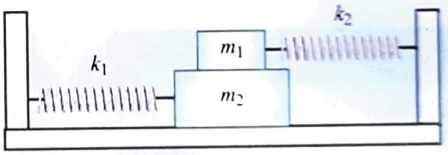
$$T_2 = 2\pi\sqrt{\frac{\rho l}{\rho_1 g}}$$

Total time period of oscillation of the cylinder is

$$T = \frac{T_1}{2} + \frac{T_2}{2} = \pi\left[\sqrt{\frac{\rho l}{\rho_1 g}} + \sqrt{\frac{\rho l}{\rho_2 g}}\right]$$

EXAMPLE 5.17

In the shown arrangement, both the springs are in their natural lengths. The coefficient of friction between m_2 and m_1 is μ. There is no friction between m_1 and the surface. If the blocks are displaced slightly, they together perform simple harmonic motion.



Obtain

(a) the frequency of such oscillations.

(b) the condition if the frictional force on block m_2 is to act in the direction of its displacement from mean position.

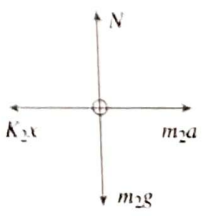
(c) the if the condition obtained in (b) is met, what can be the maximum amplitude of their oscillations?

Sol.

(a) For the oscillations of the blocks together, the equivalent force constant equals $K_1 + K_2$ and hence the frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{K_1 + K_2}{m_1 + m_2}}$$

(b) Assuming that the blocks $(m_1 + m_2)$ are displaced towards right, the various forces (except the frictional force) on m_2, in the reference frame of m_1 are shown in the following free body diagram.



where $a = \dfrac{(K_1 + K_2)x}{m_1 + m_2}$

For the frictional force to act on m_2 in the direction of its displacement

$$K_2 x > m_2\left[\frac{K_1 + K_2}{m_1 + m_2}\right]x$$

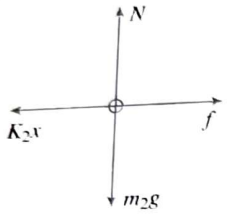
$$m_1 K_2 + m_2 K_2 - m_2 K_1 - m_2 K_2 > 0$$

$$m_1 K_2 - m_2 K_1 > 0$$

$$\frac{m_1}{m_2} > \frac{K_1}{K_2}$$

The above inequality is the desired condition.

(c) Assuming $\frac{m_1}{m_2} > \frac{K_1}{K_2}$ the FBD of m_2 with m_1 and m_2 together displaced towards right can be shown as



If A be the amplitude of oscillations of m_2, $K_2A - f = m_2\omega^2 A$ (for the extreme position)

$$f = K_2A - m_2\omega^2 A \Rightarrow f = A(K_2 - m_2\omega^2)$$

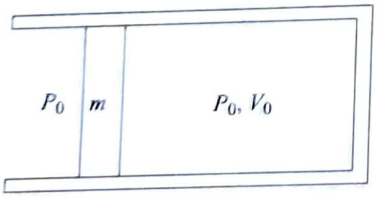
$$f \le \mu m_2 g \Rightarrow \mu m_2 g \ge A(K_2 - m_2\omega^2)$$

$$A \le \frac{\mu m_2 g}{(K_2 - m_2\omega^2)} \Rightarrow A_{max} = \frac{\mu m_2 g}{(K_2 - m_2\omega^2)}$$

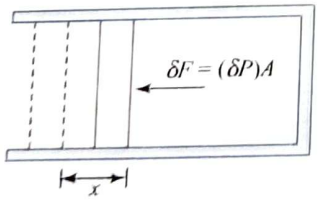
Here $\omega^2 = \frac{(K_1 + K_2)}{(m_1 + m_2)}$; $A_{max} = \frac{\mu m_2 g(m_1 + m_2)}{m_1K_2 - m_2K_1}$

EXAMPLE 5.18

A smooth piston of mass m, area of cross-section A is in equilibrium in a gas jar when the pressure of the gas is P_0. Find the angular frequency of oscillation of the piston, assuming adiabatic change of state of the gas.



Sol.



n adiabatic procedure, $PV^\gamma = C$

aking log of both sides, we have $\log P + \gamma \log V = \log C$

aking differentials of both sides, we have $\frac{\delta P}{P} + \frac{\gamma \delta V}{V} = 0$

his gives the excess pressure ΔP (when we disturb the piston)

$$\delta P = \frac{-\gamma P}{V}\delta V$$

where ΔV = change in volume of the gas = $A\Delta x$.

Substituting $\delta P = \frac{\delta F}{A}$, $V = V_0$, $P = P_0$ and $\delta V = A\delta x$

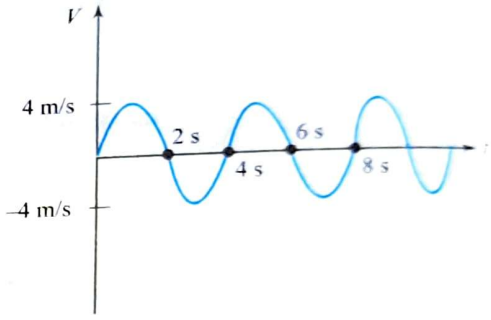
We have $k_{eff} = \left|\frac{\delta F}{\delta x}\right| = \frac{\gamma P_0 A^2}{V_0}$

Then, using $\omega = \sqrt{\frac{k_{eff}}{m}}$

We have $\omega = \sqrt{\frac{\gamma P_0 A^2}{mV_0}}$

EXAMPLE 5.19

If velocity of a particle moving along a straight line changes sinusoidally with time as shown in the given graph. Find the average speed over time interval $t = 0$ to $t = 2(2n - 1)$ seconds, n being any positive integer.



Sol. Method 1:

$$\text{Average speed} = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{S}{2(2n-1)}$$

Here $\Delta t = 2(2n - 1) = 4n - 2 = 4(n - 1) + 2$

From the graph it is clear that time period T is 4 s

$$\therefore \quad \Delta t = (n - 1)\,T + \frac{T}{2}$$

Total distance travelled in one time period = $4A$, where A = amplitude.

Therefore, total distance travelled in Δt is

$$S = (n - 1)\,4A + 2A = (2n - 1)\,2A$$

$$\therefore \quad \langle v \rangle = \frac{(2n-1)2A}{2(2n-1)} = A \quad \text{and} \quad \omega A = v_{max} = 4$$

$$\Rightarrow \quad \frac{2\pi}{4}A = 4$$

$$\Rightarrow \quad A = \frac{8}{\pi} \qquad \therefore \quad \langle v \rangle = \frac{8}{\pi}\ \text{m/s}$$

Method 2: It can be observed from the graph that average speed in time interval $t = 0$ to $t = 2$ s is same as that in intervals $t = 0$ to $t = 4$ s, $t = 0$ to $t = 8$ s, $t = 0$ to $t = 12$ s

or $t = 0$ to $t = 2(2n - 1)$ s

The speed as function of time is

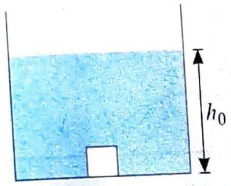
$$v = \left|4\sin\frac{2\pi}{T}t\right| = \left|4\sin\frac{2\pi}{4}t\right| = \left|4\sin\frac{\pi t}{2}\right|$$

average speed in time interval $t = 0$ to $t = 2$ s is

$$v = \frac{\int_0^2 v\,dt}{\int_0^2 dt} = \frac{\int_0^2 4\sin\frac{\pi t}{2}\,dt}{2} = \frac{8}{\pi}\text{ m/s}$$

EXAMPLE 5.20

The shown container contains liquid of variable density which varies as $d = d_0\left(4 - \frac{3y}{h_0}\right)$ where h_0 and d_0 are constants and y is measured from the bottom of container. A small solid block density is $\frac{5d_0}{2}$ and mass m is released from the bottom of the container. Show that the block will execute SHM and find its time period.



Sol. The volume of the block, $V = \frac{m}{\rho} \Rightarrow V = \frac{m}{5d_0/2} = \frac{2m}{(5d_0)}$

Buoyant force acting on the block,

$$B = Vdg = \frac{2m}{5d_0} \times d_0\left(4 - \frac{3y}{h_0}\right) \times g$$

At equilibrium of the block, $mg = \frac{2m}{5}\left(4 - \frac{3y}{h_0}\right)g \Rightarrow y = \frac{h_0}{2}$

i.e., when body is at height $h_0/2$, force acting on it is zero.

Now displace the block slightly beyond equilibrium position.

If we replace y by $\frac{h_0}{2} + x$ we get the equation of force acting upwards as

$$F = \frac{2m}{5}\left[4 - \frac{3}{h_0}\left(\frac{h_0}{2} + x\right)\right]g - mg = -\frac{2m}{5} \times \frac{3x}{h_0}g = -\frac{6mgx}{5h_0}$$

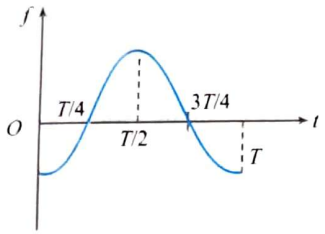
$$F = -\frac{6mg}{5h_0}x \Rightarrow a = -\frac{6g}{5h_0}x$$

comparing with $\vec{a} = -\omega^2\vec{x}$, we find $\omega^2 = \frac{6g}{5h_0} \Rightarrow T = 2\pi\sqrt{\frac{5h_0}{6g}}$

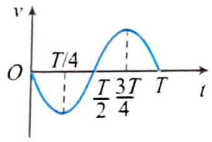
Exercises

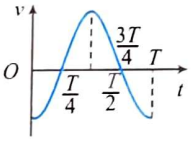
Single Correct Answer Type

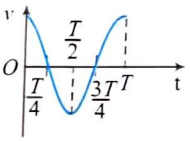
1. A body is performing simple harmonic motion with amplitude a and time period T. Variation of its acceleration (f) with time (t) is shown in the given figure.

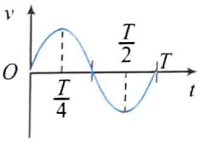


If at time t, velocity of the body is v, which of the following graphs is correct?

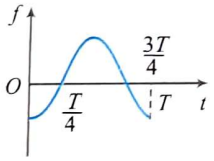
(1) 

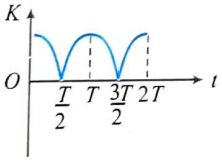
(2) 

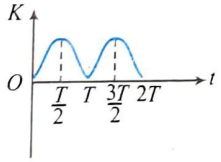
(3) 

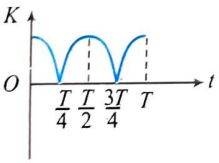
(4) 

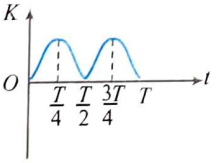
2. A particle is performing SHM with amplitude a and time period T. Its acceleration f varies with time as shown in the following figure. If at time t, kinetic energy of the particle is K, which of the following graphs is correct?



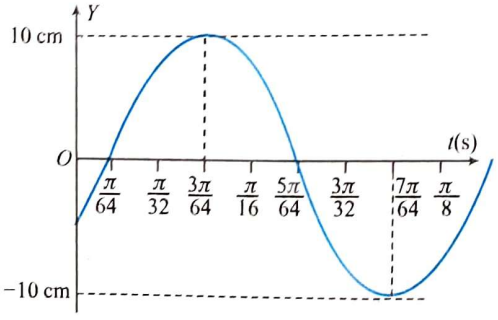
(1) 

(2) 

(3) 

(4) 

3. The diagram below shows a sinusoidal curve. The equation of the curve will be



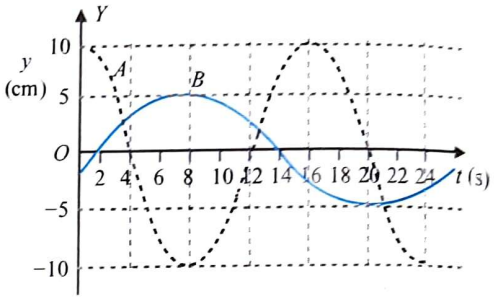
(1) $y = 10\sin\left(16t + \frac{\pi}{4}\right)$ cm

(2) $y = 10\sin\left(16t + \frac{\pi}{3}\right)$ cm

(3) $y = 10\sin\left(16t - \frac{\pi}{4}\right)$ cm

(4) $y = 10\cos\left(16t + \frac{\pi}{4}\right)$ cm

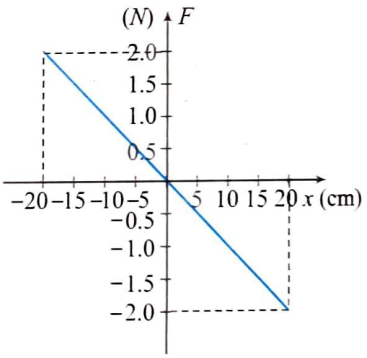
4. The following figure shows the displacement versus time graph for two particles A and B executing simple harmonic motions. The ratio of their maximum velocities is



(1) 3:1 (2) 1:3

(3) 1:9 (4) 9:1

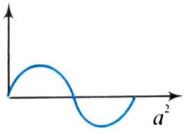
5. The figure shows the variation of force acting on a particle of mass 400 g executing simple harmonic motion. The frequency of oscillation of the particle is

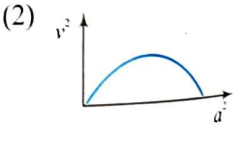


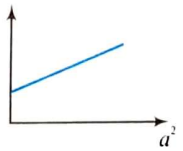
(1) 4 s^{-1} (2) $(5/2\pi)$ s^{-1}

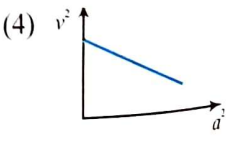
(3) $(1/8\pi)$ s^{-1} (4) $(1/2\pi)$ s^{-1}

6. A graph of the square of the velocity against the square of the acceleration of a given simple harmonic motion is

(1) 

(2) 

(3) 

(4) 

7. A block is resting on a piston which executes simple harmonic motion in vertical plain with a period of 2.0 s in vertical plane at an amplitude just sufficient for the block to separate from the piston. The maximum velocity of the piston is

(1) $\frac{5}{\pi}$ m/s (2) $\frac{10}{\pi}$ m/s

(3) $\frac{\pi}{2}$ m/s (4) $\frac{20}{\pi}$ m/s

8. A spring mass system preforms SHM. If the mass is doubled keeping amplitude same, then the total energy of SHM will become

(1) double (2) half

(3) unchanged (4) 4 times

9. A particle moves with a simple harmonic motion in a straight line. In the first second starting from rest it travels a distance a and in the next second it travels a distance b in the same direction. The amplitude of the motion is

(1) $\frac{2a^2}{3b-a}$ (2) $\frac{3a^2}{3a-b}$

(3) $\frac{2a^2}{3a-b}$ (4) $\frac{3a^2}{3b-a}$

10. A particle executes SHM with time period 8 s. Initially, it is at its mean position. The ratio of distance travelled by it in the 1st second to that in the 2nd second is

(1) $\sqrt{2}:1$ (2) $1:(\sqrt{2}-1)$

(3) $(\sqrt{2}+1):\sqrt{2}$ (4) $(\sqrt{2}-1):1$

11. The instantaneous displacement x of a particle executing simple harmonic motion is given by $x = a_1 \sin \omega t + a_2 \cos(\omega t + \pi/6)$. The amplitude A of oscillation is given by

(1) $\sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos\frac{\pi}{6}}$

(2) $\sqrt{a_1^2 + a_2^2 + 2a_1a_2 \cos\frac{\pi}{3}}$

(3) $\sqrt{a_1^2 + a_2^2 - 2a_1a_2 \cos\frac{\pi}{6}}$

(4) $\sqrt{a_1^2 + a_2^2 - 2a_1a_2 \cos\frac{\pi}{3}}$

12. A simple harmonic motion along the x-axis has the following properties: amplitude = 0.5 m, the time to go from one extreme position to other is 2 s and $x = 0.3$ m at $t = 0.5$ s. The general equation of the simple harmonic motion is

(1) $x = (0.5\text{ m}) \sin\left[\frac{\pi t}{2} + 8^\circ\right]$

(2) $x = (0.5\text{ m}) \sin\left[\frac{\pi t}{2} - 8^\circ\right]$

(3) $x = (0.5\text{ m}) \cos\left[\frac{\pi t}{2} + 8^\circ\right]$

(4) $x = (0.5\text{ m}) \cos\left[\frac{\pi t}{2} - 8^\circ\right]$

13. A block of mass m is suspended from the ceiling of an elevator (at rest) through a light spring of spring constant k. Suddenly, the elevator starts falling down with acceleration g. Then

(1) the block executes simple harmonic motion with time period $2\pi\sqrt{\frac{m}{k}}$

(2) the block executes simple harmonic motion with amplitude $\frac{mg}{k}$

(3) the block executes simple harmonic motion about its mean position and the mean position is the position when the spring acquires its natural length

(4) all of the above

14. A mass m attached to a spring of spring constant k is stretched a distance x_0 from its equilibrium position and released with no initial velocity. The maximum speed attained by mass in its subsequent motion and the time at which this speed would be attained are, respectively,

(1) $\sqrt{\frac{k}{m}}x_0, \pi\sqrt{\frac{m}{k}}$ (2) $\sqrt{\frac{k}{m}}\frac{x_0}{2}, \frac{\pi}{2}\sqrt{\frac{m}{k}}$

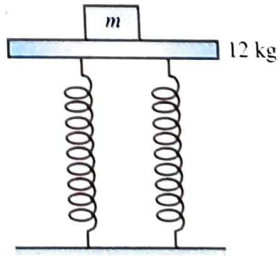
(3) $\sqrt{\frac{k}{m}}x_0, \frac{\pi}{2}\sqrt{\frac{m}{k}}$ (4) $\sqrt{\frac{k}{m}}\frac{x_0}{2}, \pi\sqrt{\frac{m}{k}}$

15. The displacement of a body executing SHM is given by $x = A \sin(2\pi t + \pi/3)$. The first time from $t = 0$ when the velocity is maximum is

(1) $\frac{1}{3}$s (2) $\frac{1}{12}$s

(3) $\frac{5}{6}$s (4) $\frac{5}{12}$s

16. A plank of mass 12 kg is supported by two identical springs as shown in the given figure. The plank always remains horizontal. When the plank is pressed down and released, it performs simple harmonic motion with time period 3 s. When a block of mass m is attached to the plank the time period changes to 6 s. The mass of the block is



(1) 48 kg (2) 36 kg

(3) 24 kg (4) 12 kg

17. The time taken by a particle executing simple harmonic motion to pass from point A to B where its velocities are same, is 2 s. After another 2 s, it returns to (B). The time period of oscillation is

(1) 2 s (2) 4 s

(3) 6 s (4) 8 s

18. Two springs are made to oscillate simple harmonically due to the same mass individually. The time periods obtained are T_1 and T_2. If both the springs are connected in series and then made to oscillate by the same mass, the resulting time period will be

(1) $T_1 + T_2$ (2) $\dfrac{T_1 T_2}{T_1 + T_2}$

(3) $\sqrt{T_1^2 + T_2^2}$ (4) $\dfrac{T_1 + T_2}{2}$

19. A particle performs simple harmonic motion about O with amplitude A and time period T. The magnitude of its acceleration at $t = T/8$ s after the particle reaches the extreme position would be

(1) $\dfrac{4\pi^2 A}{\sqrt{2T^2}}$ (2) $\dfrac{4\pi^2 A}{T^2}$

(3) $\dfrac{2\pi^2 A}{\sqrt{2T^2}}$ (4) None of these

20. In the previous question, the magnitude of velocity of particle at the mentioned instant is

(1) $\dfrac{\pi A}{T}$ (2) $\dfrac{\sqrt{2}\pi A}{T}$

(3) zero (4) $\sqrt{\dfrac{7}{8}} \times \dfrac{2\pi A}{T}$

21. A particle performs SHM of amplitude A along a straight line. When it is at a distance $\sqrt{3}/2\ A$ from mean position, its kinetic energy gets increased by an amount $1/2\ m\omega^2 A^2$ due to an impulsive force. Then its new amplitude becomes

(1) $\dfrac{\sqrt{5}}{2}A$ (2) $\dfrac{\sqrt{3}}{2}A$

(3) $\sqrt{2}A$ (4) $\sqrt{5}A$

22. A horizontal spring–block system of mass 2 kg executes SHM. When the block is passing through its equilibrium position, an object of mass 1 kg is put on it and the two move together. The new amplitude of vibration is (A being its initial amplitude)

(1) $\sqrt{\dfrac{2}{3}}A$ (2) $\sqrt{\dfrac{3}{2}}A$

(3) $\sqrt{2}A$ (4) $\dfrac{A}{\sqrt{2}}$

23. The acceleration of a particle moving along x-axis is $a = -100x + 50$. It is released from $x = 2$. Here 'a' and 'x' are in SI units. The motion of particle will be

(1) periodic, oscillatory but not SHM

(2) periodic but not oscillatory

(3) oscillatory but not periodic

(4) simple harmonic

24. In the above question, the speed of the particle at origin will be

(1) $10\sqrt{2}$ m/s (2) 1.5 m/s

(3) 10 m/s (4) None of these

25. The potential energy of a simple harmonic oscillator of mass 2 kg in its mean position is 5 J. If its total energy is 9 J and its amplitude is 0.01 m, its time period would be

(1) $\pi/10$ s (2) $\pi/20$ s

(3) $\pi/50$ s (4) $\pi/100$ s

26. A certain simple harmonic vibrator of mass 0.1 kg has a total energy of 10 J. Its displacement from the mean position is 1 cm when it has equal kinetic and potential energies. The amplitude A and frequency n of vibration of the vibrator are

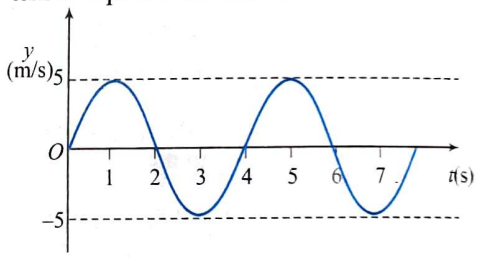
(1) $A = \sqrt{2}$ cm, $n = \dfrac{500}{\pi}$ Hz

(2) $A = \sqrt{2}$ cm, $n = \dfrac{1000}{\pi}$ Hz

(3) $A = \dfrac{1}{\sqrt{2}}$ cm, $n = \dfrac{500}{\pi}$ Hz

(4) $A = \dfrac{1}{\sqrt{2}}$ cm, $n = \dfrac{1000}{\pi}$ Hz

27. The variation of velocity of a particle executing SHM with time is shown in the figure. The velocity of the particle when a phase change of $\pi/6$ takes place from the instant it is at one of the extreme positions will be



(1) 3.53 m/s (2) 2.5 m/s

(3) 4.330 m/s (4) None of these

28. In problem 27, the displacement of the particle from the mean position corresponding to the instant mentioned is

(1) $\dfrac{5}{\pi}$ m (2) $\dfrac{5\sqrt{3}}{\pi}$ m

(3) $\dfrac{10\sqrt{3}}{\pi}$ m (4) $\dfrac{5\sqrt{3}}{2\pi}$ m

29. In problem 27, the acceleration of the particle is

(1) $\dfrac{5\sqrt{3}\,\pi}{2}$ m/s^2 (2) $\dfrac{5\pi^2}{2}$ m/s^2

(3) $\dfrac{5\sqrt{3}\,\pi}{4}$ m/s^2 (4) $5\sqrt{3}\,\pi$ m/s^2

30. In problem 27, the maximum displacement and acceleration of the particle are respectively:

(1) $\dfrac{10}{\pi}$ m and 5π m/s^2 (2) $\dfrac{5}{\pi}$ m and $\dfrac{5\pi}{2}$ m/s^2

(3) $\frac{10}{\pi}$ m and $\frac{5\pi}{2}$ m/s^2 (4) $\frac{5}{\pi}$ m and $\frac{5\pi}{4}$ m/s^2

31. A block of mass 1 kg hangs without vibrating at the end of a spring whose force constant is 200 N/m and which is attached to the ceiling of an elevator. The elevator is rising with an upward acceleration of g/3 when the acceleration suddenly ceases. The angular frequency of the block after the acceleration ceases is

(1) 13 rad/s (2) 14 rad/s

(3) 15 rad/s (4) None of these

32. A vertical spring carries a 5 kg body and is hanging in equilibrium, an additional force is applied so that the spring is further stretched. When released from this position, it performs 50 complete oscillations in 25 s, with an amplitude of 5 cm. The additional force applied is

(1) 80 N (2) 80 π^2 N

(3) 4 π^2 N (4) 4 N

33. A particle, free to move along the x-axis, has potential energy given by $U_{(x)} = k\,[1 - \exp(-x^2)]$ for $-\infty < x < +\infty$ where k is a positive constant of appropriate dimensions. Then

(1) for small displacement from $x = 0$, the motion is simple harmonic

(2) if its total mechanical energy is $k/2$, it has its minimum kinetic energy at the origin

(3) for any finite non-zero value of x, there is a force directed away from the origin

(4) at points away from the origin, the particle is in unstable equilibrium

34. Two particles move parallel to the x-axis about the origin with same amplitude 'a' and frequency ω. At a certain instant they are found at a distance $a/3$ from the origin on opposite sides but their velocities are in the same direction. What is the phase difference between the two?

(1) $\cos^{-1}\frac{7}{9}$ (2) $\cos^{-1}\frac{5}{9}$

(3) $\cos^{-1}\frac{4}{9}$ (4) $\cos^{-1}\frac{1}{9}$

35 The potential energy of a particle executing SHM along the x-axis is given by $U = U_0 - U_0 \cos ax$. What is the period of oscillation?

(1) $2\pi\sqrt{\frac{ma}{U_0}}$ (2) $2\pi\sqrt{\frac{U_0}{ma}}$

(3) $\frac{2\pi}{a}\sqrt{\frac{m}{U_0}}$ (4) $2\pi\sqrt{\frac{m}{aU_0}}$

36. A particle executing SHM of amplitude 'a' has a displacement $a/2$ at $t = T/4$ and a negative velocity. The epoch of the particle is

(1) $\frac{\pi}{3}$ (2) $\frac{2\pi}{3}$

(3) π (4) $\frac{5\pi}{3}$

37. A block of mass 4 kg hangs from a spring of spring constant $k = 400$ N/m. The block is pulled down through 15 cm below and released. What is its kinetic energy when the block is 10 cm above the equilibrium position?

(1) 5 J (2) 2.5 J

(3) 1 J (4) 1.9 J

38. A body of mass 100 g attached to a spring executes SHM of period 2 s and amplitude 10 cm. How long a time is required for it to move from a point 5 cm below its equilibrium position to a point 5 cm above it, when it makes simple harmonic vertical oscillations (take $g = 10$ m/s^2)?

(1) 0.6 s (2) 1/3 s

(3) 1.5 s (4) 2.2 s

39. A particle executing SHM has velocities u and v and accelerations a and b in two of its positions. Find the distance between these two positions.

(1) $\frac{u^2 - v^2}{a+b}$ (2) $\frac{v^2 + u^2}{a-b}$

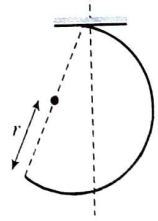
(3) $\frac{v^2 + u^2}{a+b}$ (4) $\frac{v^2 - u^2}{a-b}$

40. Two particles are executing identical simple harmonic motions described by the equations, $x_1 = a\cos(\omega t + (\pi/6))$ and $x_2 = a\cos(\omega t + \pi/3)$. The minimum interval of time between the particles crossing the respective mean positions is

(1) $\frac{\pi}{2\omega}$ (2) $\frac{\pi}{3\omega}$

(3) $\frac{\pi}{4\omega}$ (4) $\frac{\pi}{6\omega}$

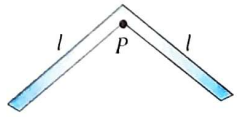
41. A uniform semicircular ring having mass m and radius r is hanging at one of its ends freely as shown in the given figure. The ring is slightly disturbed so that it oscillates in its own plane. The time period of oscillation of the ring is



(1) $2\pi\sqrt{\frac{r}{g\left(1+\frac{1}{\pi^2}\right)}}$ (2) $2\pi\sqrt{\frac{r}{g\left(1-\frac{4}{\pi^2}\right)^{1/2}}}$

(3) $2\pi\sqrt{\frac{r}{g\left(1-\frac{2}{\pi^2}\right)^{1/2}}}$ (4) $2\pi\sqrt{\frac{2r}{g\left(1+\frac{4}{\pi}\right)^{1/2}}}$

42. A system of two identical rods (L–shaped) of mass m and length l are resting on a peg P as shown in the figure. If the system is displaced in its plane by a small angle θ, find the period of oscillations.



(1) $2\pi\sqrt{\frac{\sqrt{2}l}{3g}}$ (2) $2\pi\sqrt{\frac{2\sqrt{2}l}{3g}}$

(3) $2\pi\sqrt{\frac{2l}{3g}}$ (4) $3\pi\sqrt{\frac{l}{3g}}$

43. A particle is subjected to two mutually perpendicular simple harmonic motions such that its x and y coordinates are given by

$$x = 2\sin\omega t;\ y = 2\sin\left(\omega t + \frac{\pi}{4}\right)$$

The parth of the particle will be

(1) an ellipse (2) a straight line

(3) a parabola (4) a circle

44. A particle of mass m moves in a one-dimensional potential energy $U(x) = -ax^2 + bx^4$, where a and b are positive constants. The angular frequency of small oscillations about the minima of the potential energy is equal to

(1) $\pi\sqrt{\frac{a}{2b}}$ (2) $2\sqrt{\frac{a}{m}}$

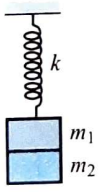
(3) $\sqrt{\frac{2a}{m}}$ (4) $\sqrt{\frac{a}{2m}}$

45. Two particles P and Q describe SHM of same amplitude a and frequency v along the same straight line. The maximum distance between the two particles is $\sqrt{2}\,a$. The initial phase difference between them is

(1) zero (2) $\pi/2$

(3) $\pi/6$ (4) $\pi/3$

46. Two masses m_1 and m_2 are suspended together by a massless spring of constant k. When the masses are in equilibrium, m_1 is removed without disturbing the system; the amplitude of vibration is:



(1) m_1g/k (2) m_2g/k

(3) $\frac{(m_1+m_2)g}{k}$ (4) $\frac{(m_2-m_1)g}{k}$

47. A body of mass m is released from a height h to a scale pan hung from a spring. The spring constant of the spring is k, the mass of the scale pan is negligible and the body does not bounce relative to the pan; then the amplitude of vibration is

(1) $\frac{mg}{k}\sqrt{1-\frac{2hk}{mg}}$ (2) $\frac{mg}{k}$

(3) $\frac{mg}{k}+\frac{mg}{k}\sqrt{1+\frac{2hk}{mg}}$ (4) $\frac{mg}{k}-\frac{mg}{k}\sqrt{1-\frac{2hk}{mg}}$

48. Frequency of a particle executing SHM is 10 Hz. The particle is suspended from a vertical spring. At the highest point of its oscillation the spring is unstretched. Maximum speed of the particle is (g = 10 m/s^2)

(1) 2π m/s (2) π m/s

(3) $1/\pi$ m/s (4) $1/2\pi$ m/s

49. An object of mass 0.2 kg executes simple harmonic motion along the x-axis with a frequency of $25/\pi$ Hz. At the position x = 0.04 m, the object has a kinetic energy of 0.5 J and potential energy of 0.4 J. The amplitude of oscillation is

(1) 0.05 m (2) 0.06 m

(3) 0.01 m (4) none of these

50. The string of a simple pendulum is replaced by a uniform rod of length L and mass M while the bob has a mass m. It is allowed to make small oscillations. Its time period is

(1) $2\pi\sqrt{\left(\frac{2M}{3m}\right)\frac{L}{g}}$ (2) $2\pi\sqrt{\frac{2(M+3m)L}{3(M+2m)g}}$

(3) $2\pi\sqrt{\left(\frac{M+m}{M+3m}\right)\frac{L}{g}}$ (4) $2\pi\sqrt{\left(\frac{2m+M}{3(M+2m)}\right)\frac{L}{g}}$

51. A particle performs simple harmonic motion with amplitude A and time period T. The mean velocity of the particle over the time interval during which it travels a distance of $A/2$ starting from executing position is

(1) A/T (2) $2A/T$

(3) $3A/T$ (4) $A/2T$

52. An object of mass 4 kg is attached to a spring having spring constant 100 N/m. It performs simple harmonic motion on a smooth horizontal surface with an amplitude of 2 m. A 6 kg object is dropped vertically onto the 4 kg object when it crosses the mean position, and sticks to it. The change in amplitude of oscillation due to collision is

(1) 1 m (2) zero

(3) $2\left[1-\sqrt{\frac{2}{5}}\right]$ (4) $2\left[1-\frac{1}{\sqrt{5}}\right]$

53. A particle of mass m is present in a region where the potential energy of the particle depends on the x-coordinate according to the expression $U = \frac{a}{x^2} - \frac{b}{x}$, where a and b are positive constants. The particle will perform

(1) oscillatory motion but not simple harmonic motion about its mean position for small displacements

(2) simple harmonic motion with time period $2\pi\sqrt{\frac{8a^2m}{b^4}}$ about its mean position for small displacements

(3) neither simple harmonic motion nor oscillatory about its mean position for small displacements

(4) none of the above

54. A particle performing simple harmonic motion having time period 3 s is in phase with another particle which also undergoes simple harmonic motion at t = 0. The time period of second particle is T (less than 3 s). If they are again in the same phase for the third time after 45 s, then the value of T will be

(1) 2.8 s (2) 2.7 s

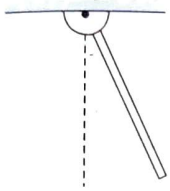
(3) 2.5 s (4) None of these

56. A particle performs SHM on the x-axis with amplitude A and time period T. The time taken by the particle to travel a distance $A/5$ starting from rest is

(1) $\frac{T}{20}$ (2) $\frac{T}{2\pi}\cos^{-1}\left(\frac{4}{5}\right)$

(3) $\frac{T}{2\pi}\cos^{-1}\left(\frac{1}{5}\right)$ (4) $\frac{T}{2\pi}\sin^{-1}\left(\frac{1}{5}\right)$

57. A metre stick swinging in vertical plane about a fixed horizontal axis passing through its one end undergoes small oscillation of frequency f_0. If the bottom half of the stick were cut off, then its new frequency of small oscillation would become



(1) f_0 (2) $\sqrt{2}\,f_0$

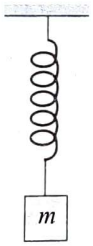
(3) $2f_0$ (4) $2\sqrt{2}\,f_0$

58. A physical pendulum is positioned so that its centre of gravity is above the suspension point. When the pendulum is released it passes the point of stable equilibrium with an angular velocity ω. The period of small oscillations of the pendulum is

(1) $\frac{4\pi}{\omega}$ (2) $\frac{2\pi}{\omega}$

(3) $\frac{\pi}{\omega}$ (4) $\frac{\pi}{2\omega}$

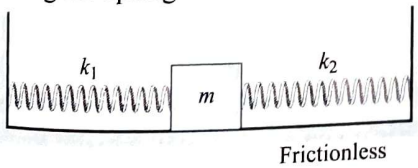
59. A block of mass 'm' is suspended from a spring and executes vertical SHM of time period T as shown in the figure. The amplitude of the SHM is A and spring is never in compressed state during the oscillation. The magnitude of minimum force exerted by spring on the block is



(1) $mg-\frac{4\pi^2}{T^2}mA$ (2) $mg+\frac{4\pi^2}{T^2}mA$

(3) $mg-\frac{\pi^2}{T^2}mA$ (4) $mg+\frac{\pi^2}{T^2}mA$

60. Two springs with negligible masses and force constants $k_1 = 200$ N/m and $k_2 = 160$ N/m are attached to the block of mass $m = 10$ kg as shown in the given figure. Initially the block is at rest, at the equilibrium position in which both springs are neither stretched nor compressed. At time $t = 0$, sharp impulse of 50 Ns is given to the block with a hammer along the spring.



Frictionless

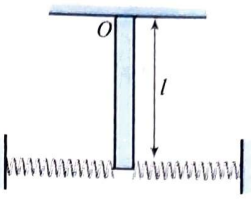
(1) Period of oscillations for the mass m is $\pi/6$ s.

(2) Maximum velocity of the mass m during its oscillation is 10 m/s.

(3) Data are insufficient to determine maximum velocity.

(4) Amplitude of oscillation is 0.83 m.

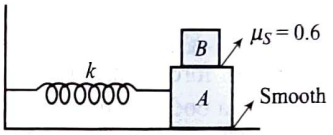
60. A thin uniform vertical rod of mass m and length l pivoted at point O is shown in the given figure. The system is placed on smooth horizontal table. The combined stiffness of the springs is equal to k. The mass of the spring is negligible. The frequency of small oscillation is



(1) $\sqrt{\frac{3k}{2m}+\frac{g}{l}}$ (2) $\sqrt{\frac{3k}{2m}+\frac{3g}{l}}$

(3) $\sqrt{\frac{3k}{m}+\frac{3g}{2l}}$ (4) $\sqrt{\frac{3k}{m}+\frac{2g}{3l}}$

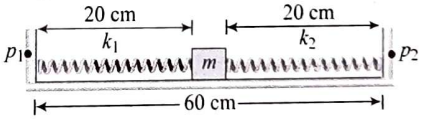
61. A block A is connected to spring and performs simple harmonic motion with a time period of 2 s. Another block B rests on A. The coefficient of static friction between A and B is $\mu_s = 0.6$. the maximum amplitude of oscillation which the system can have so that there is no relative motion between A and B is (take $\pi^2 = g = 10$)



(1) 0.3 m (2) 0.6 m

(3) 0.4 m (4) 0.52 m

62. Two springs, each of unstretched length 20 cm but having different spring constants $k_1 = 1000$ N/m and $k_2 = 3000$ N/m, are attached to two opposite faces of a small block of mass $m = 100$ g kept on a smooth horizontal surface as shown in the given figure.



The outer ends of the two springs are now attached to two pins P_1 and P_2 whose locations are shown in the figure. As a result of this, the block acquires a new equilibrium position. The block has been displaced by small amount from its equilibrium position and released to perform simple harmonic motion; then

(1) new equilibrium position is at 35 cm from P_1 and time period of simple harmonic motion is $\pi/100$ s

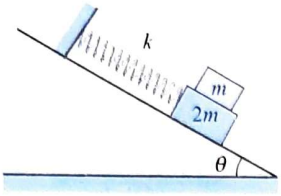
(2) new equilibrium position is at 20 cm from P_1 and time period of simple harmonic motion is $\pi/100$ s

(3) new equilibrium position is at 35 cm from P_1 and time period of simple harmonic motion is $\pi/26$ s

(4) new equilibrium position is at 30 cm from P_1 and time period of simple harmonic motion is $\pi/26$ s

63. The coefficient of friction between block of mass m and $2m$ is $\mu = 2\tan\theta$. There is no friction between block of mass $2m$ and inclined plane. The maximum amplitude of the two

block system for which there is no relative motion between both the blocks is



(1) $g\sin\theta\sqrt{\frac{k}{m}}$ (2) $\frac{mg\sin\theta}{k}$

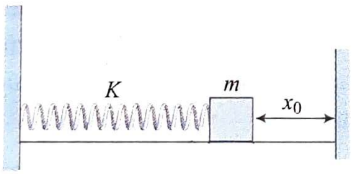
(3) $\frac{3mg\sin\theta}{k}$ (4) None of these

64. A simple pendulum of length l and a mass m of the bob is suspended in a car that is travelling with a constant speed v around a circle of radius R. If the pendulum undergoes small oscillations about its equilibrium position, the frequency of its oscillation will be

(1) $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$ (2) $\frac{1}{2\pi}\sqrt{\frac{g}{R}}$

(3) $\frac{1}{2\pi}\sqrt{\frac{\left(g^2+\frac{v^4}{R^2}\right)^{1/2}}{l}}$ (4) $\frac{1}{2\pi}\sqrt{\frac{v^2}{Rl}}$

65. One end of a spring of force constant K is fixed to a vertical wall and the other to a body of mass m resting on a smooth horizontal surface. There is another wall at a distance x_0 from the body. The spring is then compressed by $3x_0$ and released. The time taken to strike the wall from the instant of release is (given $\sin^{-1}(1/3) = (\pi/9)$)



(1) $\frac{\pi}{6}\sqrt{\frac{m}{K}}$ (2) $\frac{2\pi}{3}\sqrt{\frac{m}{K}}$

(3) $\frac{\pi}{4}\sqrt{\frac{m}{K}}$ (4) $\frac{11\pi}{9}\sqrt{\frac{m}{K}}$

66. A street car moves rectilinearly from station A (here car stops) to the next station B (here also car stops) with an acceleration varying according to the law $f = a - bx$, where a and b are positive constants and x is the distance from station A. The distance between the two stations and the maximum velocity are

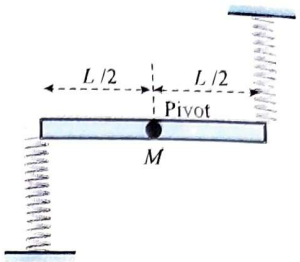
(1) $x = \frac{2a}{b}; v_{max} = \frac{a}{\sqrt{b}}$

(2) $x = \frac{b}{2a}; v_{max} = \frac{a}{b}$

(3) $x = \frac{a}{2b}; v_{max} = \frac{b}{\sqrt{a}}$

(4) $x = \frac{a}{b}; v_{max} = \frac{\sqrt{a}}{b}$

67. A uniform stick of mass M and length L is pivoted at its centre. Its ends are tied to two springs each of force constant K. In the position shown in figure, the springs are in their natural length. When the string is displaced through a small angle θ and released, the stick



(1) executes non-periodic motion

(2) executes periodic motion which is not simple harmonic

(3) executes SHM of frequency $\frac{1}{2\pi}\sqrt{\frac{6K}{M}}$

(4) executes SHM of frequency $\frac{1}{2\pi}\sqrt{\frac{K}{2M}}$

68. The velocity v of a particle of mass m moving along a straight line changes with time t as $d^2v/dt^2 = -Kv$ where K is a positive constant. Which of the following statements is correct?

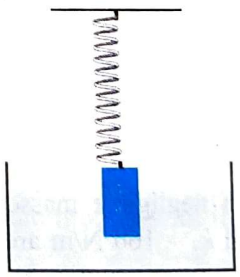
(1) The particle does not perform SHM.

(2) The particle performs SHM with time period $2\pi\sqrt{\frac{m}{K}}$

(3) The particle performs SHM with frequency $\frac{\sqrt{K}}{2\pi}$.

(4) The particle performs SHM with time period $\frac{2\pi}{K}$.

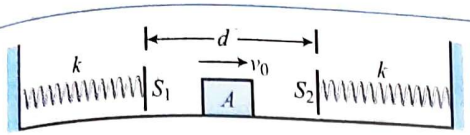
69. A solid right circular cylinder of weight 10 kg and cross-sectional area 100 cm^2 is suspended by a spring, where k = 1 kg/cm, and hangs partially submerged in water of density 1000 kg/m^3 as shown in the given figure. What is its period when it makes simple harmonic vertical oscillations? (Take g = 10 m/s^2)



(1) 0.6 s (2) 1 s

(3) 1.5 s (4) 2.2 s

70. A block 'A' of mass m is placed on a smooth horizontal platform P and between two elastic massless springs S_1 and S_2 fixed horizontally to two fixed vertical walls. The elastic constants of the two springs are equal to k and the equilibrium distance between the two springs both in relaxed states is d. The block is given a velocity v_0 initially towards one of the springs and it then oscillates between the springs. The time period T of oscillations and the minimum separation d_m of the springs will be



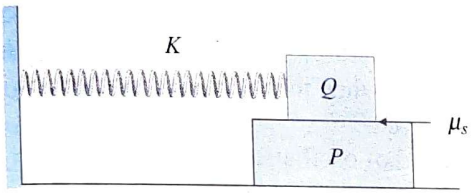
(1) $T = 2\left(\frac{d}{v} + \pi\sqrt{\frac{m}{k}}\right)$, $d_m = d$

(2) $T = 2\left(\frac{d}{v} + 2\pi\sqrt{\frac{m}{k}}\right)$, $d_m = d - v\sqrt{\frac{m}{k}}$

(3) $T = 2\left(\frac{d}{v} + 2\pi\sqrt{\frac{m}{k}}\right)$, $d_m = d - 2v\sqrt{\frac{m}{k}}$

(4) $T = 2\pi\sqrt{\frac{m}{k}}$, $d_m = d$

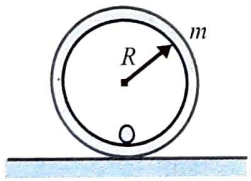
71. A block P of mass m is placed on a smooth horizontal surface. A block Q of same mass is placed over the block P and the coefficient of static friction between them is μ_s. A spring of spring constant K is attached to block Q. The blocks are displaced together to a distance A and released. The upper block oscillates without slipping over the lower block. The maximum frictional force between the block is



(1) zero (2) K

(3) $KA/2$ (4) μg

72. A thin-walled tube of mass m and radius R has a rod of mass m and very small cross section soldered on its inner surface. The side-view of the arrangement is as shown in the given figure.

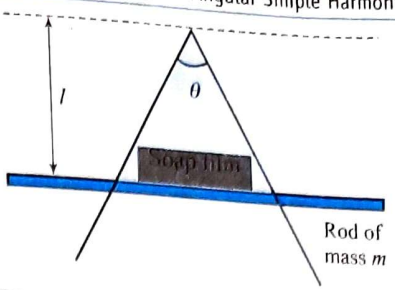


The entire arrangement is placed on a rough horizontal surface. The system is given a small angular displacement from its equilibrium position, as a result, the system performs oscillations. The time period of resulting oscillations if the tube rolls without slipping is

(1) $2\pi\sqrt{\frac{4R}{g}}$ (2) $2\pi\sqrt{\frac{2R}{g}}$

(3) $2\pi\sqrt{\frac{R}{g}}$ (4) None of these

73. A wire is bent at an angle θ. A rod of mass m can slide along the bent wire without friction as shown in the given figure. A soap film is maintained in the frame kept in a vertical position and the rod is in equilibrium as shown in the figure. If rod is displaced slightly in vertical direction, then the time period of small oscillation of the rod is



(1) $2\pi\sqrt{\frac{l}{g}}$ (2) $2\pi\sqrt{\frac{l\cos\theta}{g}}$

(3) $2\pi\sqrt{\frac{l}{g\cos\theta}}$ (4) $2\pi\sqrt{\frac{l}{g\tan\theta}}$

Multiple Correct Answers Type

1. A simple pendulum is oscillating between extreme positions P and Q about the mean position O. Which of the following statements are true about the motion of pendulum?
   (1) At point O, the acceleration of the bob is different from zero.
   (2) The acceleration of the bob is constant throughout the oscillation.
   (3) The tension in the string is constant throughout the oscillation.
   (4) The tension is maximum at O and minimum at P or Q.

2. The displacement (x) of a particle as a function of time (t) is given by
   $x = a \sin(bt + c)$
   where a, b and c are constants of motion. Choose the correct statements from the following.
   (1) The motion repeats itself in a time interval of $2\pi/b$
   (2) The energy of the particle remains constant.
   (3) The velocity of the particle is zero at $x = \pm a$
   (4) The acceleration of the particle is zero at $x = \pm a$

3. When the point of suspension of pendulum is moved, its period of oscillation
   (1) decreases when it moves vertically upwards with an acceleration a
   (2) decreases when it moves vertically downwards with acceleration greater than $2g$
   (3) increases when it moves horizontally with acceleration a
   (4) all of the above

4. A particle moves in the x-y plane such that its position vector changes with time according to the equation, $\vec{r} = (\hat{i} + 2\hat{j})A\cos\omega t$. The motion of the particle is
   (1) on a straight line (2) on an ellipse
   (3) periodic (4) simple harmonic

5. For a body executing SHM with amplitude A, time period T, maximum velocity v_{max} and phase constant zero, which of the following statements are correct for $0 \le t \le T/4$ (y is displacement from mean position)?

(1) At $y = (A/2)$, $v > (v_{max}/2)$
(2) For $v = (v_{max}/2)$, $y > (A/2)$
(3) For $t = (T/8)$, $y > (A/2)$
(4) For $y = (A/2)$, $t < (T/8)$

6. At two particular closest instants of time t_1 and t_2 the displacements of a particle performing SHM are equal. At these instants
(1) instantaneous speeds are equal
(2) instantaneous accelerations are equal
(3) phase of the motion are unequal
(4) kinetic energies are equal

7. For the spring pendulum shown in the given figure, the value of spring constant is 3×10^4 N/m and amplitude of oscillation is 0.1 m. The total mechanical energy of oscillating system is 200 J. Mark out the correct option(s).

(1) Minimum PE of the oscillating system is 50 J.
(2) Maximum PE of the oscillating system is 200 J.
(3) Maximum KE of the oscillating system is 200 J.
(4) Minimum KE of the oscillating system is 150 J.

8. A coin is placed on a horizontal platform, which undergoes vertical simple harmonic motion of angular frequency ω. The amplitude of oscillation is gradually increased. The coin will leave contact with the platform for the first time
(1) at the highest position of the platform
(2) at the mean position of the platform
(3) for an amplitude of g/ω^2
(4) for an amplitude of $\sqrt{g/\omega}$

9. For a simple harmonic motion with given angular frequency ω, two arbitrary initial conditions are necessary and sufficient to determine the motion completely. These initial conditions may be
(1) initial position and initial velocity
(2) amplitude and initial phase
(3) total energy of oscillation and amplitude
(4) total energy of oscillation and initial phase

10. The potential energy U of a body of unit mass moving in one dimensional conservative force field is given by $U = x^2 - 4x + 3$. All units are in SI. For this situation mark out the correct statement(s).
(1) The body will perform simple harmonic motion about $x = 2$ units.
(2) The body will perform oscillatory motion but not simple harmonic motion.
(3) The body will perform simple harmonic motion with time period $\sqrt{2}\pi$ s.
(4) If speed of the body at equilibrium position is 4 m/s, then the amplitude of oscillation would be $2\sqrt{2}$ m.

11. A particle performing simple harmonic motion undergoes initial displacement of $A/2$ (where A is the amplitude of simple harmonic motion) in 1 s. At $t = 0$, the particle may be at the extreme position or mean position. The time period of the simple harmonic motion can be
(1) 6 s
(2) 2.4 s
(3) 12 s
(4) 1.2 s

12. A particle is subjected to two simple harmonic motions along x and y directions according to $x = 3 \sin 100\pi t$, $y = 4 \sin 100\pi t$.
(1) Motion of particle will be on ellipse travelling in clockwise direction.
(2) Motion of particle will be on a straight line with slope 4/3.
(3) Motion will be simple harmonic motion with amplitude 5.
(4) Phase difference between two motions is $\pi/2$.

13. The speed v of a particle moving along a straight line, when it is at a distance (x) from a fixed point of the line is given by $v^2 = 108 - 9x^2$ (all equations are in CGS units):
(1) the motion is uniformly accelerated along the straight line
(2) the magnitude of the acceleration at a distance 3 cm from the point is 27 cm/s^2
(3) the motion is simple harmonic about the given fixed point
(4) the maximum displacement from the fixed point is 4 cm

14. A horizontal plank has a rectangular block placed on it. The plank starts oscillating vertically and simple harmonically with an amplitude of 40 cm. The block just loses contact with the plank when the latter is at momentary rest. Then
(1) the period of oscillation is $\left(\frac{2\pi}{5}\right)$
(2) the block weighs double its actual weight, then the plank is at one of the positions of momentary rest.
(3) the block weighs 0.5 times its weight on the plank halfway up
(4) the block weighs 1.5 times its weight on the plank halfway down
(5) the block weighs its true weight on the plank when the later moves fastest

15. A 20 g particle is subjected to two simple harmonic motions $x_1 = 2\sin 10t$, $x_2 = 4\sin\left(10t + \frac{\pi}{3}\right)$, where x_1 and x_2 are in metres and t is in seconds.
(1) The displacement of the particle at $t = 0$ will be $2\sqrt{3}$ m
(2) Maximum speed of the particle will be $20\sqrt{7}$ m/s.
(3) Magnitude of maximum acceleration of the particle will be $200\sqrt{7}$ m/s^2.
(4) Energy of the resultant motion will be 28 J.

16. The potential energy of a particle of mass 0.1 kg, moving along the x–axis, is given by $U = 5x(x - 4)$ J, where x is in meter. It can be concluded that
(1) the particle is acted upon by a constant force
(2) the speed of the particle is maximum at $x = 2$ m
(3) the particle executes SHM
(4) the period of oscillation of the particle is $(\pi/5)$ s

The time period of a particle in simple harmonic motion is T. Assume potential energy at mean position to be zero. After a time of $T/6$ it passes its mean position, its

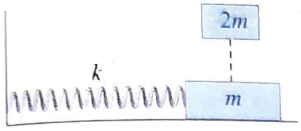
(1) velocity will be half its maximum velocity

(2) displacement will be half its amplitude

(3) acceleration will be nearly 86% of its maximum acceleration

(4) KE = PE

An object of mass m is performing simple harmonic motion on a smooth horizontal surface as shown in the figure. Just as the oscillating object reaches its extreme position, another object of mass $2m$ is dropped on to oscillating object, which sticks to it. For this situation mark out the correct statement(s).



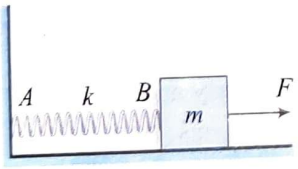
(1) Amplitude of oscillation remains unchanged.

(2) Time period of oscillation remains unchanged.

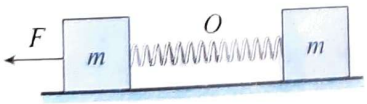
(3) The total mechanical energy of the system does not change.

(4) The maximum speed of the oscillating object changes.

The given Fig. (a) shows a spring of force constant k fixed at one end and carrying a mass m at the other end placed on a horizontal frictionless surface. The spring is stretched by a force F. Figure (b) shows the same spring with both ends free and a mass m fixed at each free end. Each of the spring is stretched by the same force F. The mass in case (a) and the masses in case (b) are then released. Which of the following statements are true?



(a)



(b)

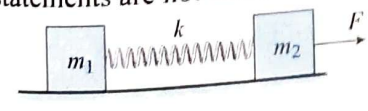
(1) While oscillating, the maximum extension of the spring is more in case (a) than in case (b).

(2) The maximum extension of the spring is same in both cases.

(3) The time period of oscillation is the same in both cases.

(4) The time period of oscillation in case (a) is $\sqrt{2}$ times that in case (b).

Two blocks connected by a spring rest on a smooth horizontal plane as shown in the given figure. A constant force F starts acting on block m_2 as shown in the figure. Which of the following statements are not correct?



(1) Length of the spring increases continuously if $m_1 > m_2$.

(2) Blocks start performing SHM about centre of mass of the system, which moves rectilinearly with constant acceleration.

(3) Blocks start performing oscillations about centre of mass of the system with increasing amplitude.

(4) Acceleration of m_2 is maximum at initial moment of time only.

21. A block of mass m is suspended by a rubber cord of natural length $l = mg/k$, where k is force constant of the cord. The block is lifted upwards so that the cord becomes just tight and then block is released suddenly. Which of the following will not be true?

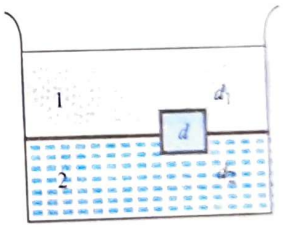
(1) Block performs periodic motion with amplitude greater than l.

(2) Block performs SHM with amplitude equal to l.

(3) Block will never return to the position from where it was released.

(4) Angular frequency ω is equal to 1 rad/s.

22. A cylindrical block of density d stays fully immersed in a beaker filled with two immiscible liquids of different densities d_1 and d_2. The block is in equilibrium with half of it in liquid 1 and the other half in liquid 2 as shown in the given figure. If the block is given a displacement downwards and released, then neglecting friction study the following statements.



(1) It executes simple harmonic motion.

(2) Its motion is periodic but not simple harmonic.

(3) The frequency of oscillation is independent of the size of the cylinder.

(4) The displacement of the centre of the cylinder is symmetric about its equilibrium position.

23. A mass of 0.2 kg is attached to the lower end of a massless spring of force constant 200 N/m, the upper end of which is fixed to a rigid support. Study the following statements.

(1) In equilibrium the spring will be stretched by 1 cm.

(2) If the mass is raised till the spring becomes unstretched and then released, it will go down by 2 cm before moving upwards.

(3) The frequency of oscillation will be nearly 5 Hz.

(4) If the system is taken to the moon, the frequency of oscillation will be the same as that on the earth.

24. A spring of spring constant k stores 5 J of energy when stretched by 25 cm. It is kept vertical with one end fixed. A mass m is attached to the other end. It makes 5 oscillations per second. Then

(1) $m = 0.16$ kg (2) $m = 0.32$ kg

(3) $k = 160$ N/m (4) $k = 320$ N/m

25. Three simple harmonic motions in the same direction, each of amplitude 'a' and periodic time 'T', are superposed. The first and second, and the second and third differ in phase from each other by $\pi/4$, with the first and third not being identical. Then

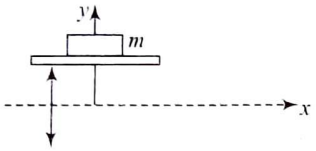
(1) the resultant motion is not simple harmonic

(2) the resultant amplitude is $(\sqrt{2}+1)a$

(3) the phase difference between the second SHM and the resultant motion is zero

(4) the energy in the resultant motion is three times the energy in each separate SHM

26. A horizontal platform with a mass m placed on it is executing SHM along y-axis. If the amplitude of oscillation is 2.5 cm, the minimum period of the motion for the mass not to be detached from the platform is ($g = 10$ m/s^2)



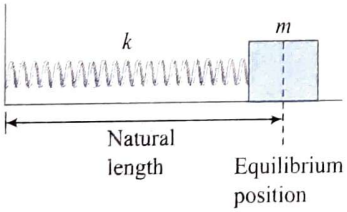
(1) $\frac{10}{\pi}$s (2) $\frac{\pi}{10}$s

(3) $\frac{\pi}{\sqrt{10}}$s (4) $\frac{1}{\sqrt{10}}$s

Linked Comprehension Type

For Problems 1–2

A block of mass m is connected to a spring of spring constant k as shown in the figure. The block is found at its equilibrium position at $t = 1$ s and it has a velocity of +0.25 m/s at $t = 2$ s. The time period of oscillation is 6 s.



Based on the given information, answer the following questions:

1. The amplitude of oscillation is

(1) $\frac{3}{2\pi}$ m (2) 3 m

(3) $\frac{1}{\pi}$ m (4) 1.5 m

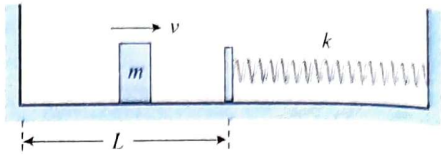
2. Determine the velocity of particle at $t = 5$ s.

(1) –0.4 m/s (2) 0.5 m/s

(3) –0.25 m/s (4) None of these

For Problems 3–4

A spring having a spring constant k is fixed to a vertical wall as shown in the following figure. A block of mass m moves with velocity v towards the spring from a parallel wall opposite to this wall. The mass hits the free end of the spring compressing it and is decelerated by the spring force and comes to rest and then turns back till the spring acquires its natural length and contact with the spring is broken. In this process, it regains its angular speed in the opposite direction and makes a perfect elastic collision on the opposite left wall and starts moving with same speed as before towards right. The above processes are repeated and there is periodic oscillations.



3. What is the maximum compression produced in the spring?

(1) $v\sqrt{\frac{m}{k}}$ (2) $\sqrt{\frac{m}{k}}$

(3) $v\sqrt{mk}$ (4) $v\sqrt{\frac{k}{m}}$

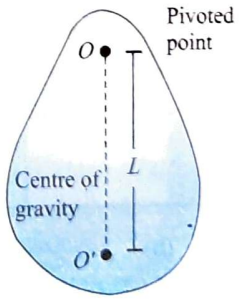
4. What is the time period of oscillations?

(1) $\pi\sqrt{\frac{m}{k}}$ (2) $\sqrt{\frac{\pi m}{k}}+\frac{L}{v}$

(3) $\pi\sqrt{\frac{m}{k}}+\frac{2L}{v}$ (4) $\pi\sqrt{\frac{m}{k}}+\frac{L}{v}$

For Problems 5–7

In a physical pendulum, the time period for small oscillation is given by, $T = 2\pi\sqrt{I/Mgd}$ where I is the moment of inertia of the body about an axis passing through a pivoted point O and perpendicular to the plane of oscillation and d is the separation point between centre of gravity and the pivoted point.



In the physical pendulum, a special point exists where if we concentrate the entire mass of body, the resulting simple pendulum (w.r.t. pivot point O) will have the same time period as that of physical pendulum. This point is termed centre of oscillation.

$$T = 2\pi\sqrt{\frac{I}{Mgd}} = 2\pi\sqrt{\frac{L}{g}}$$

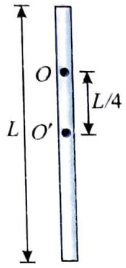
Moreover, this point possesses two other important remarkable properties:

Property I: Time period of physical pendulum about the centre of oscillation (if it would be pivoted) is same as in the original case.

Property II: If an impulse is applied at the centre of oscillation in the plane of oscillation, the effect of this impulse at pivoted point is zero. Because of this property, this point is also known as the centre of percussion.

From the given information answer the following questions:

5. A uniform rod of mass M and length L is pivoted about point O as shown in the given figure. It is slightly rotated from its mean position so that it performs angular simple harmonic motion. For this physical pendulum, determine the time period of oscillation.



(1) $2\pi\sqrt{\frac{L}{g}}$ (2) $\pi\sqrt{\frac{7L}{3g}}$

(3) $2\pi\sqrt{\frac{2L}{3g}}$ (4) None of these

6. For the above question, locate the centre of oscillation.

(1) $\frac{L}{4}$ from O (down) (2) $\frac{L}{4}$ from O (up)

(3) $\frac{2L}{3}$ from O (down) (4) $\frac{7L}{12}$ from O (down)

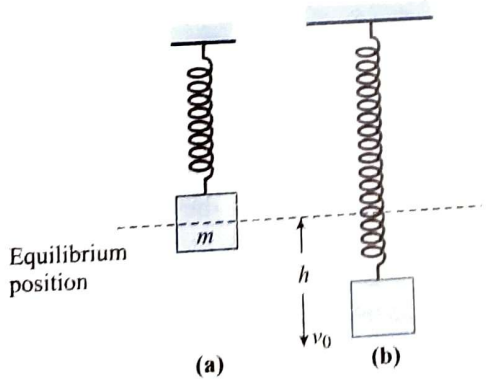
7. If an impulse J is applied at the centre of oscillation in the plane of oscillation, then angular velocity of the rod will be

(1) $\frac{4J}{ML}$ (2) $\frac{2J}{ML}$

(3) $\frac{3J}{2ML}$ (4) $\frac{J}{ML}$

For Problems 8–10

A block of mass m is connected to a spring of spring constant k and is at rest in equilibrium as shown. Now, the block is displaced by h below its equilibrium position and imparted a speed v_0 towards down as shown in the given figure.



As a result of the jerk, the block executes simple harmonic motion about its equilibrium position. Based on this information, answer the following questions:

8. The amplitude of oscillation is

(1) h (2) $\sqrt{\frac{mv_0^2}{k}+h^2}$

(3) $\sqrt{\frac{m}{k}}v_0+h$ (4) None of these

9. The equation for the simple harmonic motion is

(1) $y=-A\sin\left[\sqrt{\frac{k}{m}}t+\sin^{-1}\left(\frac{h}{A}\right)\right]$

(2) $y=-A\cos\left[\sqrt{\frac{k}{m}}t+\sin^{-1}\left(\frac{h}{A}\right)\right]$

(3) $y=A\sin\left[\sqrt{\frac{k}{m}}t+\cos^{-1}\left(\frac{h}{A}\right)+\frac{\pi}{2}\right]$

(4) $y=A\sin\left[\sqrt{\frac{k}{m}}t+\cos^{-1}\left(\frac{h}{A}\right)+\frac{\pi}{4}\right]$

10. Find the time taken by the block to cross the mean position for the first time.

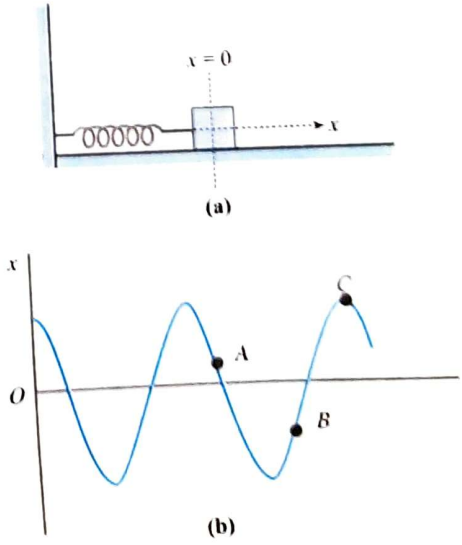
(1) $\frac{2\pi-\cos^{-1}\left(\frac{h}{A}\right)}{\sqrt{\frac{k}{m}}}$ (2) $\frac{\frac{\pi}{2}-\cos^{-1}\left(\frac{h}{A}\right)}{\sqrt{\frac{k}{m}}}$

(3) $\frac{\pi-\sin^{-1}\left(\frac{h}{A}\right)}{\sqrt{\frac{k}{m}}}$ (4) $\frac{\pi-\sin^{-1}\left(\frac{h}{A}\right)}{2\sqrt{\frac{k}{m}}}$

For Problems 11–15

A 100 g block is connected to a horizontal massless spring of force constant 25.6 N/m. As shown in Fig. (a), the block is free to oscillate on a horizontal frictionless surface. The block is displaced 3 cm from the equilibrium position and, at $t = 0$, it is released from rest at $x = 0$. It executes simple harmonic motion with the positive x-direction indicated in Fig. (a).

The position–time ($x - t$) graph of motion of the block is as shown in Fig. (b)



11. When the block is at position A on the graph, its
(1) position and velocity both are negative
(2) position is positive and velocity is negative
(3) position is negative and velocity is positive
(4) position and velocity both the positive

12. When the block is at position B on the graph, its
(1) position and velocity are positive
(2) position is positive and velocity is negative
(3) position in negative and velocity is positive
(4) position and velocity are negative

13. When the block is at position C on the graph, its
(1) velocity is maximum and acceleration is zero
(2) velocity is minimum and acceleration is zero
(3) velocity is zero and acceleration is negative
(4) velocity is zero and acceleration is positive

Let us now make a slight change to the initial conditions. At $t = 0$, let the block be released from the same position, i.e., from a displacement 3 cm along positive x-direction but with an initial velocity $v_i = -16\sqrt{3}$ cm/s.

14. Position of the block as a function of time can now be expressed as

(1) $x = 3\cos\left(16t + \frac{\pi}{2}\right)$ cm

(2) $x = 3\cos\left(16t + \frac{\pi}{3}\right)$ cm

(3) $x = 2\sqrt{3}\cos\left(16t + \frac{\pi}{6}\right)$ cm

(4) $x = 2\sqrt{3}\cos\left(16t + \frac{\pi}{4}\right)$ cm

15. Velocity of the block as a function of time can be expressed as

(1) $v = 48\sin\left(16t + \frac{\pi}{2}\right)$ cm/s

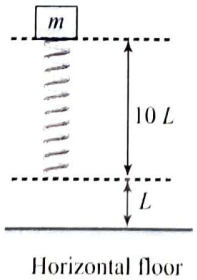
(2) $v = -48\sin\left(16t + \frac{\pi}{3}\right)$ cm/s

(3) $v = -32\sqrt{3}\sin\left(16t + \frac{\pi}{4}\right)$ cm/s

(4) $v = -32\sqrt{3}\sin\left(16t + \frac{\pi}{6}\right)$ cm/s

For Problems 16–18

A small block of mass m is fixed at upper end of a massive vertical spring of spring constant $k = 4mg/L$ and natural length '$10L$'. The lower end of spring is free and is at a height L from fixed horizontal floor as shown. The spring is initially unstressed and the spring–block system is released from rest in the shown position.



16. At the instant the speed of block is maximum, the magnitude of force exerted by the spring on the block is

(1) $\frac{mg}{2}$ (2) mg

(3) zero (4) None of these

17. As the block is coming down, the maximum speed attained by the block is

(1) $\sqrt{gL}$ (2) $\sqrt{3gL}$

(3) $\frac{3}{2}\sqrt{gL}$ (4) $\sqrt{\frac{3}{2}gL}$

18. Till the block reaches its lowest position for the first time, the time duration for which the spring remains compressed is

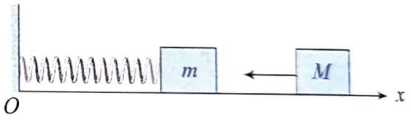
(1) $\pi\sqrt{\frac{L}{2g}} + \sqrt{\frac{L}{4g}}\sin^{-1}\frac{1}{3}$ (2) $\frac{\pi}{4}\sqrt{\frac{L}{g}} + \sqrt{\frac{L}{4g}}\sin^{-1}\frac{1}{3}$

(3) $\pi\sqrt{\frac{L}{2g}} + \sqrt{\frac{L}{4g}}\sin^{-1}\frac{2}{3}$ (4) $\frac{\pi}{2}\sqrt{\frac{L}{2g}} + \sqrt{\frac{L}{4g}}\sin^{-1}\frac{2}{3}$

For Problems 19–21

One end of an ideal spring is fixed to a wall at origin O and the axis of spring is parallel to the x-axis. A block of mass $m = 1$ kg is attached to free end of the spring and it is performing SHM. Equation of position of the block in coordinate system shown in the following figure is $x = 10 + 3\sin(10t)$, where t is in seconds and x in cm.

Another block of mass $M = 3$ kg, moving towards the origin with velocity 30 cm/s collides with the block performing SHM at $t = 0$ and gets stuck to it.



19. Angular frequency of oscillation after collision is
(1) 20 rad/s (2) 5 rad/s
(3) 100 rad/s (4) 50 rad/s

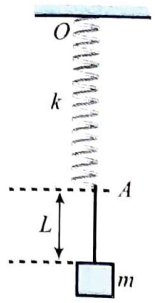
20. New amplitude of oscillation is
(1) 3 cm (2) 20 cm
(3) 10 cm (4) 100 cm

21. New equation for position of the combined body is
(1) $(10 + 3\sin 5t)$ cm
(2) $(10 - 3\sin 5t)$ cm
(3) $(10 + 3\cos 10t)$ cm
(4) $(10 - 3\cos 10t)$ cm

For Problems 22–24

A body of mass m is attached by an inelastic string to a suspended spring of spring constant k. Both the string and the spring have negligible mass and the string is inextensible and of length L. Initially, the mass m is at rest.

22. If the mass m is now raised up to point A (the top end of the string, see the given figure and allowed to fall from rest, the maximum extension of the spring in the subsequent motion will be



(1) L (2) $\frac{mg}{k}$

(3) $\frac{mg}{k}\sqrt{1+\frac{2kL}{mg}}$ (4) $\frac{mg}{k}\left[1+\sqrt{1+\frac{2kL}{mg}}\right]$

23. If the mass m, from the initial position of rest, is pulled down a distance A and then released, assuming that the string remains taut throughout the motion, the maximum (downward) acceleration of the oscillating body will be

(1) $\frac{kA}{m}$ (2) $\frac{kA}{2m}$

(3) $\frac{g}{2}$ (4) g

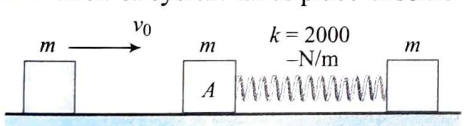
24. The largest amplitude A_{max}, for which the string will remain taut throughout the motion is

(1) $\frac{mg}{2k}$ (2) $\frac{mg}{k}$

(3) $\frac{2mg}{3k}$ (4) L

For Problems 25–27

Two identical blocks A and B, each of mass $m = 3$ kg, are connected with the help of an ideal spring and placed on a smooth horizontal surface as shown in the given figure. Another identical block C moving with velocity $v_0 = 0.6$ m/s collides with A and sticks to it, as a result, the motion of system takes place in some way.



Based on this information, answer the following questions:

25. After the collision of C and A, the combined body and block B would

(1) oscillate about centre of mass of system and centre of mass is at rest.

(2) oscillate about centre of mass of system and centre of mass is moving.

(3) oscillate but about different locations other than the centre of mass.

(4) not oscillate.

26. Oscillation energy of the system, i.e., part of the energy which is oscillating (changing) between potential and kinetic forms, is

(1) 0.27 J (2) 0.09 J

(3) 0.18 J (4) 0.45 J

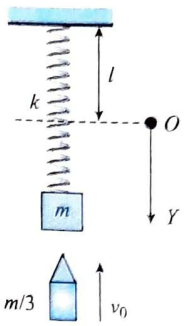
27. The maximum compression of the spring is

(1) $3\sqrt{30}$ mm (2) $3\sqrt{20}$ mm

(3) $3\sqrt{10}$ mm (4) $3\sqrt{50}$ mm

For Problems 28–30

A block of mass m is suspended from one end of a light spring as shown. The origin O is considered at distance equal to natural length of the spring from the ceiling and vertical downward direction as positive y-axis. When the system is in equilibrium, a bullet of mass $m/3$ moving in vertical upward direction with velocity v_0 strikes the block and embeds into it. As a result, the block (with bullet embedded into it) moves up and starts oscillating.



Based on the given information, answer the following questions:

28. Mark out the correct statement(s).

(1) The block–bullet system performs SHM about $y = mg/k$.

(2) The block–bullet system performs oscillatory motion but not SHM about $y = mg/k$.

(3) The block–bullet system performs SHM about $y = 4mg/3k$.

(4) The block-bullet system performs oscillatory motion but not SHM about $y = 4mg/3k$.

29. The amplitude of oscillation would be

(1) $\sqrt{\left(\frac{4mg}{3k}\right)^2+\frac{mv_0^2}{12k}}$ (2) $\sqrt{\frac{mv_0^2}{12k}+\left(\frac{mg}{3k}\right)^2}$

(3) $\sqrt{\frac{mv_0^2}{6k}+\left(\frac{mg}{k}\right)^2}$ (4) $\sqrt{\frac{mv_0^2}{6k}+\left(\frac{4mg}{3k}\right)^2}$

30. The time taken by the block–bullet system to move from $y = mg/k$ (initial equilibrium position) to $y = 0$ (natural length of spring) is (A represents the amplitude of motion)

(1) $\sqrt{\frac{4m}{3k}}\left[\cos^{-1}\left(\frac{mg}{3kA}\right)-\cos^{-1}\left(\frac{4mg}{3kA}\right)\right]$

(2) $\sqrt{\frac{3k}{4m}}\left[\cos^{-1}\left(\frac{mg}{3kA}\right)-\cos^{-1}\left(\frac{4mg}{3kA}\right)\right]$

(3) $\sqrt{\frac{4m}{6k}}\left[\sin^{-1}\left(\frac{4mg}{3kA}\right)-\sin^{-1}\left(\frac{mg}{3kA}\right)\right]$

(4) None of the above

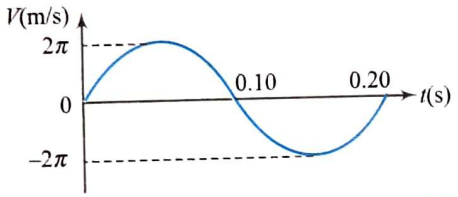
Matrix Match Type

1. For a particle executing SHM along a straight line, match the statements in column I with statement in column II. (Note that displacement given is column I is to be measured from mean position).

Column I	Column II
i. Velocity–displacement graph may be	a. straight line
ii. Acceleration–velocity graph may be	b. circle
iii. Acceleration–displacement graph will be	c. ellipse
iv. Acceleration–time graph will be	d. sinusoidal curve

2. A simple harmonic oscillator consists of a block attached to a spring with $k = 200$ N/m. The block slides on a frictionless horizontal surface, with equilibrium point $x = 0$. A graph of the block's velocity v as a function of time t is shown.

Correctly match the required information in the left column with the values given in the right column. (use $\pi^2 = 10$)

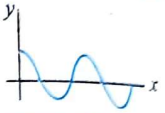
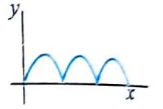


Column I	Column II
i. The block's mass in kg	a. – 0.20
ii. The block's displacement at $t = 0$ in metres	b. – 200
iii. The block's acceleration at $t = 0.10$ s in m/s^2	c. 0.20
iv. The block's maximum kinetic energy in Joule	d. 4.0

3. In the column I, a system is described in each option and corresponding time period is given in the column II. Suitably match them.

Column I	Column II
i. A simple pendulum of length 'l' oscillating with small amplitude in a lift moving down with retardation $g/2$.	a. $T = 2\pi\sqrt{\frac{2l}{3g}}$
ii. A block attached to an end of a vertical spring, whose other end is fixed to the ceiling of a lift, stretches the spring by length 'l' in equilibrium. Its time period when lift moves up with an acceleration $g/2$ is	b. $T = 2\pi\sqrt{\frac{l}{g}}$
iii. The time period of small oscillation of a uniform rod of length 'l' smoothly hinged at one end. The rod oscillates in vertical plane.	c. $T = 2\pi\sqrt{\frac{2l}{g}}$
iv. A cubical block of edge 'l' and specific density $\rho/2$ is in equilibrium with some volume inside water filled in a large fixed container. Neglect viscous forces and surface tension. The time period of small oscillations of the block in vertical direction is	d. $T = 2\pi\sqrt{\frac{l}{2g}}$

4. In simple harmonic motion, match the following graphs:

Column I	Column II
i. Position (y) vs. time (x)	a.
ii. Velocity (y) vs. time (x)	b. 
iii. Potential energy (y) vs. time (x)	c. 
iv. Total energy (y) vs. time (x)	d.

5. Match the following:

Column I	Column II
i. A constant force acting along the line of SHM affects	a. the time period
ii. A constant torque acting along the arc of angular SHM affects	b. the frequency
iii. A particle falling on the block executing SHM when the later crosses the mean position affects	c. the mean position
iv. A particle executing SHM when kept on a uniformly accelerated car affects	d. the amplitude

6. In Column I equations describing the motion of a particle are given and in Column II possible nature of the motions. Match the entries of Column I with the entries of Column II.

Column I	Column II
i. $y = Ae^{i(\omega t + \phi)}$	a. SHM
ii. $y = B \sin \omega t + C \cos \omega t$	b. Periodic
iii. $y = A \sin(\omega t + kx)$	c. Oscillatory
iv. $y = kx$	d. Rectilinear

7. A spring pendulum executes SHM in such a way that the block is having velocity v when it crosses the mean position. Now the changes have been made in such a way that the velocity while crossing the mean position gets doubled without changing mass of the block. In Column I some statements (incomplete) are given and corresponding completions are given in Column II. Match the entries of Column I with the entries of Column II. Assume the horizontal configuration of pendulum.

Column I	Column II
i. The frequency of oscillation will change by a factor of	a. 2
ii. The amplitude of oscillation will change by a factor of	b. $\sqrt{2}$
iii. The magnitude of maximum acceleration will change by a factor of	c. 1
iv. Maximum PE increases by a factor of	d. 4

8. Two particles 'A' and 'B' start SHM at $t = 0$. Their positions as function of time are given by

$X_A = A \sin \omega t$

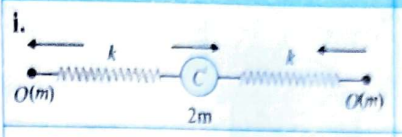
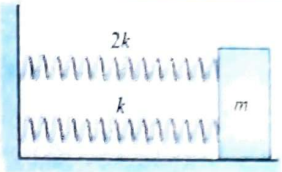
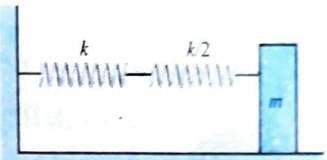
$X_B = A \sin (\omega t + \pi/3)$

Column I	Column II
i. Minimum time when x is same	a. $\frac{5\pi}{6\omega}$
ii. Minimum time when velocity is same	b. $\frac{\pi}{3\omega}$
iii. Minimum time after which $v_A < 0$ and $v_B < 0$	c. $\frac{\pi}{\omega}$
iv. Minimum time after which $x_A < 0$ and $x_B < 0$	d. $\frac{\pi}{2\omega}$

9. A particle of mass 2 kg is moving on a straight line under the action of force $F = (8 - 2x)$ N. The particle is released from rest at $x = 6$ m. For the subsequent motion, match the following (all the values in the Column II are in their SI units):

Column I	Column II
i. Equilibrium position is at x	a. $\pi/4$
ii. Amplitude of SHM is	b. $\pi/2$
iii. Time taken to go directly from $x = 2$ to $x = 4$	c. 4
iv. Energy of SHM is	d. 6
v. Phase constant of SHM assuming equation of the form $A \sin (\omega t + \phi)$	e. 2

10. Column I lists the various modes of oscillations of masses connected to springs. Column II lists the corresponding frequencies of oscillations when executing SHM.

Column I	Column II
i. 	a. $\frac{1}{2\pi}\sqrt{\frac{3k}{2m}}$
ii.	b. $\frac{1}{2\pi}\sqrt{\frac{2k}{m}}$
iii. 	c. $\frac{1}{2\pi}\sqrt{\frac{k}{3m}}$
iv. 	d. $\frac{1}{2\pi}\sqrt{\frac{3k}{m}}$

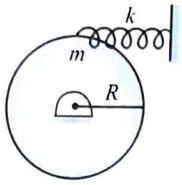
11. Match the following:

Column I	Column II
i. Linear combination of two SHMs	a. $T = \sqrt{\frac{R}{g}}$ (R is radius of the earth)
ii. $y = A \sin\omega_1 t + A \sin(\omega_2 t + \phi)$	b. SHM for equal frequencies and amplitude
iii. Time period of a pendulum of infinite length.	c. Superposition may not always be an SHM
iv. Maximum value of time period of an oscillating pendulum.	d. Amplitude will be $\sqrt{2}A$ for $\omega_1 = \omega_2$ and phase difference of $\pi/2$

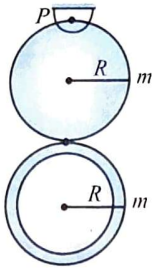
Numerical Value Type

1. A small body of mass m is connected to two horizontal springs of elastic constant k, natural length $3d/4$. In the equilibrium position both springs are stretched to length d, as shown in the given figure. What wil be the ratio of period of the motion (T_b/T_a) if the body is displaced horizontally by a small distance where T_a is the time period when the particle oscillates along the line of springs and T_b is time period when the particle oscillates perpendicular to the plane of the figure? Neglect effects of gravity.

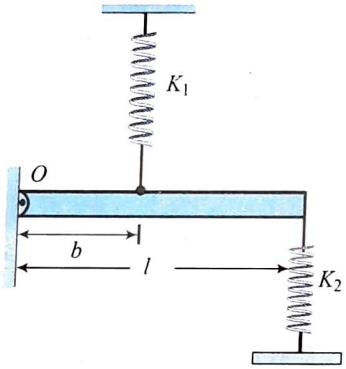
2. A uniform disc of mass m and radius R is pivoted smoothly at its centre of mass. A light spring of stiffness k is attached with the disc tangentially as shown in the following figure. Find the angular frequency in rad/s of torsional oscillations of the disc. (Take $m = 5$ kg and $K = 10$ N/m.)



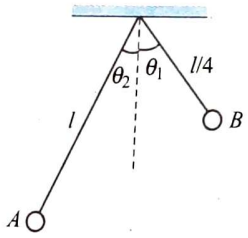
3. A uniform disc of mass m and radius $R = \frac{80}{23\pi^2}m$ is pivoted smoothly at P. If a uniform disc of mass m and radius R is welded at the lowest point of the disc, find the period of SHM of the system (disc + ring). (in seconds)



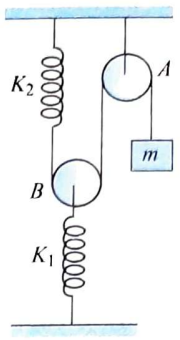
4. A rod of mass m and length l hinged at one end is connected by two springs of spring constants k_1 and k_2 so that it is horizontal at equilibrium. What is the angular frequency of the system? (in rad/s) (Take $l = 1$ m, $b = 1/4$ m, $K_1 = 16$ N/m, $K_2 = 63$ N/m and $m = 3$ kg)



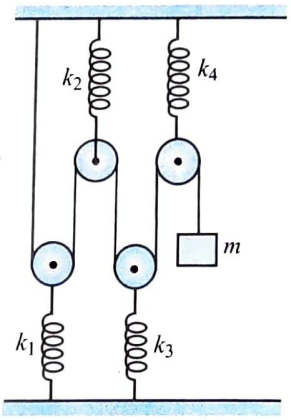
5. Two simple pendulums A and B having lengths l and $l/4$, respectively are released from the position as shown in the following figure. Calculate the time (in seconds) after which the two strings become parallel for the first time. (Take $l = \frac{90}{\pi^2}m$ and $g = 10$ m/s^2)



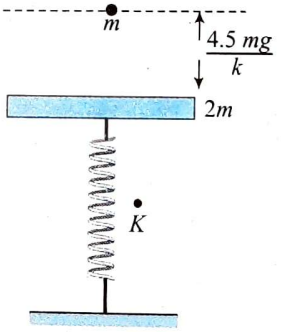
6. A block of mass m is tied to one end of a spring which passes over a smooth fixed pulley A and under a light smooth movable pulley B. The other end of the string is attached to the lower end of a spring of spring constant K_2. Find the period of small oscillation of mass m about its equilibrium position (in second). (Take $m = \pi^2$ kg, $K_2 = 4K_1$ and $K_1 = 17$ N/m.)



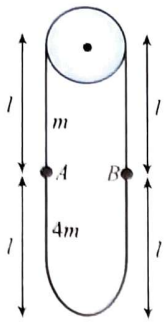
7. In the arrangement shown in the given figure, pulleys are small and light and springs are ideal and $K_1 = 25\ \pi^2$ N/m, $K_2 = 2K_1$, $K_3 = 3K_1$ and $K_4 = 4K_1$ are the force constants of the springs. Calculate the period of small vertical oscillations of block of mass $m = 3$ kg.



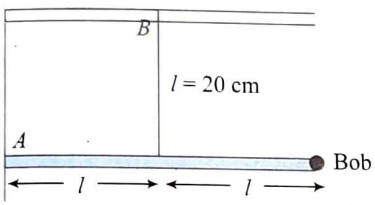
8. In the figure shown a plate of mass 60 g is at rest and in equilibrium. A particle of mass $m = 30$ g is released from height 4.5 mg/k from the plate. The particle sticks to the plate. Neglecting the duration of collision find the time from the collision of the particle and plate to the moment when the spring has maximum compression. Spring has force constant 1 N/m. Calculate the value of time in the form π/x and find the value of x.



9. Two uniform ropes having linear mass densities m and $4m$ are joined to form a closed loop. The loop is hanging over a fixed frictionless small pulley with the lighter rope above as shown in the following figure (in the figure equilibrium position is shown). Now if point A (joint) is slightly displaced in downward direction and released, It is found that the loop performs SHM with the period of oscillation equal to N. Find the value of N (take $l = \frac{150m}{4\pi^2}$, $g = 10$ m/s^2).



10. A weightless rigid rod with a small iron bob at the end is hinged at point A to the wall so that it can rotate in all directions. The rod is kept in the horizontal position by a vertical inextensible string of length 20 cm, fixed at its midpoint. The bob is displaced slightly perpendicular to the plane of the rod and string. Find period of small oscillations of the system in the form $\frac{\pi X}{10}$ s and fill the value of X.

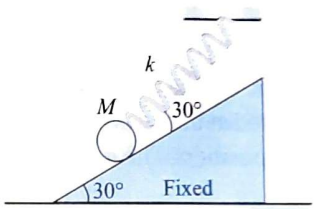


11. A force $F = -4x + 8$ (in N) is acting on block where x is the position of block in metres. The energy of oscillation is 32 J. The block oscillates between two points, Out of which value of position of one point (in m) is an integer from 0 to 9. Find it.

12. A particle executes SHM along x-axis with a frequency $\left(\frac{25}{\pi}\right)$ Hz. The mass of the particle is 0.20 kg. At the position $x = 0.04$ m, the kinetic energy is 0.50 J and the potential energy is 0.40 J. The potential energy is zero at mean position. Find the amplitude of oscillations (in cm).

13. A particle is moving on X-axis and has potential energy $U = 2 - 20x + 5x^2$ joule, where x is position. The particle is released at $x = -3$. If the mass of the particle is 0.1 kg, then the maximum velocity (in ms^{-1}) of the particle is 25 β. If amplitude is 5 m, then find the value of β.

14. A sphere of mass M and radius R is on a smooth fixed inclined plane in equilibrium as shown in the figure. If now the sphere is displaced through a small distance along the plane, what will be the angular frequency (in $rads^{-1}$) of the resulting SHM? (Given, $k = \frac{4M}{3}$)



Archives

JEE MAIN

Single Correct Answer Type

1. If x, v, and a denote the displacement, velocity, and acceleration of a particle executing SHM of time period T, then which of the following does not change with time?

(1) $a^2T^2 + 4\pi^2v^2$ (2) $\frac{aT}{x}$

(3) $aT + 2\pi v$ (4) $\frac{aT}{v}$ **(AIEEE 2009)**

2. A mass M, attached to a horizontal spring, executes SHM with a amplitude A_1. When the mass M passes through its mean position then a smaller mass m is placed over it and both of them move together with amplitude A_2. The ratio of (A_1/A_2) is

(1) $\frac{M}{M+m}$ (2) $\frac{M+m}{M}$

(3) $\left(\frac{M}{M+m}\right)^{1/2}$ (4) $\left(\frac{M+m}{M}\right)^{1/2}$

(AIEEE 2011)

3. Two particles are executing simple harmonic motion of the same amplitude A and frequency ω along the x-axis. Their mean position is separated by distance $X_0 (X_0 > A)$. If the maximum separation between them is $(X_0 + A)$, the phase difference between their motion is

(1) $\pi/2$ (2) $\pi/3$

(3) $\pi/4$ (4) $\pi/6$ **(AIEEE 2011)**

4. An ideal gas enclosed in a vertical cylindrical container supports a freely moving piston of mass M. The piston and cylinder have equal cross sectional area A. When the piston is in equilibrium, the volume of the gas is V_0 and its pressure is P_0. The piston is slightly displaced from the equilibrium position and released. Assuming that the system is completely isolated from its surrounding, the piston executes a simple harmonic motion with frequency

(1) $\frac{1}{2\pi}\frac{V_0MP_0}{A^2\gamma}$ (2) $\frac{1}{2\pi}\sqrt{\frac{A^2\gamma P_0}{MV_0}}$

(3) $\frac{1}{2\pi}\sqrt{\frac{MV_0}{A\gamma P_0}}$ (4) $\frac{1}{2\pi}\frac{A\gamma P_0}{V_0M}$

(JEE Main 2013)

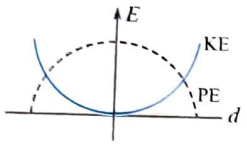
5. The amplitude of a damped oscillator decreases to 0.9 times its original magnitude is 5 s. In another 10 s it will decrease to α times its original magnitude, where α equals

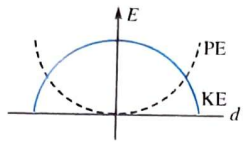
(1) 0.81 (2) 0.729

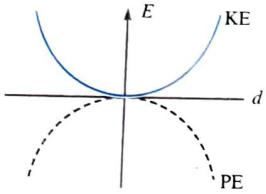
(3) 0.6 (4) 0.7 **(JEE Main 2013)**

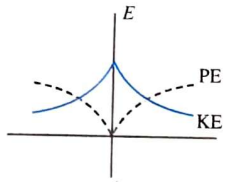
6. A particle moves with simple harmonic motion in a straight line. In first τ s, after starting from rest it travels a distance a, and in next τ s it travels $2a$, in same direction, then:
(1) amplitude of motion is $4a$
(2) time period of oscillations is 6τ
(3) amplitude of motion is $3a$
(4) time period of oscillations is 8τ **(JEE Main 2014)**

7. For a simple pendulum, a graph is plotted between its kinetic energy (KE) and potential energy (PE) against its displacement d. Which one of the following represents these correctly? (Graphs are schematic and not drawn to scale)

(1) 

(2) 

(3) 

(4) 

(JEE Main 2015)

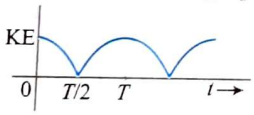
8. A particle performs simple harmonic motion with amplitude A. Its speed is tripled at the instant that it is at a distance $2A/3$ from equilibrium position. The new amplitude of the motion is

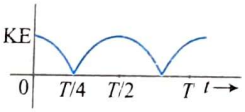
(1) $\frac{A}{3}\sqrt{41}$ (2) $3A$

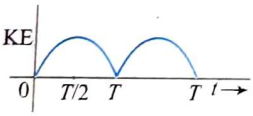
(3) $A\sqrt{3}$ (4) $\frac{7A}{3}$

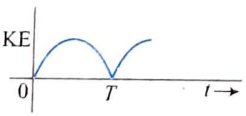
(JEE Main 2016)

9. A particle is executing simple harmonic motion with a time period T. At time $t = 0$, it is at its position of equilibrium. The kinetic energy–time graph of the particle will look like

(1) 

(2) 

(3) 

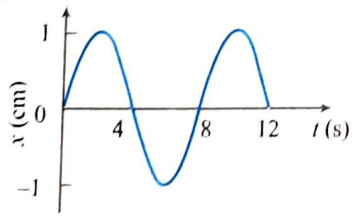
(4) 

(JEE Main 2017)

JEE ADVANCED

Single Correct Answer Type

1. The x–t graph of a particle undergoing simple harmonic motion is shown below. The acceleration of the particle at $t = 4/3$ s is

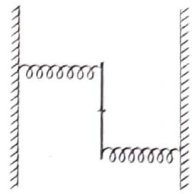


(1) $\frac{\sqrt{3}}{32}\pi^2$ cm/s^2 (2) $\frac{-\pi^2}{32}$ cm/s^2

(3) $\frac{\pi^2}{32}$ cm/s^2 (4) $-\frac{\sqrt{3}}{32}\pi^2$ cm/s^2

(IIT-JEE 2009)

2. A uniform rod of length L and mass M is pivoted at the centre. Its two ends are attached to two springs of equal spring constants k. The springs are fixed to rigid supports as shown in the figure, and rod is free to oscillate in the horizontal plane. The rod is gently pushed through a small angle θ in one direction and released. The frequency of oscillation is



(1) $\frac{1}{2\pi}\sqrt{\frac{2k}{M}}$ (2) $\frac{1}{2\pi}\sqrt{\frac{k}{M}}$

(3) $\frac{1}{2\pi}\sqrt{\frac{6k}{M}}$ (4) $\frac{1}{2\pi}\sqrt{\frac{24k}{M}}$

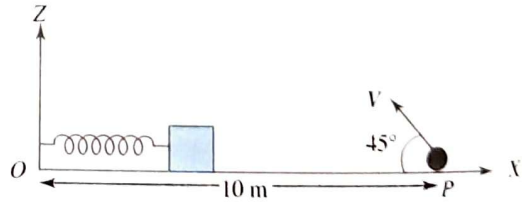
(IIT-JEE 2009)

3. A point mass is subjected to two simultaneous sinusoidal displacements in x-direction, $x_1(t) = A \sin \omega t$ and $x_2(t) = A \sin (\omega t + 2\pi/3)$. Adding a third sinusoidal displacement $x_3(t) = B \sin (\omega t + \phi)$ brings the mass to a complete rest. The values of B and ϕ are

(1) $\sqrt{2}A, \frac{3\pi}{4}$ (2) $A, \frac{4\pi}{3}$

(3) $\sqrt{3}A, \frac{5\pi}{6}$ (4) $A, \frac{\pi}{3}$ **(IIT-JEE 2011)**

4. A small block is connected to one end of a massless spring of un-stretched length 4.9 m. The other end of the spring (see the figure) is fixed. The system lies on a horizontal frictionless surface. The block is stretched by 0.2 m and released from rest at $t = 0$. It then executes simple harmonic motion with angular frequency $\omega = \pi/3$ rad/s. Simultaneously at $t = 0$, a small pebble is projected with speed v from point P at an angle of 45° as shown in the figure. Point P is at a horizontal distance of 10 m from O. If the pebble hits the block at $t = 1$ s, the value of v is (take $g = 10$ m/s^2)



(1) $\sqrt{50}$ m/s (2) $\sqrt{51}$ m/s

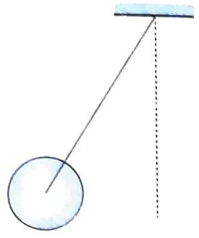
(3) $\sqrt{52}$ m/s (4) $\sqrt{53}$ m/s

(IIT-JEE 2012)

Multiple Correct Answers Type

1. A metal rod of length 'L' and mass 'm' is pivoted at one end. A thin disk of mass 'M' and radius 'R' ($<L$) is attached at its

centre to the free end of the rod. Consider two ways the disc is attached: (case A). The disc is not free to rotate about its centre and (case B) the disc is free to rotate about its centre. The rod–disc system performs SHM in vertical plane after being released from the same displaced position. Which of the following statement(s) is (are) true?

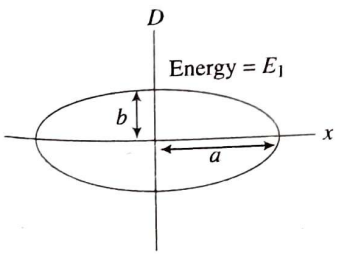


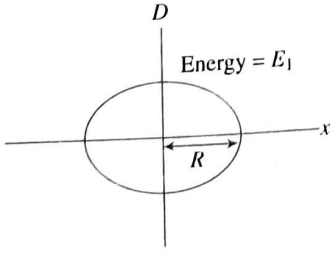
(1) Restoring torque in case A = Restoring torque in case B
(2) Restoring torque in case A < Restoring torque in case B
(3) Angular frequency for case A > Angular frequency for case B
(4) Angular frequency for case A < Angular frequency for case B **(IIT-JEE 2011)**

2. A particle of mass m is attached to one end of a mass-less spring of force constant k, lying on a frictionless horizontal plane. The other end of the spring is fixed. The particle starts moving horizontally from its equilibrium position at time $t = 0$ with an initial velocity u_0. When the speed of the particle is 0.5 u_0. It collides elastically with a rigid wall. After this collision,
(1) the speed of the particle when it returns to its equilibrium position is u_0.
(2) the time at which the particle passes through the equilibrium position for the first time is $t = \pi\sqrt{\frac{m}{k}}$
(3) the time at which the maximum compression of the spring occurs is $t = \frac{4\pi}{3}\sqrt{\frac{m}{k}}$
(4) the time at which the particle passes through the equilibrium position for the second time is $t = \frac{5\pi}{3}\sqrt{\frac{m}{k}}$

(JEE Advanced 2013)

3. Two independent harmonic oscillators of equal mass are oscillating about the origin with angular frequencies ω_1 and ω_2 and have total energies E_1 and E_2, respectively. The variations of their momenta p with positions x are shown in figures. If $\frac{a}{b} = n^2$ and $\frac{a}{R} = n$, then the correct equation(s) is (are)





(1) $E_1\omega_1 = E_2\omega_2$ (2) $\frac{\omega_2}{\omega_1} = n^2$
(3) $\omega_1\omega_2 = n^2$ (4) $\frac{E_1}{\omega_1} = \frac{E_2}{\omega_2}$

(JEE Advanced 2015)

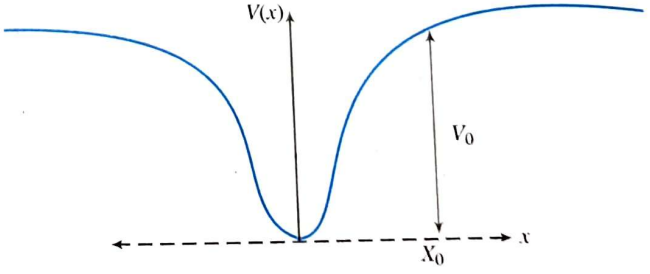
4. A block with mass M is connected by a massless spring with stiffness constant k to a rigid wall and moves without friction on a horizontal surface. The block oscillates with small amplitude A about an equilibrium position x_0. Consider two cases: (1) when the block is at x_0; and (2) when the block is at $x = x_0 + A$. In both the cases, a particle with mass m ($< M$) is softly placed on the block after which they stick to each other. Which of the following statement(s) is (are) true about the motion after the mass m is placed on the mass M?
(1) The amplitude of oscillation in the first case changes by a factor of $\sqrt{\frac{M}{m+M}}$, whereas in the second case it remains unchanged
(2) The final time period of oscillation in both the cases is same
(3) The total energy decreases in both the cases
(4) The instantaneous speed at x_0 of the combined masses decreases in both the cases **(JEE Advanced 2016)**

Linked Comprehension Type

For Problems 1–3

When a particle of mass m moves on the x-axis in a potential of the form $V(x) = kx^2$ it performs simple harmonic motion. The corresponding time period is proportional to m, k as can be seen easily using dimensional analysis. However, the motion of a particle can be periodic even when its potential energy increases on both sides of $x = 0$ in a way different from kx^2 and its total energy is such that the particle does not escape to infinity. Consider a particle of mass m moving on the x-axis. Its potential energy is $V(x) = \alpha x^4$ ($\alpha > 0$) for $|x|$ near the origin and becomes a constant equal to V_0 for $|x| \geq X_0$ (see figure).

(IIT-JEE 2010)



1. If the total energy of the particle is E, it will perform periodic motion only if
(1) $E < 0$ (2) $E > 0$
(3) $V_0 > E > 0$ (4) $E > V_0$

2. For periodic motion of small amplitude A, the time period T of this particle is proportional to

(1) $A\sqrt{\frac{m}{\alpha}}$ (2) $\frac{1}{A}\sqrt{\frac{m}{\alpha}}$

(3) $A\sqrt{\frac{\alpha}{m}}$ (4) $A\sqrt{\frac{2\alpha}{m}}$

3. The acceleration of this particle for $|x| > X_0$ is

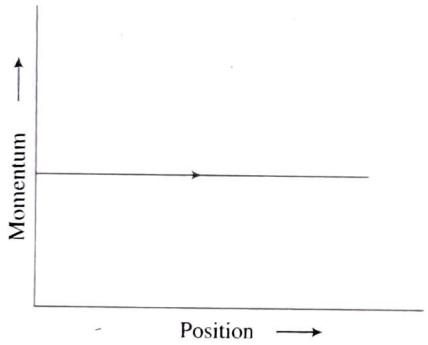
(1) proportional to V_0 (2) proportional to $\frac{V_0}{mX_0}$

(3) proportional to $\sqrt{\frac{V_0}{mX_0}}$ (4) zero

For Problems 4–6

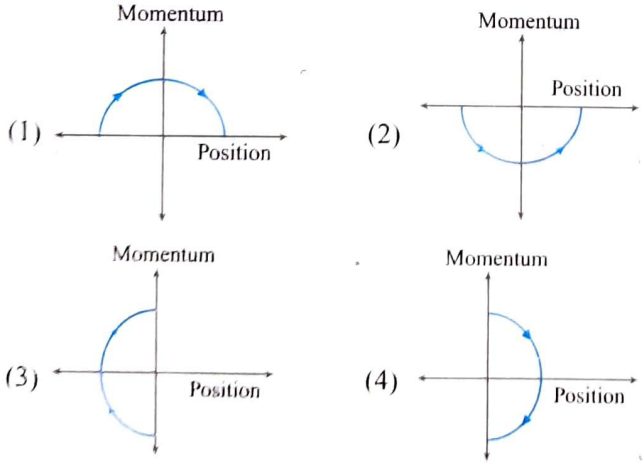
Phase space diagrams are useful tools in analysing all kinds of dynamical problems. They are especially useful in studying the changes in motion as initial position and momentum are changed. Here we consider some simple dynamical systems in one dimension. For such systems, phase space is a plane in which position is plotted along horizontal axis and momentum is plotted along vertical the axis.

The phase space diagram is $x(t)$ vsersus $p(t)$ curve in this plane. The arrow on the curve indicates the time flow. For example, the phase space diagram for a particle moving with constant velocity is a straight line as shown in figure. We use the sign convention in which position or momentum upwards (or to right) is positive and downwards (or to left) is negative.

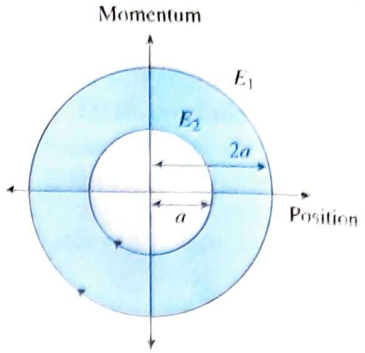


(IIT-JEE 2011)

4. The phase space diagram for a ball thrown vertically up from ground is



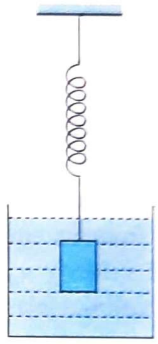
5. The phase space diagram for simple harmonic motion is a circle centred at the origin. In the figure, the two circles represent the same oscillator but for different initial conditions, and E_1 and E_2 are the total mechanical energies respectively. Then

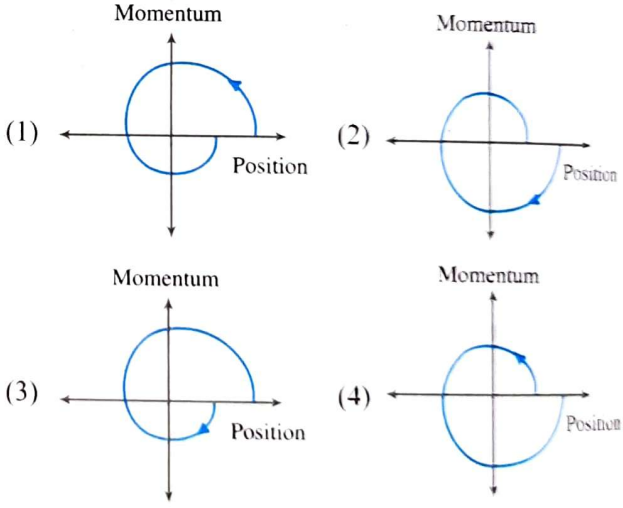


(1) $E_1 = \sqrt{2}\,E_2$ (2) $E_1 = 2E_2$

(3) $E_1 = 4E_2$ (4) $E_1 = 16E_2$

6. Consider the spring–mass system, with the mass submerged in water, as shown in the figure. The phase space diagram for one cycle of this system is





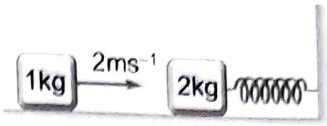
Numerical Value Type

1. A 0.1 kg mass is suspended from a wire of negligible mass. The length of the wire is 1 m and its cross-sectional area is 4.9×10^{-7} m^2. If the mass is pulled a little in the vertically downward direction and released, it performs simple harmonic motion of angular frequency 140 rad s^{-1}. If the Young's modulus of the material of the wire is $n \times 10^9$ Nm^{-2}, the value of n is.

(IIT-JEE 2010)

2. A spring–block system is resting on a frictionless floor as shown in the figure. The spring constant is 2.0 Nm^{-1} and the mass of the block is 2.0 kg. Ignore the mass of the spring. Initially the spring is in an unstretched condition.

Another block of mass 1.0 kg moving with a speed of 2.0 ms^{-1} collides elastically with the first block. The collision is such that the 2.0 kg block does not hit the wall. The distance, in metres, between the two blocks when the spring returns to its unstretched position for the first time after the collision is _____.



(JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (4)	3. (3)	4. (1)	5. (2)
6. (4)	7. (2)	8. (3)	9. (3)	10. (2)
11. (4)	12. (2)	13. (4)	14. (3)	15. (1)
16. (2)	17. (4)	18. (3)	19. (1)	20. (2)
21. (3)	22. (1)	23. (4)	24. (1)	25. (4)
26. (1)	27. (2)	28. (2)	29. (3)	30. (3)
31. (2)	32. (3)	33. (1)	34. (1)	35. (3)
36. (1)	37. (2)	38. (2)	39. (1)	40. (4)
41. (4)	42. (2)	43. (1)	44. (2)	45. (2)
46. (1)	47. (3)	48. (4)	49. (2)	50. (2)
51. (3)	52. (3)	53. (2)	54. (3)	55. (2)
56. (2)	57. (1)	58. (1)	59. (4)	60. (3)
61. (2)	62. (1)	63. (3)	64. (3)	65. (4)
66. (1)	67. (3)	68. (3)	69. (1)	70. (1)
71. (3)	72. (1)	73. (1)		

Multiple Correct Answers Type

1. (1),(4)	2. (1),(2),(3)	3. (1),(2)
4. (1),(3),(4)	5. (1),(2),(3),(4)	6. (1),(2),(3),(4)
7. (1),(2),(4)	8. (1),(3)	9. (1),(2),(4)
10. (1),(3),(4)	11. (1),(3)	12. (2),(3)
13. (2),(3)	14. (1),(2),(3),(4),(5)	
15. (1),(2),(3),(4)	16. (2),(3),(4)	17. (1),(3)
18. (1),(2),(3),(4)	19. (2),(4)	20. (1),(3),(4)
21. (1),(3),(4)	22. (1),(4)	23. (1),(2),(3) (4)
24. (1),(3)	25. (2),(3)	26. (2),(4)

Linked Comprehension Type

1. (1)	2. (3)	3. (1)	4. (3)	5. (2)
6. (4)	7. (1)	8. (2)	9. (1)	10. (3)
11. (2)	12. (3)	13. (3)	14. (3)	15. (4)
16. (2)	17. (3)	18. (2)	19. (2)	20. (1)
21. (2)	22. (4)	23. (1)	24. (2)	25. (2)
26. (1)	27. (3)	28. (3)	29. (2)	30. (1)

Matrix Match Type

1. i. → b., c.; ii. → b., c.; iii. → a.; iv. → d.
2. i. → c.; ii. → a.; iii. → b.; iv. → d.
3. i. → a.; ii. → b.; iii. → a.; iv. → d.
4. i. → a., b.; ii. → a., b.; iii. → c.; iv. → d.
5. i. → c., d.; ii. → c., d.; iii. → a., b., d.; iv. → c., d.
6. i. → a., b., c.; ii. → a., b., c.; iii. → a.; iv. → d.
7. i. → c.; ii. → a.; iii. → a.; iv. → d.
8. i. → b.; ii. → a.; iii. → d.; iv. → c.
9. i. → c.; ii. → e.; iii. → b.; iv. → c.; v. → b.
10. i. → b.,; ii. → a.; iii. → d.; iv. → c.
11. i. → b., c.; ii. → b., c., d.; iii. → a.; iv. → a.

Numerical Value Type

1. (2)	2. (2)	3. (2)	4. (8)	5. (1)
6. (1)	7. (2)	8. (5)	9. (5)	10. (4)
11. (6)	12. (6)	13. (2)	14. (1)	

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2)	2. (4)	3. (2)	4. (2)	5. (2)
6. (2)	7. (2)	8. (4)	9. (2)	

JEE Advanced

Single Correct Answer Type

1. (4)	2. (3)	3. (2)	4. (1)

Multiple Correct Answers Type

1. (1),(4) 2. (1),(4) 3. (2),(4)
4. (1),(2),(4)

Linked Comprehension Type

1. (3)	2. (2)	3. (4)	4. (4)	5. (3)
6. (2)				

Numerical Value Type

1. (4) 2. (2.09)

6 Travelling Waves

WAVE MOTION

When an object moves through space, it carries energy with itself. Wherever the object goes, the energy goes with it. So energy can be transported in space by the motion of objects or particles. There is another way (wave motion) to transport energy from one part of space to other without any bulk motion of material together with it. It is through wave motion. When you speak, you create some disturbance in the part of the air close to your lips. Energy is transferred to these air particles either by pushing them ahead or pulling them back. The density of the air in this part temporarily increases or decreases. These disturbed particles exert force on the next layer of air, transferring the disturbance to that layer. In this way, the disturbance proceeds in air and finally the air near the ear of the listener gets disturbed. Now we can say that a wave is a disturbance that propagates in space, transports energy and momentum from one point to another without the transport of matter.

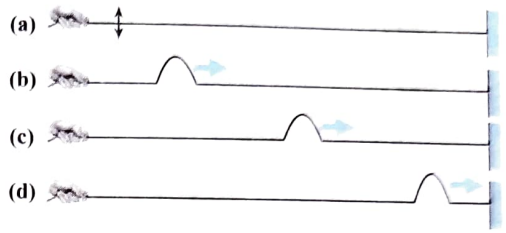
MECHANICAL AND NON-MECHANICAL WAVES

We are all familiar with waves, like water waves and sound waves. All such waves are produced and propagated in a material medium, *e.g.*, water or air. Also for the onward propagation of momentum and energy, the material medium in which waves are produced must possess inertia and elasticity. Such waves as are produced in an elastic material medium are called **mechanical waves or elastic waves**.

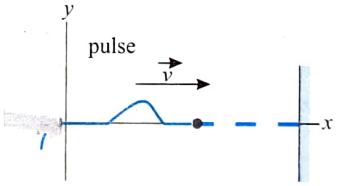
There is another category of waves which requires no material medium for their propagation. They can travel even in vacuum. Such kind of waves are called **non-mechanical or electromagnetic waves**. Examples of such non-mechanical waves are : radio waves, micro waves, light waves, X-rays and γ-rays. They travel through vacuum with the common speed of 3×10^8 m/s. Electromagnetic waves are produced due to changes of electric and magnetic fields associated with moving charges.

PULSE AND WAVE

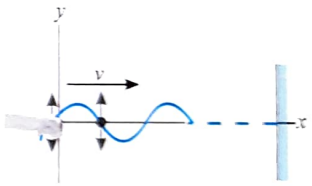
If we flip your wrist at one end of a long string that is under tension and has its opposite end fixed as shown in figure. In this manner, a single bump is generated as deformation and travels along the string. This bump is called "wave pulse". In the given figure, you can observe four consecutives 'snapshots' of the creation and propagation of the travelling pulse. Here the string is the medium through which the pulse travels. The pulse has definite shape and a definite speed of propagation along the medium (the string). The shape of the pulse does not change as it travels along the string.



A transverse wave pulse on a string is generated by rapidly displacing an end of a string up and down, but just once. Flipping the string continuously at a fixed rate produces a series of propagating pulse called wave train. If the string is struck in exactly the same way each time, then all the pulses are identical, and the resulting wave is periodic. If the end of string is struck in such a way that it performs simple harmonic motion, the disturbance so produced is called a sinusoidal wave train.



Formation of a pulse



Formation of sinusoidal wave train

TYPES OF WAVES

Before discussing the formation of the mechanical waves, let us know the essential conditions for the same.

- The medium should possess elasticity so that the particles after being displaced tend to regain their original positions.
- The medium should possess the property of inertia so that it can store energy in it.
  The frictional resistance amongst the various particles of the medium should be negligibly small so that the particles continue vibrating for a sufficiently long time and the wave travels a sufficient distance through the medium.

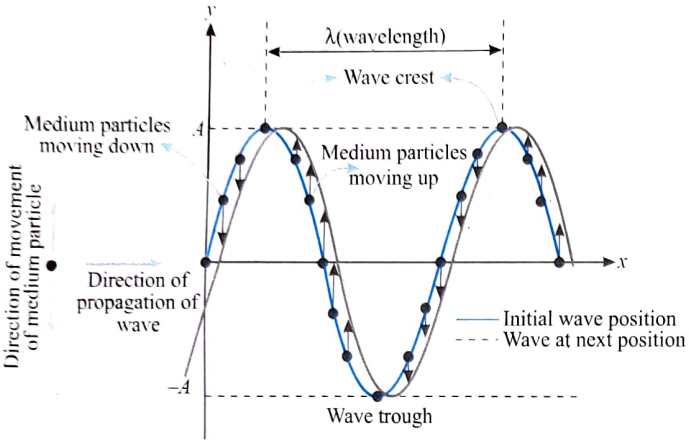
Based on the basis of motion of particles of the medium. There are two types of wave motion for mechanical waves namely (i) transverse wave motion and (ii) longitudinal wave motion.

TRANSVERSE WAVES

In a transverse wave the particle displacement is perpendicular to the direction of wave propagation. We cannot propagate transverse wave in a gas or in a liquid because there is no mechanism for driving the motion perpendicular to the direction of propagation of wave. In case of liquids transverse waves can be generated only on the surface of the liquids because of surface tension.

In case of a transverse wave propagating in a taut string, a **crest** is a position of the string, which is raised temporarily above the normal position of rest of particles of the string, when a transverse wave passes.

And **trough** is a position of the string, which is depressed temporarily below the normal position of rest of particles of the string, when a transverse wave passes. The displacement of a particle of string is maximum above the line of mean position, at the position of crest of the wave, while the displacement of a string particle is maximum below the mean position at the trough of the wave.



The displacement–position graph of a longitudinal wave at an instant

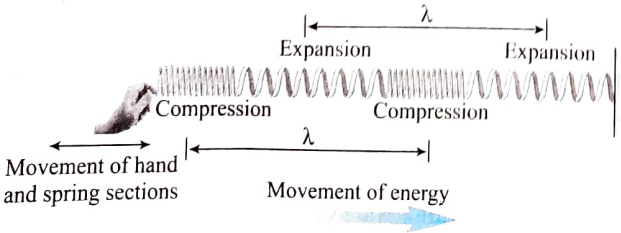
Important Points:

- A particle either at the crest or at the trough has a tendency to move towards the mean position.
- A particle at the crest or the trough has zero velocity, and the distance of the particle from the mean position is termed as amplitude of the wave.
- Distance between two consecutive crests/troughs is equal to the wavelength of the wave.
- Distance between a consecutive pair of crest and trough is half the wavelength of the wave.

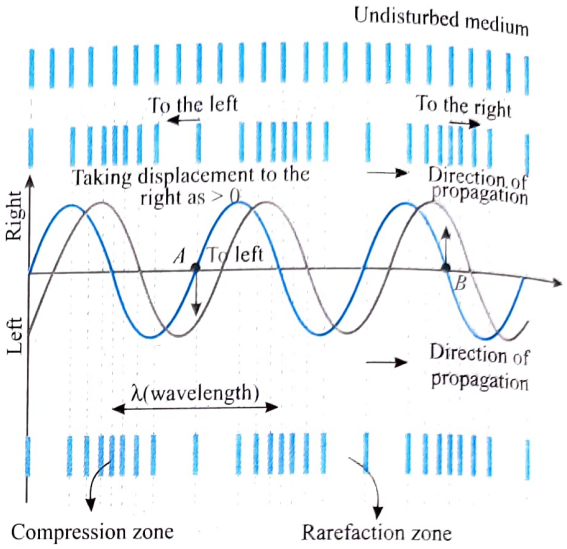
LONGITUDINAL WAVES

A wave motion is said to be longitudinal if the particles of the medium through which the wave is travelling vibrate in a direction parallel to the direction of propagation of the wave.

For example, a pulse moving down a long-stretched spring. If the left end of the spring given in figure is pushed briefly to the right and then pulled briefly to the left. This movement creates a sudden compression at the end of the spring called a pulse. This pulse travels along the spring (to the right in figure). Here you can observe that the direction of the displacement of the coils is parallel to the direction of propagation of the compressed region. Here longitudinal wave along a stretched spring is formed.



In this type of waves, oscillatory motion of the medium particles produces regions of compression (high pressure) and rarefaction (low pressure) which propagated in space with time. Sound waves are longitudinal waves. In case of sound wave the disturbance is a series of high-pressure and low-pressure regions that travels through air.



The displacement–position graph of a longitudinal wave at an instant.

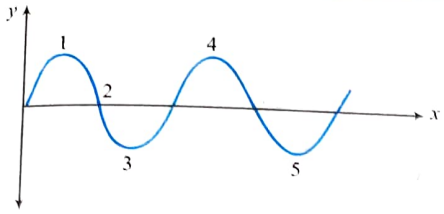
Important Points:

- A longitudinal wave moves by the phenomena of compression and rarefaction in the medium.
- In compression, distance between any two consecutive particles of medium is less than their normal distance. Therefore, density is more than the normal density. In a rarefaction, distance between any two consecutive particles of medium is more than the normal distance. therefore, density is less than the normal density.
- At an instant of time a particle in compression phase and another in rarefaction phase are exactly out of phase from each other.
- All particles in compression phase are in phase (at a given instant of time).
- All particles in rarefaction phase are in phase (at a given instant of time).
- Distance between two consecutive compressions or rarefactions is equal to the wavelength of the wave. Longitudinal waves can propagate through any state of matter.

WAVE PARAMETERS

Phase

Phase defines the position (in terms of distance from mean position) and velocity of a particle oscillating under the influence of a wave. The particles of the medium which are in the same state of motion (at the same displacement from their respective mean positions moving in the same direction) are said to be in phase or differing in phase by $2n\pi$, where $n = 1, 2, 3, \ldots$ and the particles of which state of motion are exactly opposite (displacements from the mean position and velocities are exactly opposite) are said to be out of phase or differing in phase by $n\pi$ where $n = 1, 3, 5, \ldots$

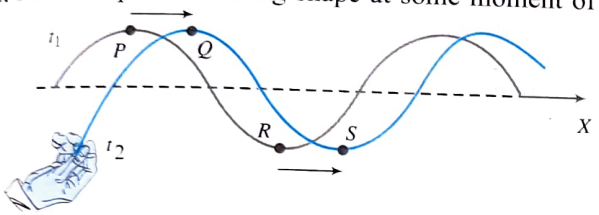


- Oscillations of the particles 1 and 4 are in phase.
- Oscillations of the particles 1 and 2 are not in phase.
- Oscillations of the particles 3 and 5 are in phase.
- Oscillations of the particles 4 and 5 are out of phase.

Wave Speed

Wave speed is the distance travelled by the wave in unit time. In wave motion, the disturbance created at a point of space travels and has a speed. The distance covered per unit time by the wave is known as wave speed.

If we plot the disturbance against position along wave motion at a fixed time, the graph so obtained is called waveform. For a string wave, it is a snapshot of string shape at some moment of time.



The figure shows waveforms at two nearby moments of time t_1 and t_2. The disturbance (say, peak of string) at P has moved to Q in time $t_2 - t_1$. The displacement of wave is $\overrightarrow{PQ}$ in time $\Delta t = t_2 - t_1$. Then

$$v = \text{wave speed} = \frac{PQ}{t_2 - t_1} = \left(\frac{RS}{t_2 - t_1}\right)$$

We see that waveform of t_1 has just translated by PQ, acquiring the new position at t_2.

Amplitude

Amplitude A of oscillations of a particle of the medium propagating the wave is the maximum displacement of the particle from its mean position on either side.

Wavelength

Wavelength λ is the distance between two nearest particles (along the direction of propagation of wave) which are in the same phase of vibration. It can also be defined as the distance travelled by the wave in one time period of oscillation.

Wave Frequency

Wave frequency f is the number of times an oscillating particle is at its maximum displacement on one side, during 1 s of motion. It is expressed in Hertz (1 Hz = 1 cycle/s).

Time Period

Time period T is the time taken by an oscillating particle to move from mean position to one extreme, then move on to the other extreme position and come back to the mean position. (It is the time the oscillating particle takes to make one complete cycle.) It is the time after which the particle repeats its motion.

Intensity of the Wave

Intensity of the wave is the energy transmitted per unit area per second in the form of the wave in the direction of the propagation of the wave by the source. This energy is carried forward by the medium particles which while oscillating transfer the energy to the next particles.

In our further discussion, we assume that as a wave proceeds forward, it does so without any dissipation of energy, i.e., neglecting the effects of air drag, internal resistances, etc. which cause loss of energy as wave progresses. In other words, the amplitude of a simple progressive wave remains same as it progresses forward.

ILLUSTRATION 6.1

A sinusoidal wave is travelling along a rope. The oscillator that generates wave completes 40.0 vibrations in 30.0 s. Also, a given 'maximum' travels 425 cm along the rope in 10.0 s. What is the wavelength of the wave?

Sol. The given information comprises things that can be measured directly in laboratory. A high speed photograph would reveal the wavelength which can be computed from $v = f\lambda$.

The frequency is $f = \dfrac{40.0 \text{ waves}}{30.0 \text{ s}} = 1.33 \text{ s}^{-1}$

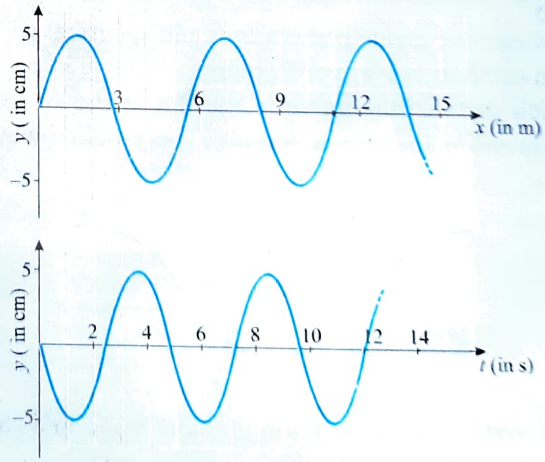
And the wave speed is $v = \dfrac{425 \text{ cm}}{10.0 \text{ s}} = 42.5 \text{ cm/s}$

Since $v = \lambda f$, the wavelength is $\lambda = \dfrac{v}{f} = \dfrac{42.5 \text{ cm/s}}{1.33 \text{ s}^{-1}} = 0.319 \text{ m}$

If we turn up the oscillator to a higher frequency, the wavelength would get shorter because speed is a constant factor. It is determined just by the properties of the string itself.

ILLUSTRATION 6.2

A transverse wave is travelling along a horizontal string. The first figure is the shape of the string at an instant of time. The second picture is a graph of the vertical displacement of a point on the string as a function of time. Find the wave speed.



Sol. From x-y graph, the length of two waves is 11 m.

Hence wavelength $\lambda = \dfrac{11}{2} = 5.5 \text{ m}$

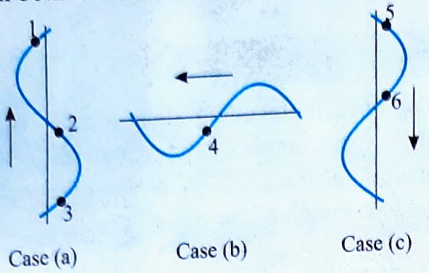
From y-t graph it is clear that a point completes two vibrations in 12 sec.

Hence time period $T = \dfrac{12}{2} = 6 \text{ sec}$

It means the more speed, $v = \dfrac{\lambda}{T} = \dfrac{5.5}{6} = \dfrac{11}{12} \text{ m/s}$

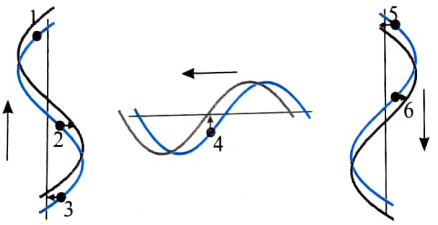
ILLUSTRATION 6.3

The figures of case (a), case (b) and case (c) shows the shape of three strings on which sinusoidal transverse waves are propagating. The arrows in the diagram indicate the direction of wave propagation. Out of the 6 particles marked (1, 2, 3, 4, 5, 6), which particles have their instantaneous velocity and acceleration both directed towards their mean position.



Case (a) Case (b) Case (c)

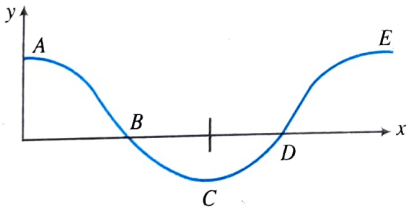
Sol. When a sinusoidal wave travels through a medium, all particles perform SHM. Their acceleration is always towards their mean position. This can be observed by looking at the direction of wave and predicting the position of a particle a moment later. From figure it is clear that the Velocities of points 3, 4 and 5 are towards mean position.



Case (a) Case (b) Case (c)

CONCEPT APPLICATION EXERCISE 6.1

1. Transverse waves are possible in solids but not in fluids. Why?
2. How can one create plane waves and spherical waves?
3. Is an oscillation a wave? Explain.
4. Which parts of the curve in the figure shown represent compression and rarefaction for a longitudinal wave?



5. How would you create a longitudinal wave in a stretched spring? Would it be possible to create a transverse wave in a spring?
6. A longitudinal wave is produced on a toy slinky. The wave travels at a speed of 30 cm/s and the frequency of the wave is 20 Hz. What is the minimum separation between two consecutive compressions of the slinky?
7. A narrow pulse (for example, a short pip by a whistle) is sent across a medium. If the pulse rate is 1 after every 20 s (that is the whistle is blown for a split of second after every 20 s), is the frequency of the note produced by the whistle equal to 1/20 or 0.05 Hz?
8. Waves are generated on a water surface. Calculate the phase difference between two points A and B, when
   (a) A and B lie on the same wavefront at a distance of 2λ between them
   (b) A and B lie on successive crests separated by 1 m
   (c) A and B lie on successive troughs separated by 1.5 m
9. Transverse waves are not produced in liquids and gases. Why?

ANSWERS

4. ABC represents compression, CDE represents rarefaction. **6.** 1.5 cm **8.** (a) 0 (zero) (b) 2π radian (c) 2π radian

WAVE FUNCTION

A function of one variable is such as $x(t)$ is sufficient to describe the motion of a point mass moving along a line. To give a mathematical description of an extended moving object such as a moving pulse, functions which depend on two variables such as x and t are required. Functions which can mathematically represent a moving wave pulse are called **wave function**

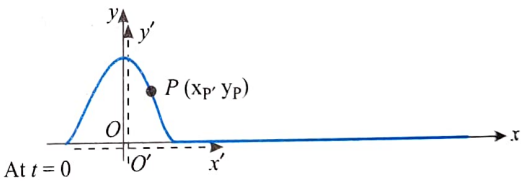
Let us consider a pulse is moving on a long tight string along $+x$ direction. Figure (a) shows a pulse at $t = 0$.

The shape of the string at this time can be mathematically assign by the function.

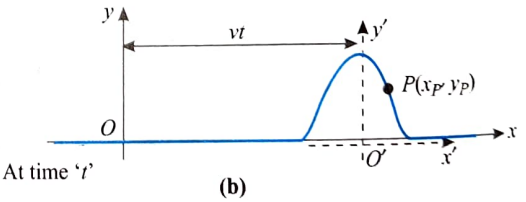
$$y = f(x) \quad \text{...(i)}$$

The function $f(x)$ represents the displacement y of a particle P at $x = 0$ as a function of time

$$y(x, t = 0) = f(x) \quad \text{...(ii)}$$



(a)



(b)

Here we can introduce a new coordinate system with origin O'. This new coordinate system moves with the speed v of the pulse. In this reference frame the pulse is stationary. Let the equation of pulse in this reference frame is given by

$$y' = f(x') \quad \text{...(iii)}$$

Now consider any time 't', at this time the displacement of the pulse will be vt as shown in Fig. (b). As the shape of the pulse does not change, the coordinate of point P on the pulse in the frame of reference moving with pulse will remain unchanged. At this position

$$y_p = f(x_p) \quad \text{...(iv)}$$

The coordinate of point P in two reference frames are related as

$$y = y_P \text{ and } x = x_P + vt \text{ or } x_P = x - vt$$

Substituting the value of x_p and y_p in Eq. (iv) we get

$$y = f(x - vt) \quad \ldots(v)$$

Equation (v) is called wave function, it represents a wave travelling in the positive x-direction with a constant speed v. Such a wave is called a travelling wave or a progressive wave.

The wave travelling in positive x-direction (Eq. (v)) can also be written as

$$y = f(vt - x) = f\left(t - \frac{x}{v}\right) \quad \ldots(vi)$$

The function f is arbitrary and depends on how the source moves. The time t and the position x must appear in the wave equation in the combination $t - x/v$ only. For example,

$$y = A \sin \frac{(t - x/v)}{T}, \quad y = Ae^{\frac{(t-x/v)}{T}}$$

etc. are valid wave equations. They represent waves travelling in positive x-direction with constant speed.

The equation $y = A \sin \dfrac{(x^2 - v^2t^2)}{T}$ does not represent a wave travelling in x-direction with a constant speed.

If a wave travels in negative x-direction with speed v, its general equation may be written as

$$y = f(x + vt) \text{ or } y = f\left(t + \frac{x}{v}\right) \quad \ldots(vii)$$

Important Point:

Here it is important to understand the meaning of y. If we consider an element of the string at point P, having a particular value of its x coordinate. When the pulse passes through P, its y coordinate of this element changes. The wave function $y(x, t)$ represents the y coordinate—the transverse position—of any element located at position x at any time t. Furthermore, if t is fixed (as, for example, if we have taken a snapshot of the pulse), then the wave function $y(x)$ is called the waveform, it defines a curve representing the geometric shape of the pulse at that time.

ILLUSTRATION 6.4

A wave pulse is travelling along $+x$ direction on a string at 2 m/s. Displacement y (in cm) of the particle at $x = 0$ at any time t is given by $2/(t^2 + 1)$. Find

(a) Expression of the function $y = (x, t)$, i.e., displacement of a particle at position x and time t.

(b) Draw the shape of the pulse at $t = 0$ and $t = 1$ s.

Sol.

(a) By replacing t by $t - (x/v)$, we can get the desired wave function, i.e.,

$$y = \frac{2}{\left(t - \frac{x}{2}\right)^2 + 1}$$

(b) We can use wave function at a particular instant to find the shape of the wave pulse using different values of x.

at $t = 0$, $\quad y = \dfrac{2}{\frac{x^2}{4} + 1}$

at $x = 0$, $\quad y = 2$ cm

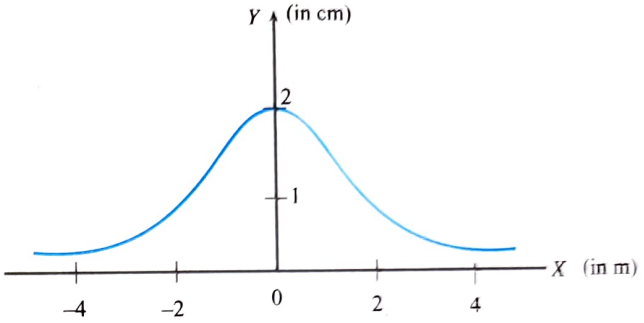
$x = 2$ cm, $\quad y = 1$ cm

$x = -2$ cm, $\quad y = 1$ cm

$x = 4$ cm, $\quad y = 0.4$ cm

$x = -4$ cm, $\quad y = 0.4$ cm

Using these values, shape of the pulse is drawn.



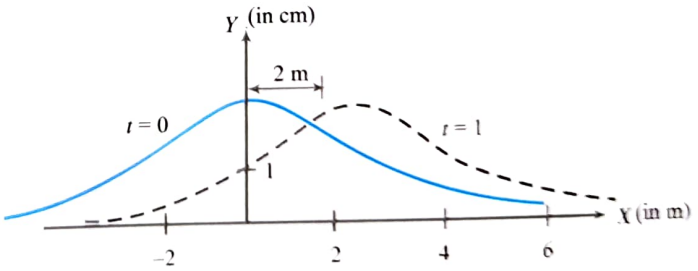
Similarly for $t = 1$ s, shape can be drawn. What do you conclude about direction of motion of the wave from the graphs? Also check how much the pulse has moved in 1 s time interval. This is equal to wave speed. Here is the procedure:

at $t = 1$ s, $\quad y = \dfrac{2}{\left(1 - \frac{x}{2}\right)^2 + 1}$

at $x = 2$ m $\quad y = 2$ cm (maximum value)

at $x = 0 \quad y = 1$ cm

at $x = 4$ m $\quad y = 1$ cm



The pulse has moved to the right by 2 in 1 s interval.

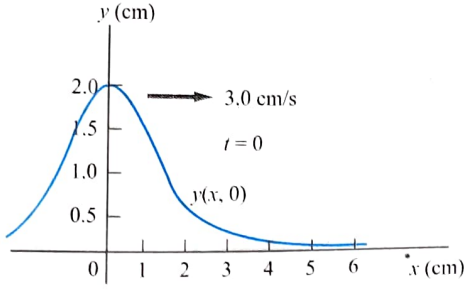
ILLUSTRATION 6.5

A pulse moving to the right along the x-axis is represented by the wave function

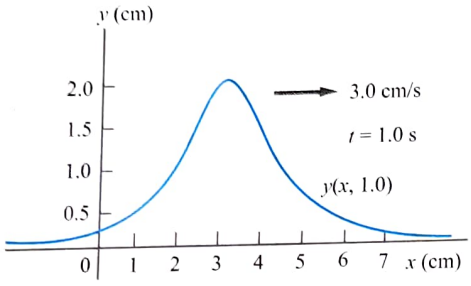
$$y(x, t) = \frac{2}{(x - 3.0t)^2 + 1}$$

where x and y are measured in centimetres and t is measured in seconds. Find expression for the wave function at $t = 0$, $t = 1.0$ s, and $t = 2.0$ s and plot the shape of pulse at these lines.

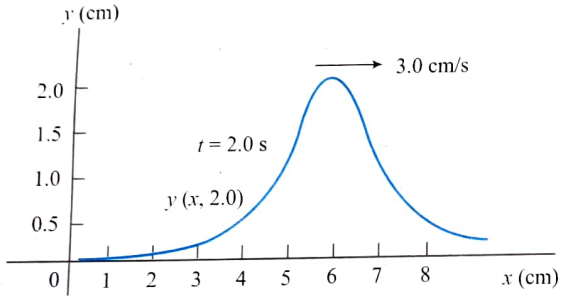
Sol. Figure (a) shows the pulse represented by this wave function at $t = 0$. Imagine this pulse moving to the right and maintaining its shape as suggested by Figs. (b) and (c).



(a)



(b)



(c)

We categorize this example as a relatively simple analysis problem in which we interpret the mathematical representation of a pulse.

The wave function is of the form $y = f(x - vt)$. Inspection of the expression for $y(x, t)$ reveals that the wave speed is $v = 3.0$ cm/s. Furthermore, by letting $x - 3.0t = 0$, we find that the maximum value of y given by $A = 2.0$ cm.

Write the wave function expression at $t = 0$: $y(x, 0) = \dfrac{2}{x^2 + 1}$

Write the wave function expression at $t = 1.0$ s:

$$y(x, 1.0) = \frac{2}{(x - 3.0)^2 + 1}$$

Write the wave function expression at $t = 2.0$ s:

$$y(x, 2.0) = \frac{2}{(x - 6.0)^2 + 1}$$

For each of these expressions we can have various values of x and plot the wave function. This procedure yields the wave functions shown in the three parts of figure.

These snapshots show that the pulses move to the right without changing its shape and that it has a constant speed of 3.0 cm/s.

ILLUSTRATION 6.6

At $t = 0$, transverse pulse in a wire is described by the function

$$y = \frac{6}{x^2 + 3}$$

where x and y are in metres. Write the function $y(x, t)$ that describes this pulse if it is travelling in the positive x-direction with a speed of 4.50 m/s.

Sol. At $t = 0$, the wave pulse looks like a bump centred at $x = 0$. As time goes on, the wave function will be function of t as well as x. The point about which the bump is centred will be $x_0 = 4.5t$.

We obtain a function of the same shape by writing

$$y(x, t) = \frac{6}{[(x - x_0)^2 + 3]}$$

where the centre of the pulse is at $x_0 = 4.5\ t$. Thus we have

$$y(x, t) = \frac{6}{(x - 4.5t)^2 + 3}$$

Note that for y to stay constant as t increases, x must increase by $4.5\ t$, as it describes the wave moving at 4.5 m/s.

In general, we can cause any waveform to move along the x-axis at a velocity v, by substituting $(x - v_x t)$ for x in the wave function $y(x)$ at $t = 0$. A wave function that depends on t through $(x + v_x t)$ describes a wave moving in the negative x-direction.

DIFFERENTIAL EQUATION OF WAVE FUNCTION

Consider a wave function of a wave moving towards $+x$ direction:

$$y = f(x - vt) \quad \ldots(i)$$

Taking derivative of y with respect to t keeping x constant

$$\frac{\partial y}{\partial t} = v.f'(x - vt) \quad \ldots(ii)$$

$\dfrac{\partial y}{\partial t}$ is the velocity of an element of the string on which the wave is moving. Differentiating (ii) again w.r.t. time

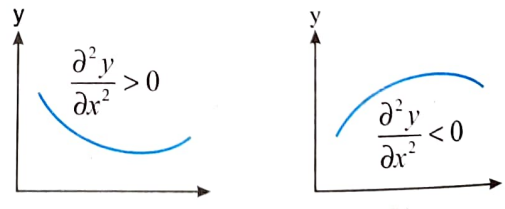
$$\frac{\partial^2 y}{\partial t^2} = v^2.f''(x - vt) \quad \ldots(iii)$$

$\dfrac{\partial^2 y}{\partial t^2}$ is the acceleration of an element of the string on which the wave is moving.

Now taking second derivative of y with respect to x keeping t constant

$$\frac{\partial^2 y}{\partial x^2} = f''(x - vt) \quad \ldots(iv)$$

$\dfrac{\partial^2 y}{\partial x^2}$ is the measure of bending of the string on which the wave is moving. If $\dfrac{\partial^2 y}{\partial x^2}$ is positive the shape of string concave up or otherwise. As shown in Figs. (a) and (b).



(a) (b)

From (iii) and (iv) we get, $\dfrac{\partial^2 y}{\partial t^2} = v^2.\dfrac{\partial^2 y}{\partial x^2}$ $\quad \ldots(v)$

Equation (v) is the differential wave equation.

Important Points:

- The differential wave equation (v) shows that the acceleration of an element of the string is related to the amount of bending of the string at that element. At points where the bending is large ($|\partial^2 y/\partial x^2|$ is large). The acceleration magnitude is large, ($|\partial^2 y/\partial t^2|$ is large). The acceleration is zero $\frac{\partial^2 y}{\partial t^2}=0$ at points where the string is straight $\frac{\partial^2 y}{\partial x^2}=0$.
- If $\frac{\partial^2 y}{\partial x^2}$ is positive, the string bends upward, the acceleration is directed upward.
- If $\frac{\partial^2 y}{\partial x^2}$ is negative, the string bends downward, the acceleration is directed downward.

ILLUSTRATION 6.7

Show that the function $y(x,t)=Ae^{-B(x-vt)^2}$ represents a travelling wave.

Sol. A wave must satisfy the one dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad ..(i)$$

Differentiating the given function w.r.t. x, keeping t constant.

$$\frac{\partial y}{\partial x}=-2AB(x-vt)e^{-B(x-vt)^2} \quad ...(ii)$$

and $$\frac{\partial^2 y}{\partial x^2}=\left[-2AB+4AB^2(x-vt)^2\right]e^{-B(x-vt)^2} \quad ...(iii)$$

Differentiating the given function w.r.t. t keeping t constant,

we have $$\frac{\partial y}{\partial x}=2AB(x-vt)e^{-B(x-vt)^2} \quad ...(iv)$$

nd $$\frac{\partial^2 y}{\partial t^2}=v^2\left[-2AB+4AB^2(x-vt)^2\right]e^{-B(x-vt)^2} \quad ...(v)$$

n comparing Eqs. (iii) and (v), we have $\frac{\partial^2 y}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

hus the given function represents a travelling wave.

LLUSTRATION 6.8

A transverse pulse moving on a long string along x-axis is given as, $y=\frac{2}{(5x-3t)^2+1}$, where x is measured in meter y in centimeter and t is measured in seconds. Find the

(i) wave speed, height of the pulse, shape of the pulse at $t=0$.

(ii) particle velocity as the function of x and t.

(iii) particle velocity at $x=\pm 1$ m, $t=0$ sec.

ol.

) We can write the equation of the pulse at a time t is given as

$$y=\frac{2}{25(x-\frac{3}{5}t)^2+1} \quad ...(i)$$

Hence the wave speed should be, $v=\frac{3}{5}$ m/s

The height of the pulse is $y_{max}=2$ cm

The shape of the pulse at $t=0$ is $y=\frac{2}{25x^2+1}$

(ii) The particle velocity, $v_p=\frac{\partial y}{\partial t}=\frac{\partial}{\partial t}\left(\frac{2}{(5x-3t)^2+1}\right)$

$$\Rightarrow v_P=\frac{-2}{[(5x-3t)^2+1]^2}\frac{\partial}{\partial t}[(5x-3t)^2+1]$$

$$=\frac{-2}{(5x-3t)^2+1}2(5x-3t)(-3)$$

Hence particle velocity as the function of x and t is,

$$v_p=\frac{12(5x-3t)}{(5x-3t)^2+1}$$

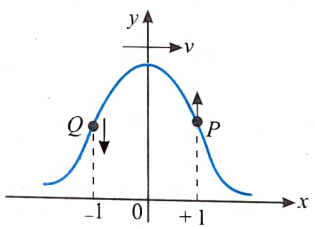
(iii) At $x=+1$ m and $t=0$ sec, the velocity of the particle P is

$$v_p=\frac{12[5(1)-0]}{[5(1)-3(0)]^2+1}=\frac{30}{13}=2.3 \text{ m/s}$$

The particle P is moving up at $x=+1$ m and $t=0$, the velocity of the particle Q is

$$v_p=\frac{12\{5(-1)-0\}}{[5(-1)-3(0)]^2+1}\text{cm/s}=-\frac{30}{13}=-2.3 \text{ cm/s}$$

Then, the particle Q is moving down as shown in figure.



CONCEPT APPLICATION EXERCISE 6.2

1. The equation of a travelling wave is given by

$$y=+\frac{b}{a}\sqrt{a^2-(x-ct)^2} \quad \text{where} \quad -a<x<a$$

$=0$, otherwise

Find the amplitude and the wave velocity for the wave. What is the initial particle velocity at the position $x=a/2$?

2. Show that **(a)** $y=(x+vt)^2$, **(b)** $y=(x+t)^2$, **(c)** $y=(x-vt)^2$, and **(d)** $y=2\sin x\cos vt$ are each a solution of one-dimensional wave equation but not (e) $y=x^2-v^2t^2$ and (*f*) $y=\sin 2x\cos vt$.

3. A wave is propagating on a long stretched string along its length taken as the positive x-axis. The wave equation is given as $y=y_0e^{-\left(\frac{t}{T}-\frac{x}{\lambda}\right)^2}$ where $y_0=4$ mm, $T=1.0$ s and $\lambda=4$ cm.

(a) Find the velocity of the wave.

(b) Find the function $f(t)$ giving the displacement of the particle at $x=0$.

(c) Find the function $g(x)$ giving the shape of the string at $t = 0$.

(d) Plot the shape $g(x)$ of the string at $t = 0$.

(e) Plot the shape of the string at $t = 5$ s.

4. A travelling wave pulse is given by $y = \dfrac{10}{5+(x+2t)^2}$

In which direction and with what velocity is the pulse propagating? What is the amplitude of the pulse?

5. A travelling wave pulse is given by

$$y = \frac{0.8}{(3x^2+12xt+12t^2+4)}$$

where x and y are in m and t is in seconds. Find the velocity and amplitude of the wave.

6. If the displacement relation for a particle in a wave is given by $y = 5\sin\left(\dfrac{t}{0.04} - \dfrac{x}{4}\right)$, determine the maximum speed of the particle in SI units.

7. Verify that wave function $y = \dfrac{2}{(x-3t)^2+1}$ is a solution to the linear wave equation, x and y are in centimetres.

ANSWERS

1. $b, c, \dfrac{bc\sqrt{3}}{3a}$

3. (a) 4 cm/s (b) $f(t) = y_0 e^{-(t/T)^2}$ (c) $g(x) = y_0 e^{-(x/\lambda)^2}$

4. Negative x-axis with speed 2 m/s, 2 m

5. 2 m/s, 0.2 m 6. 125 ms^{-1}

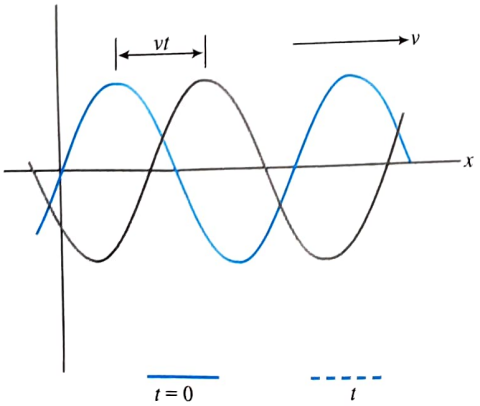
EQUATION OF A PLANE PROGRESSIVE WAVE (TRAVELLING WAVE)

DEFINITION OF A PROGRESSIVE WAVE

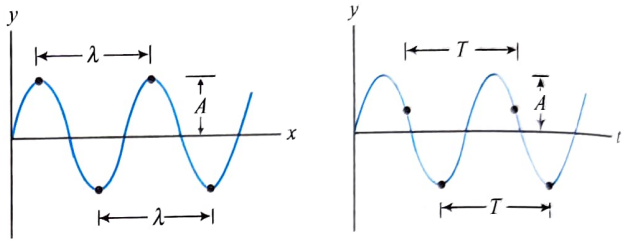
A wave which travels in a given direction with constant amplitude (in other words without attenuation) is known as a progressive wave.

TRAVELLING WAVE MODEL

In this section, we introduce an important wave function whose shape is shown in figure. The wave represented by this curve is called a sinusoidal wave because the curve is the same as that of the function sin θ plotted against θ. A sinusoidal wave could be established on a rope by shaking the end of the rope up and down in simple harmonic motion.



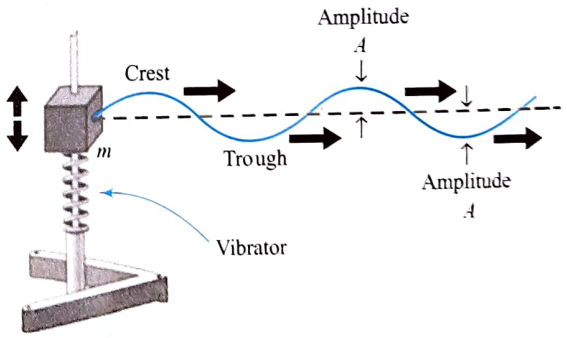
The sinusoidal wave is the simplest example of a periodic continuous wave and can be used to build more complex waves. The bold curve in figure represents a snapshot of a travelling sinusoidal wave at $t = 0$, and the dotted curve represents a snapshot of the wave at some later time t. Imagine two types of motion that can occur. First, the entire waveform in figure moves to the right so that the bold curve moves toward the right and eventually reaches the position of the dotted curve. This movement is the motion of the wave. If we focus on one element of the medium, such as the element at $x = 0$, we see that each element moves up and down along the y-axis in simple harmonic motion. This movement is the motion of the elements of the medium. It is important to differentiate between the motion of the wave and the motion of the elements of the medium.



(a) (b)

SINE WAVE TRAVELLING ON A STRING

Let us consider a long string along positive x-direction is held under tension. The left end at $x = 0$ being vibrated in a simple harmonic motion by a vibrator. The vibrator is doing work on the string and the energy is continuously supplied to the string. Any part of the string continues to vibrate up and down once the first disturbance has reached it. It receives energy from the left, transmits it to the right and the process continues. The nature of vibration of any particle is similar to that of the left end, the only difference being that the motion is repeated after a time delay of x/v.



A very important special case arises when the person vibrates the left end $x = 0$ in a simple harmonic motion. The equation of motion of this end may then be written as

$$y = f(t) = A\sin\omega t \quad \text{...(i)}$$

where A represents the amplitude and ω the angular frequency. The time period of oscillation is $T = \dfrac{2\pi}{\omega}$ and the frequency of oscillation is $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$. The wave produced by such a vibrating source is called a sine wave or sinusoidal wave.

Since the displacement of the particle at $x = 0$ is given by Eq. (i) the displacement of the particle at x at time t will be

$$y = f\left(t - \frac{x}{v}\right)$$

or, $$y = A\sin\omega\left(t - \frac{x}{v}\right) \quad \ldots(ii)$$

This follows from the fact that the wave moves along the string with a constant speed v and the displacement of the particle at x at time; originated at $x = 0$ at time $t - x/v$.

Traditionally, a number of parameters (v, λ, T, ν and ω) are used to describe a progressive wave. Another important parameter is k (called wave number, angular wave number or propagation constant) and is defined as

$$k = \frac{2\pi}{\lambda} \quad \ldots(iii)$$

Also, $$\frac{\omega}{k} = \frac{2\pi/T}{2\pi/\lambda} = \frac{\lambda}{T} = v \text{ or } \omega = kv \quad \ldots(iv)$$

From Eqs. (ii), (iii) and (iv)

$$y = A\sin\left[k(vt - x)\right] = A\sin(\omega t - kx) \quad \ldots(v)$$

Since y is a function of both space (x) and time (t), it is denoted by $y(x,t)$. Thus, from Eq. (v), the equation of a progressive wave travelling towards right (i.e., towards positive direction of x-axis) is given as

$$y(x,t) = A\sin(\omega t - kx) \quad \ldots(vi)$$

In case, the wave is travelling towards left (i.e., towards negative direction of x-axis),

$$y(x,t) = A\sin(\omega t + kx) \quad \ldots(vii)$$

In case, there is an initial phase ϕ_0, Eqs. (vi) and (vii) are written as

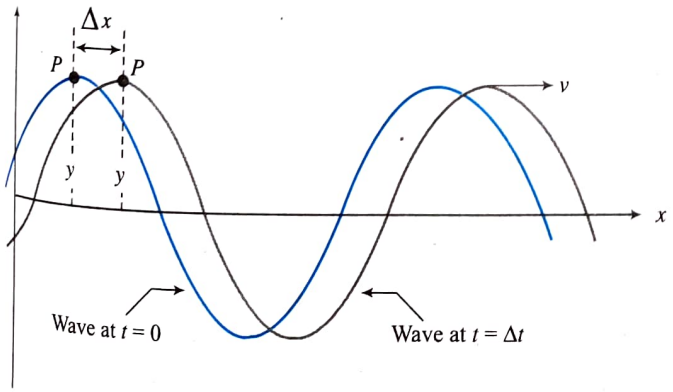
$$y(x,t) = A\sin(\omega t - kx + \phi_0) = A\sin\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + \phi_0\right] \quad \ldots(viii)$$

and $$y(x,t) = A\sin(\omega t + kx + \phi_0) = A\sin\left[2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) + \phi_0\right] \quad \ldots(ix)$$

Speed of Sinusoidal Wave

Consider a wave travelling along positive x-direction. Figure shows two snapshots of the wave at a small interval of time Δt. Let x be the movement of entire wave pattern in time Δt, then wave speed is defined as

$$v = \frac{\Delta x}{\Delta t} \quad \ldots(i)$$



A travelling wave in the positive x-direction is represented by

$$y(x,t) = A\sin(\omega t - kx + \phi_0) \quad \ldots(ii)$$

The phase of the wave is given by

$$\phi = (\omega t - kx) + \phi_0 \quad \ldots(iii)$$

As long as the shape of the wave does not change, ϕ remains constant. In such a case, as t increases, x increases in such a way that $(\omega t - kx)$ is a constant, It is to be noted that ϕ_0 (initial phase angle) remains constant.

Since $(\omega t - kx) = a$ constant, $\frac{d}{dt}(\omega t - kx) = 0$

or $$\omega - k\frac{dx}{dt} = 0 \text{ or } \frac{dx}{dt} = \frac{\omega}{k}$$

But as $\frac{dx}{dt} = v$ (speed of the travelling wave),

$$v = \frac{\omega}{k} \quad \ldots(iv)$$

As $\omega = 2\pi f$ and $k = 2\pi/\lambda$, from Eq. (iv), we obtain

$$v = \frac{2\pi f}{2\pi/\lambda} \text{ or } v = f\lambda \quad \ldots(v)$$

Equation (iv) is a general relation which is valid for all progressive waves.

General Expression for a Sinusoidal Wave

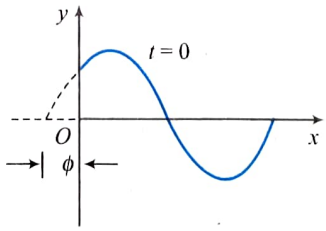
The general equation of simple harmonic progressive wave is given by

$$y = A\sin 2\pi\left(\frac{t}{T} \pm \frac{x}{\lambda}\right)$$

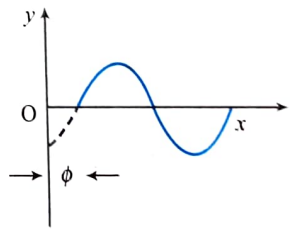
If an initial phase ϕ exists at origination point of wave, equation is changed to $y = A\sin\left[2\pi\left(\frac{t}{T} \pm \frac{x}{\lambda}\right) + \phi\right]$

Positive and Negative Initial Phase Constants

In general, the equation of a harmonic wave travelling along the positive x-axis is expressed as $y = A\sin(kx - \omega t \pm \phi)$, where ϕ is called the initial phase constant. It determines the initial displacement of the particle at $x = 0$ when $t = 0$.



Positive initial phase constant
$y = A\sin(kx - \omega t + \phi)$
The sine curve starts from the left of the origin.



Negative initial phase constant
$y = A\sin(kx - \omega t - \phi)$
The sine curve starts from the right of the origin.

Change in Phase with Time for a Constant x, i.e., at a Fixed Point in the Medium

$$[\phi]_{t_1} = 2\pi\left(\frac{t_1}{T} - \frac{x}{\lambda}\right) + \phi; \quad [\phi]_{t_2} = 2\pi\left(\frac{t_2}{T} - \frac{x}{\lambda}\right) + \phi$$

(For the wave travelling in positive x-direction)

$$\Delta\phi = \phi_{t_2} - \phi_{t_1} = \frac{2\pi}{T} \times (t_2 - t_1) = \frac{2\pi}{T} \times \Delta t \Rightarrow \Delta\phi = \frac{2\pi \times \Delta t}{T}$$

Phase difference $= \frac{2\pi}{T} \times$ Time difference

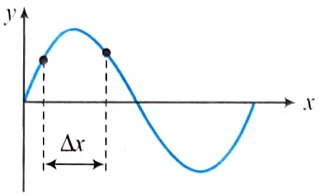
For $\Delta t = T$, i.e., after one time period, $\Delta\phi = \frac{2\pi}{T} \times T = 2\pi$

A phase change of 2π ($2n\pi$ in general) gives the points oscillating in phase.

VARIATION OF PHASE WITH DISTANCE

At a given instant of time $t = t$, phase at $x = x_1$,

$$[\phi]_{x_1} = 2\pi\left(\frac{t}{T} - \frac{x_1}{\lambda}\right) + \phi$$



For the wave travelling in positive x-direction and phase at $x = x_2$,

$$[\phi]_{x_2} = 2\pi\left(\frac{t}{T} - \frac{x_2}{\lambda}\right) + \phi$$

$$\Rightarrow \quad \Delta\phi = [\phi]_{x_1} - [\phi]_{x_2} = \frac{2\pi}{\lambda}(x_2 - x_1) = \frac{2\pi}{\lambda}\Delta x$$

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$

i.e., Phase difference $= \frac{2\pi}{\lambda} \times$ Path difference

For $\Delta x = n\lambda$, $\Delta\phi = n2\pi$, where $n = 1, 2, 3, \ldots$

Points separated by distance $n\lambda$ where n is an integer are in same phase (at any instant of time).

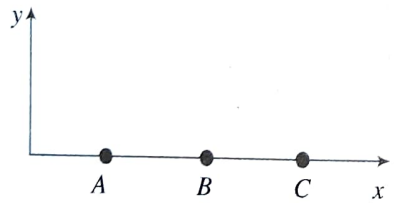
For $\Delta x = m\frac{\lambda}{2}$, where $m = 1, 3, 5, 7, \ldots$

$$\Delta\phi = m \times \pi$$

Points separated by distance $m(\lambda/2)$, where m is an odd integer are out of phase (at any instant of time).

ILLUSTRATION 6.9

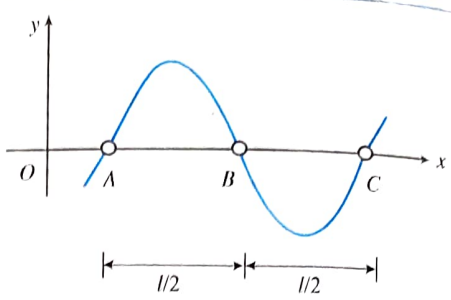
A, B and C are the three particles of a medium which are equally separated and lie along the x-axis. When a sinusoidal transverse wave of wavelength λ propagates along the x-axis, the following observations are made:



(i) A and B have the same speed.
(ii) A and C have the same velocity.
Find **(a)** the minimum distance between A and B and **(b)** the minimum distance between A and C.

Sol.

(a) Points A and B have the same speed. Point A goes downward and the point B goes upward. The distance between the points A and B is $\lambda/2$.



(b) Points A and C have the same velocity. Both the points move downward. The distance between A and C is λ.

ILLUSTRATION 6.10

When a train of plane wave traverses a medium, individual particles execute periodic motion given by the equation

$$y = 4\sin\frac{\pi}{2}\left(2t + \frac{x}{8}\right)$$

where the lengths are expressed in centimetres and time in seconds. Calculate the amplitude, wavelength, (a) the phase difference for two positions of the same particle which are occupied at time interval 0.4 s apart and (b) the phase difference at any given instant of two particles 12 cm apart.

Sol. The equation of a wave motion is given by

$$y = A\sin\frac{2\pi}{\lambda}(vt + x) \qquad \ldots(i)$$

Here, $y = 4\sin\frac{\pi}{2}\left(2t + \frac{x}{8}\right)$

This equation can be written as $y = 4\sin\frac{2\pi}{32}(16t + x)$ ...(ii)

Comparing Eq. (i) with Eq. (ii), we get,
Amplitude $A = 4$ cm; wavelength $\lambda = 32$ cm; wave velocity $v = 16$ cm/s

Here frequency is given as $f = \frac{v}{\lambda} = \frac{16}{32} = \frac{1}{2} = 0.5$ Hz

(a) Phase of a particle at instant t_1 is given by

$$\phi_1 = \frac{\pi}{2}\left(2t_1 + \frac{x}{8}\right)$$

The phase at instant t_2 is given by $\phi_2 = \frac{\pi}{2}\left(2t_2 + \frac{x}{8}\right)$

The phase difference is given as

$$\phi_1 - \phi_2 = \frac{\pi}{2}\left[\left(2t_1 + \frac{x}{8}\right) - \left(2t_2 + \frac{x}{8}\right)\right]$$

$$= \pi(t_1 - t_2) = \pi(0.4) \qquad (\text{As } t_1 - t_2 = 0.4)$$

$$= 180 \times 0.4 = 72° \qquad (\pi \text{ rad} = 180°)$$

(b) Phase difference at an instant between two particles with path difference Δ is

$$\phi = \frac{2\pi}{\lambda} \times \Delta x = \frac{2\pi}{32} \times 12 \qquad (\text{As } \Delta x = 12 \text{ cm})$$

$$= \frac{3\pi}{4}$$

ILLUSTRATION 6.11

A transverse harmonic wave of amplitude 1.0 cm and wavelength 3.0 m is travelling in positive x direction on a stretched string. At an instant, the particle at $x = 5.0$ m is at $y = +0.5$ cm and is travelling in positive y direction. Find the coordinate of the nearest particle ($x > 5.0$ m) which is at its positive extreme at this instant.

Sol. We make a wave diagram for this condition, let A be the point under consideration. The position of point A is taken at the right side of point M as it is travelling in positive y-direction. The phase difference between points M and B is 2π. Hence the phase difference between the points A and B should be

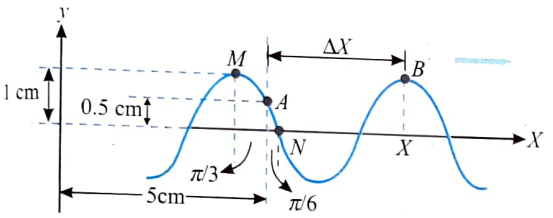
$$\Delta\phi = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

For finding x co-ordinate of particle B, first we have to find the distance between A and B say Δx.

Using a relation between phase difference and path difference,

$$\Delta\phi = \frac{2\pi}{\lambda}(\Delta x)$$

$$\frac{2\pi}{\lambda}(\Delta x) = \frac{5\pi}{3} \Rightarrow \Delta x = \frac{5\times 3.0}{6} = 2.5 \text{ m}$$



Hence, $\Delta x = \frac{5\times 3.0}{6} = 2.5$ m ; It means the coordinate of point B should be, $x_B = 5.0 + 2.5 = 7.5$ m

ILLUSTRATION 6.12

A wave is travelling along X-axis. The disturbance at $x = 0$ and $t = 0$ is $A/2$ and is increasing, where A is amplitude of the wave. If $y = A\sin(kx - \omega t + \phi)$, determine the initial phase ϕ.

Sol. At $x = 0$ and $t = 0$, the disturbance is $A/2$. Hence

$$\frac{A}{2} = A\sin\phi \quad \Rightarrow \quad \sin\phi = \frac{1}{2} \text{ which gives } \frac{\pi}{6}, \frac{5\pi}{6}$$

One of these two values is to be accepted. The basis for this is that ϕ should also satisfy the value of $\frac{\partial y}{\partial t}$ given in question.

The rate of increment of y is given by

$$\frac{\delta y}{\delta t} = -\omega A\cos(kx - \omega t + \phi)$$

At $x = 0$ and $t = 0$, $\frac{\delta y}{\delta t} = -\omega A\cos\phi$

If y increases, $\frac{\delta y}{\delta t}$ is positive. Since $\frac{\delta y}{\delta t}$ is positive, $\phi = \frac{5\pi}{6}$ will be suitable initial phase: $\frac{\delta y}{\delta t} = -\omega A\cos\frac{5\pi}{6}$

$$\frac{\delta y}{\delta t} = \omega A\left(-\frac{\sqrt{3}}{2}\right)$$

$$\frac{\delta y}{\delta t} = \frac{\sqrt{3}}{2}\omega A > 0$$

Thus, the initial phase is $\phi = 5\pi/6$.

ILLUSTRATION 6.13

A wave travelling along X-axis is given by

$$y = 2(\text{mm})\sin(3t - 6x + \pi/4)$$

where x is in centimetres and t in second. Write the phases and, hence, the find the phase difference between them at $t = 0$ for two points on X-axis, $x = x_1 = \pi/3$ cm and $x = x_2 = \pi/2$ cm.

Sol. At $t = 0$

$$y = \sin(-6x + \pi/4)$$

Phase at x_1; $\phi = \left(-6\times\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{7}{4}\pi$ and that at

x_2 is $\left(-6\times\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{11}{4}\pi$

The phase difference is $\left|\frac{7}{4}\pi - \frac{11}{4}\pi\right| = \pi$; the disturbances at the two points are out of phase.

ILLUSTRATION 6.14

Two sinusoidal waves in a string are defined by the functions $y_1 = (2.00 \text{ cm})\sin(20.0x - 32.0t)$ and $y_2 = (2.00 \text{ cm})\sin(25.0x - 40.0t)$ where y_1, y_2 and x are in centimetres and t is in seconds.

(a) What is the phase difference between these two waves at the point $x = 5.00$ cm at $t = 2.00$ s?

(b) What is the positive x value closest to the origin for which the two phases differ by $\pm\pi$ at $t = 2.00$ s? (That is a location where the two waves add to zero.)

Sol. Think of the two waves as moving sinusoidal graphs with different wavelengths and frequencies. The phase difference between them is a function of the variables x and t. It controls how the waves add up at each point.

Looking at the wave function, let us pick out the expressions for the phase as the 'angles we take the sines of'—then arguments of the sine functions. Then algebra will give particular answers.

At any time and place, the phase shift between the waves is found by subtracting the phases of the two waves, $\Delta\phi = \phi_1 - \phi_2$.

$$\Delta\phi = [(20.0 \text{ rad/cm})x - (32.0 \text{ rad/s})t] - [(25.0 \text{ rad/cm})x - (40.0 \text{ rad/s})t]$$

Collecting terms, $\Delta\phi = -(5.00 \text{ rad/cm})x + (8.00 \text{ rad/s})t$

(a) At $x = 5.00$ cm and $t = 2.00$ s, the phase difference is

$\Delta\phi = (-5.00 \text{ rad/cm})(5.00 \text{ cm}) + (8.00 \text{ rad/s})(2.00 \text{ s})$

$|\Delta\phi| = 9.00$ rad

(b) The sine functions repeat whenever their arguments change by an integer number of cycles, an integer multiple of 2π rad. Then the phase shift equals $\pm\pi$ whenever $\Delta\phi = \pi + 2n\pi$, for all integer values of n.

Substituting this into the phase equation, we have

$$\pi + 2n\pi = -(5.00 \text{ rad/cm})x + (8.00 \text{ rad/s})t$$

At $t = 2.00$ s,

$$\pi + 2n\pi = -(5.00 \text{ rad/cm})x + (8.00 \text{ rad/s})(2.00 \text{ s})$$

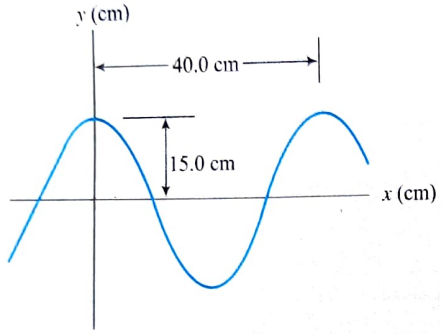
or $(5.00 \text{ rad/cm})x = (16.0 - \pi - 2n\pi)\text{rad}$

The smallest positive value of x is found when $n = 2$

$$x = \frac{(16.0 - 5\pi)\text{rad}}{5.00 \text{ rad/cm}} = 0.0584 \text{ cm}$$

ILLUSTRATION 6.15

A sinusoidal wave travelling in the positive x-direction has an amplitude of 15 cm, wavelength 40 cm and frequency 8 Hz. The vertical displacement of the medium at $t = 0$ and $x = 0$ is also 15 cm, as shown in figure.



(a) Find the angular wave number, period, angular frequency and speed of the wave.

(b) Determine the phase constant ϕ, and write a general expression for the wave function.

Sol.

(a) Wave number $k = \frac{2\pi}{\lambda} = \frac{2\pi \text{ rad}}{40 \text{ cm}} = \frac{\pi}{20}$ rad/cm

Time period $T = \frac{1}{f} = \frac{1}{8}$ s

Angular frequency $\omega = 2\pi f = 16\pi$ rad/s

Speed of wave $v = f\lambda = 320$ cm/s

(b) It is given that $A = 15$ cm and also $y = 15$ cm at $x = 0$ and $t = 0$ then using $y = A \sin(\omega t - kx + \phi)$ $15 = 15 \sin\phi$

$\Rightarrow \sin\phi = 1$

or $\phi = \frac{\pi}{2}$ rad

Therefore, the wave function is $y = A \sin(\omega t - kx + \frac{\pi}{2})$

$$= (15 \text{ cm}) \sin\left[(16\pi \text{ rad s}^{-1})t - \left(\frac{\pi}{20}\frac{\text{rad}}{\text{cm}}\right)x + \frac{\pi}{2}\right]$$

ILLUSTRATION 6.16

A wave is described by $y = (2.00 \text{ cm}) \sin(kx - \omega t)$, where $k = \pi$ rad/m, $\omega = 2\pi$ rad/s, x is in metres, and t is in seconds. Determine the amplitude, wavelength, frequency, and speed of the wave.

Sol. The wave function is a moving graph. The position of a particle of the medium, represented by y, is varying all the time and from every point to the next point at each instant. But we can pick out the parameters that characterize the whole wave and have constant values.

We compare the given wave function with the general sinusoidal wave equation $y = A \sin(kx - \omega t + \phi)$

Its functional equality to $y = (2.00 \text{ cm}) \sin(kx - \omega t)$ reveals that the amplitude is $A = 2.00$ cm

The angular wave number is $k = \pi$ rad/m so that

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\pi} = 2 \text{ m}$$

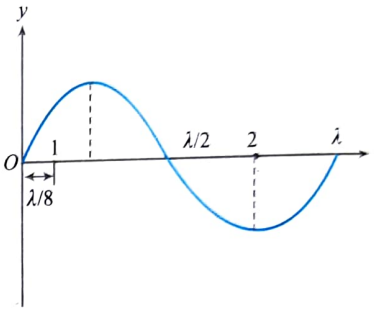
The angular frequency is $\omega = 2\pi$ rad/s so that

$$f = \frac{\omega}{2\pi} = \frac{2\pi}{2\pi} = 1 \text{ Hz}$$

The wave speed is $v = f\lambda = 1 \times 2 = 2$ m/s

CONCEPT APPLICATION EXERCISE 6.3

1. What is the phase difference between the particles 1 and 2 located as shown in figure?

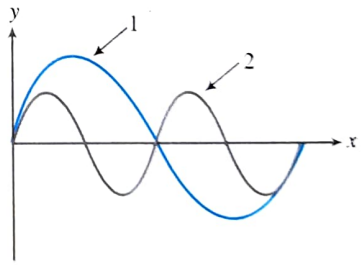


2. For a travelling harmonic wave

$y = 2.0 \cos(10t - 0.0080x + 0.35)$,

where x and y are in centimetres and t in seconds. What is the phase difference between oscillatory motion of two points separated by a distance of **(a)** 4 m **(b)** 0.5 m **(c)** $\lambda/2$ (d) $3\lambda/4$?

3. The figure shows two snapshots, each of a wave travelling along a particular string. The phase for the waves are given by

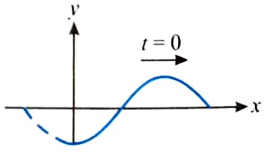


(a) $4x - 8t$

(b) $8x - 16t$. Which phase corresponds to which wave in the figure?

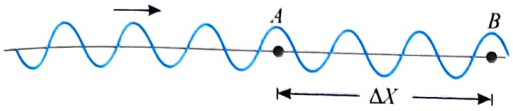
4. The particle displacement due to a sound wave is given by $y = 2.15 \times 10^{-3} \sin(100\pi t - \pi/300\, x)$, where x and y are in cm and t in seconds. Write down the amplitude, frequency, wavelength and velocity of the wave.

5. A harmonic transverse wave travelling on a string has an amplitude of 2 cm, wavelength of 1.25 m and velocity of 5 m/s in $+x$-direction. At $t = 0$, this wave has a crest at $x = 0$. Find the wave equation $y = f(x,t)$.



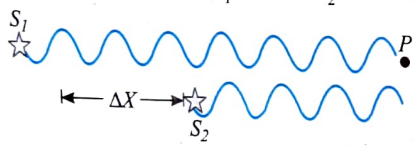
6. A progressive wave of frequency 500 Hz is travelling with a velocity of 360 m/s. Two points A and B are separated by a distance of 12 cm, the phase difference between the two points is _____.

7. There are two corks A and B at a separation Δx, lying on a water wave. Assume it as a transverse wave having wavelength λ moving along + x-direction. Find Δx if the **(a)** KE **(b)** momenta of two identical corks on the wave will be equal.



8. Two tuning forks separated by a distance l each of frequency $f = 150$ Hz with zero initial phase. If the wave speed is 10 m/s, at a distance $x = \frac{l}{4}$, find the phase difference between these two waves.

9. Two sources S_1 and S_2 emit transverse wave with wavelengths $\lambda = 1$ m and having zero phase angles. If their frequency is 10 Hz and the separation distance between the sources is $\Delta x = \frac{1}{4}$m. Find the phase difference between the waves at some point (say P) if the sources start emitting waves at **(a)** $t = 0$ simultaneously **(b)** $t_1 = 0$ and $t_2 = 1$ s respectively.



10. The equation of a progressive wave is given by $y = 0.20\sin 2\pi\left(60t - \frac{x}{5}\right)$ where x and y are in metre and t is in second. Find the

(a) phase difference between two particles separated by a distance of $\Delta x = 125$ cm,

(b) phase difference between the two instants 1/120 s and 1/40 s, for any particle.

(c) position/s of the particle/s which differ by a phase difference of $\Delta\theta = \pi/3$, with respect to that at position $x = 10$ m.

ANSWERS

1. $\frac{5\pi}{4}$ **2.** (a) 3.2 rad (b) 0.40 rad (c) π rad (d) $\frac{3\pi}{2}$ rad

3. (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$, snapshots 1 and 2 correspond to a and b respectively.

4. 2.15×10^{-3} cm, 50 s^{-1}, 600 cm, 300 ms^{-1}

5. $y = 0.02\cos(8\pi t - 1.6\pi x)$

6. $\frac{\pi}{3}$ **7.** (a) $\frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda, \frac{5\lambda}{2}, 3\lambda$ (b) $\lambda, 2\lambda, 3\lambda$

8. $15\pi l$ **9.** (a) $\frac{\pi}{2}$ rad (b) $\frac{7\pi}{10}$ rad

10. (a) $\frac{\pi}{2}$ (b) 2π (c) $x = 10 \pm \left(5n + \frac{5}{6}\right)$; where $n = 0, 1, 2$, etc....

PARTICLE VELOCITY AND ACCELERATION

Let a harmonic wave is propagating towards positive x direction, the wave function can be written as

$$y = A\sin(\omega t - kx) \quad ...(i)$$

We can analyze the motion of any element of the string using above expression. Each and every element of the string oscillate along y-direction hence its x coordinate remains constant. It means, the transverse speed v_y (not to be confused with the wave speed v, it is particle velocity) and the transverse acceleration a_y of elements of the string are given by

$$v_y = \left.\frac{dy}{dt}\right]_{x=\text{constant}} = \frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx) \quad ...(ii)$$

$$a_y = \left.\frac{dv_y}{dt}\right]_{x=\text{constant}} = \frac{\partial v_y}{\partial t} = -\omega^2 A\sin(\omega t - kx) \quad ...(iii)$$

The maximum values of the transverse speed (particle velocity) and transverse acceleration (particle acceleration) are simply the absolute values of the coefficients of the cosine and sine functions:

$$v_{y,\,\max} = \omega A \quad ...(iv)$$

$$a_{y,\,\max} = \omega^2 A \quad ...(v)$$

Here it is to be noted that the transverse speed and transverse acceleration of elements of the string do not reach their maximum values simultaneously. The transverse speed reaches to its maximum value (ωA) at $y = 0$, whereas the magnitude of the transverse acceleration reaches its maximum value ($\omega^2 A$) at $y = \pm A$. Finally, Eqs. (iv) and (v) are identical in mathematical form to the corresponding equations for simple harmonic motion.

Let us differentiate Eq. (i) with respect to displacement of wave (x), we will get the slope of displacement curve of wave as

$$\frac{\partial y}{\partial x} = -Ak\cos(\omega t - kx) \quad ...(vi)$$

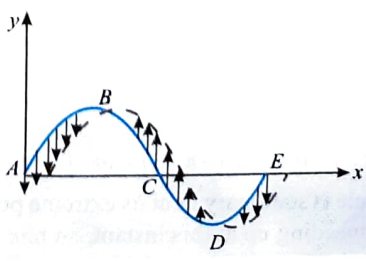
From Eqs. (ii) and (vi) we have $\frac{\partial y}{\partial x} = -\frac{k}{\omega}\frac{\partial y}{\partial t} = -\frac{1}{v}\frac{\partial y}{\partial t}$

$$\Rightarrow \quad \frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$$

Hence, particle velocity, $v_p = -v\frac{\partial y}{\partial x}$

Concept: Figure shows a snap shot of a pulse on a string that represents a transverse wave moving along + x-direction

Particle velocity is given by $v_p = -v_{\text{wave}}\frac{\partial y}{\partial x}$ ...(i)



For particles lying in part *AB* of string

As $\frac{\partial y}{\partial x} > 0$

From Eq. (i) $v_P < 0$

Hence the particles situated in part AB of the string should move down.

For particles lying in part *BD* of string

As $\frac{\partial y}{\partial x} < 0$, hence from Eq. (i) $v_P > 0$. It means the particles situated in this part, should move up.

For particles lying in part *DE* of string

As $\frac{\partial y}{\partial x} > 0$, it means $v_P < 0$

It means the particles situated in this part should move down.

ILLUSTRATION 6.17

For a wave described by, $y = A\sin(\omega t - kx)$ consider the following points **(i)** $x = 0$, **(ii)** $x = \frac{\pi}{4k}$ **(iii)** $x = \frac{\pi}{2k}$ **(iv)** $x = \frac{3\pi}{4k}$ For a particle at each of these points at $t = 0$, describe whether the particle is moving or not and in what direction and describe whether the particle is speeding up, slowing down or instantaneously not accelerating.

Sol. Displacement function of the medium particles,

$$y = A\sin(\omega t - kx)$$

Particle velocity $v_P(x, t) = \frac{\partial y}{\partial t} = \omega A\cos(\omega t - kx)$

Particle acceleration $a_P(x, t) = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$

(a) $t = 0, x = 0$: $v_P = +\omega A$ and $a_P = 0$

i.e., particle is moving upwards buts its acceleration is zero.

(b) $t = 0,\ x = \frac{\pi}{4k} \quad \therefore\ kx = \frac{\pi}{4}$

$$v_P = \omega A\cos\left(-\frac{\pi}{4}\right) = +\frac{\omega A}{\sqrt{2}}$$

and $$a_P = -\omega^2 A\sin\left(-\frac{\pi}{4}\right) = +\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is positive, i.e., the particle is moving upwards (along positive y-direction). Further v_P and a_P are in the same direction (both are positive). Hence the particle is speeding up.

(c) $t = 0,\ x = \frac{\pi}{2k} \quad \therefore \quad kx = \frac{\pi}{2}$

$$v_P = \omega A\cos(-\pi/2) = 0$$

$$a_P = -\omega^2 A\sin(-\pi/2) = \omega^2 A$$

i.e., particle is stationary or at its extreme position ($y = -A$). So, it is speeding up at this instant.

(d) $t = 0,\ x = \frac{3\pi}{4k} \quad \therefore \quad kx = \frac{3\pi}{4}$

$$v_P = \omega A\cos\left(-\frac{3\pi}{4}\right) = -\frac{\omega A}{\sqrt{2}}$$

$$a_P = -\omega^2 A\sin\left(-\frac{3\pi}{4}\right) = +\frac{\omega^2 A}{\sqrt{2}}$$

Velocity of particle is negative, i.e., the particle is moving downwards. Further v_P and a_P are in opposite directions, i.e., the particle is slowing down.

ILLUSTRATION 6.18

A plane progressive wave is given by

$x = (40\text{ cm})\cos(50\pi t - 0.02\pi y)$

where y is in centimetres and t in seconds. What will be the particle velocity at $y = 25$ cm in time $t = 1/200$ s?

Sol. Given that $x = 40\cos(50\pi t - 0.02\pi y)$

Therefore, particle velocity can be given as

$$v_p = \frac{dx}{dt} = (40\times 50\pi)\{-\sin(50\pi t - 0.02\pi y)\}$$

Putting $x = 25$ cm and $t = \frac{1}{200}$ s,

$$v_p = -(2000\pi\text{ cm/s})\sin\left[50\pi\left(\frac{1}{200}\right) - 0.02\pi(25)\right]$$

$$= 10\pi\sqrt{2}\text{ m/s}$$

ILLUSTRATION 6.19

A transverse mechanical harmonic wave is travelling on a string. Maximum velocity and maximum acceleration of a particle on the string are 3 m/s and 90 m/s^2, respectively. If the wave is travelling with a speed of 20 m/s on the string, write wave function describing the wave.

Sol. Maximum particle velocity, $u_{max} = \omega A$ ...(i)

Maximum particle acceleration, $a_{max} = \omega^2 A$ ...(ii)

Dividing Eq. (ii) by Eq. (i)

$\therefore$ Angular frequency, $\omega = \frac{a_{max}}{u_{max}} = \frac{90}{3} = 30$ rad/s

From Eq. (i), amplitude, $A = \frac{u_{max}}{\omega} = \frac{3}{30} = 0.1$ m

Propagation constant, $k = \frac{\omega}{v} = \frac{30}{20} = 1.5\text{ m}^{-1}$

Equation of wave is $y = A\sin(\omega t \pm kx)$ or wave function $y = 0.1\sin(30t \pm 1.5x)$ where x is in metres and t in seconds.

Positive sign is for wave propagation along negative x-axis and negative sign for wave propagating along positive x-axis.

ILLUSTRATION 6.20

The equation of a travelling plane sound wave has the form $y = 60\cos(1800t - 5.3x)$, where y is in micrometres, t in seconds and x in metres. Find

(a) the ratio of the displacement amplitude with which the particles of the medium oscillate to the wavelength,

(b) the velocity oscillation amplitude of particles of the medium and its ratio to the wave propagation velocity,

(c) the oscillation amplitude of relative deformation of the medium and its relation to the velocity oscillation amplitude of particles of the medium,

(d) the particle acceleration amplitude.

Sol. Comparing the given equation $y = 60\cos(1800t - 5.3x)$ with the wave equation $y = a\cos(\omega t - kx)$;

ω=1800 rad/s, $a = 60 \times 10^{-6}$ m, $k = 5.3$ per metre.

As $k = \frac{2\pi}{\lambda}$, $\lambda = \frac{2\pi}{k} = \frac{2\pi}{5.3}$ m

(a) Displacement amplitude of particles of the medium = Amplitude of the wave = 60×10^{-6} m.

$$\text{Required ratio} = \frac{\text{Displacement amplitude}}{\text{Wavelength}} = \frac{60\times10^{-6}\text{ m}}{\left(\frac{2\pi}{5.3}\right)\text{m}}$$

$$= \frac{60}{2\pi}\times10^{-6}\times5.3 = 5.06\times10^{-5}$$

(b) Velocity oscillation amplitude = Magnitude of the velocity of a particle of medium when passes through its equilibrium position = $a\omega$.

Velocity of propagation of the wave

$$= f\lambda = \frac{\omega}{2\pi}\times\lambda = \frac{\omega}{\left(\frac{2\pi}{\lambda}\right)} = \frac{\omega}{k}$$

$$\text{Required ratio } \frac{a\omega}{\left(\frac{\omega}{K}\right)} = ka = 5.3\times60\times10^{-6} = 3.18\times10^{-4}$$

(c) Deriving expression for relative deformation:

Let at $x = x$ at any instant $t = t$, displacement of particle is $y = y$ and at $x = x + dx$ displacement of particle be $y = y + dy$. (The displacement verses x graph at time t is continuous.)

Relative deformation, which is given as change in deformation with change in $x = dy/dx$ at an instant of time

$$\left(\frac{dy}{dx}\right)_t = -60\sin(1800t - 5.3x)(-5.3\times10^{-6})$$

$$= 318\times10^{-6}\sin(1800t - 5.3x)$$

For maximum relative deformation, $\left(\frac{dy}{dx}\right)_{max} = 318\times10^{-6}$

One should note that $(dy/dx)_t$ is nothing but the slope of the curve at (x, y).

Now, $\left(\frac{dy}{dx}\right)_t = \left(\frac{dy}{dt}\times\frac{dt}{dx}\right)$, $\left(\frac{dy}{dt}\right)_t = \left(\frac{dy}{dx}\times\frac{dx}{dt}\right)$

$$= \frac{dy}{dx} \times \text{wave velocity}$$

dx/dt = velocity of the wave = constant

$$\Rightarrow \left[\frac{dy}{dt}\right]_{max} = \left[\frac{dy}{dx}\right]_{max} \times \text{velocity of wave}$$

velocity oscillation amplitude = velocity of the wave × oscillation amplitude of relative deformation

(d) Velocity of particle $dy/dt = a\omega\cos(\omega t - kx)$

Acceleration of the particle

$$\frac{d^2y}{dt^2} = a\omega\times[-\sin(\omega t - kx)]\times\omega$$

$$= -a\omega^2\sin(\omega t - kx)$$

Maximum value of $|\sin(\omega t - kx)| = 1$

Thus, maximum acceleration of the particle = particle acceleration amplitude

$= a\omega^2 = 60 \times 10^{-6}(1800)^2 = 194.4$ m/s^2

ILLUSTRATION 6.21

A wave is travelling in a spring has wave equation as, $y = A\sin(kx - \omega t)$ where $A = 0.1$ m, $\lambda = \frac{1}{2}$m and $T = \frac{1}{5}$s

(i) For this wave draw the shape of the waveform (y-x graph) at $t = T$, and displacement–time graph (y-t graph) at $x = 0$, **(ii)** Find the velocity and acceleration of a particle on string at $x = \frac{\lambda}{2}$ and $t = T$, **(iii)** Draw the acceleration–displacement graph (a-y graph)

Sol. Here, wave number $k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.5} = 4\pi$/m and angular frequency, $\omega = \frac{2\pi}{T} = \frac{2\pi}{1/5} = 10\pi$ rad/s

The wave equation should be,

$$y = 0.1\sin(4\pi x - 10\pi t) = 0.1\sin 2\pi(2x - 5t) \quad \ldots(i)$$

(i) Shape of the waveform (y-x graph) at $t = T = \frac{1}{5}$s

Substituting the value of t in Eq. (i), we get

$y = 0.1\sin 4\pi x$

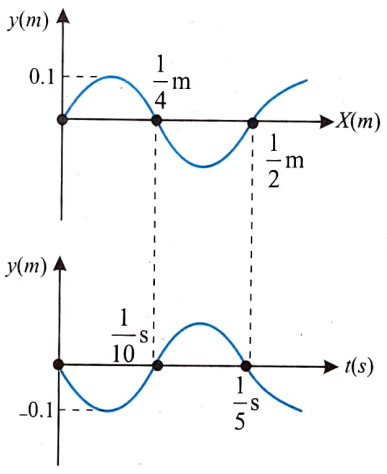
The shape of the wave is a sine curve.

Displacement-time graph (y-t graph) at $x = 0$

Substituting the value of $x = 0$ in Eq. (i), we get

$y = -0.1\sin 10\pi t$

The y-t graph is sinusoidal (but a negative sine curve).



(ii) Velocity of medium particle, $v = \left|\frac{dy}{dt}\right| = -A\omega\cos(kx - \omega t)$

At $x = \frac{\lambda}{2}$ and $t = T$

Hence $v = -(0.1)(10\pi)\cos\left\{4\pi \times \frac{1}{4} - \left(10\pi \times \frac{1}{5}\right)\right\}$

$= -\pi$ m/s (downwards)

Acceleration of medium particle,

$$a = \frac{dv}{dt} = -A\omega^2 \sin(kx - \omega t)$$

At $x = \frac{\lambda}{2}$ and $t = T$

$$a = -(0.1)(10\pi)^2 \sin(-\pi) = 0$$

(iii) The acceleration of the medium particles,

$$a = \frac{dv}{dt} = -A\omega^2 \sin(kx - \omega t) = -\omega^2 y$$

Hence the a-y graph is linear, passing through origin and having negative slope.

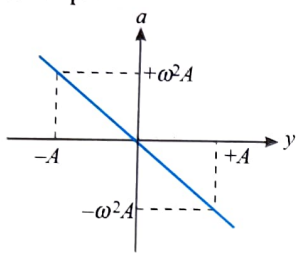
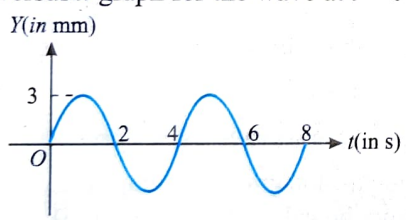


ILLUSTRATION 6.22

A harmonic wave having wavelength $\lambda = 6$ m propagates along positive x direction on a long tight string. The displacement (y) of a particle at $x = 2$ m varies with time (t) as shown in the graph

(a) Write the equation of the wave

(b) Draw y versus x graph for the wave at $t = 0$



Sol.

(a) Given, the wavelength $\lambda = 6$ m $\therefore$ $k = \frac{2\pi}{\lambda} = \frac{\pi}{3}$

From the graph $T = 4$ s $\therefore$ $\omega = \frac{2\pi}{T} = \frac{\pi}{2}$

Let the equation of the wave be,

$$y = (3\text{mm})\sin\left(\frac{\pi}{3}x - \frac{\pi}{2}t + \delta\right)$$

At $x = 2$m ; $t = 0$, y is zero

$$\therefore \quad \sin\left(\frac{\pi}{3} \times 2 + \delta\right) = 0 \quad \Rightarrow \quad \delta = -\frac{2\pi}{3} \text{ or } \frac{\pi}{3}$$

Also, $\frac{\partial y}{\partial t} > 0$ at $t = 0$, $x = 2$ m

$$\frac{\partial y}{\partial t} = -3\frac{\pi}{2}\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}t + \delta\right)$$

At $t = 0$, $x = 2$m,

$$\frac{\partial y}{\partial t} = -\frac{3\pi}{2}\cos\left(\frac{2\pi}{3} + \delta\right)$$

for this to be positive, $\cos\left(\frac{2\pi}{3} + \delta\right) < 0$,

$$\therefore \quad \delta = \frac{\pi}{3} \left[\text{and } \delta \neq -\frac{2\pi}{3}\right]$$

$$\therefore \quad y = (3\text{ mm})\sin\left(\frac{\pi}{3}x - \frac{\pi}{2}t + \frac{\pi}{3}\right) \quad \ldots(i)$$

(b) Substituting $t = 0$ in equation (i) we get, $y = 3\sin\left(\frac{\pi}{3}x + \frac{\pi}{3}\right)$

Hence y-t graph should be

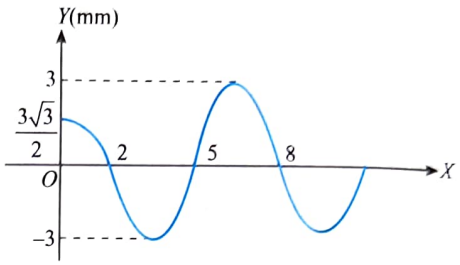


ILLUSTRATION 6.23

(a) Write the expression for y as a function of x and t for a sinusoidal wave travelling along a rope in the negative x direction with the following characteristics: $A = 8.00$ cm, $\lambda = 80.0$ cm, $f = 3.00$ Hz, and $y(0, t) = 0$ at $t = 0$.

(b) Write an expression for y as a function of x and t for the wave in part (a) assuming that $y(x, 0) = 0$ at the point $x = 10.0$ cm.

Sol. Think about the graph of y as a function of x at one instant as a smooth succession of identical crests and troughs, with the length in space of each cycle being 80.0 cm. The distance from the top of a crest to the bottom of a trough is 16.0 cm. Now think of the whole graph moving towards the left at 240 cm/s.

Using the travelling wave model, we can put constants with the right values into $y = A\sin(kx + \omega t + \phi)$ to have the mathematical representation of the wave. We have the same (positive) signs for both kx and ωt so that a point of constant phase will be at a decreasing value of x as t increases that is, so that the wave will move to the left.

The amplitude is $A = y_{max} = 8.00$ cm $= 0.0800$ m.

The wave number is $k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.800\text{ m}} = \frac{5\pi}{2}$

The angular frequency, $\omega = 2\pi f = 2\pi(3.00\text{ s}^{-1}) = 6.00\pi$ rad/s

(a) We have $y = A\sin(kx + \omega t + \phi)$, choosing $\phi = 0$ will make it true that $y(0, 0) = 0$. Then the wave function becomes upon substitution of the constant values for this wave

$$y = (0.080\text{ m})\sin\left(\frac{5\pi}{2}x + 6.00\pi t\right)$$

(b) In general, $y = (0.0800\text{ m})\sin\left(\frac{5\pi}{2}x + 6.00\pi t + \phi\right)$

If $y(x, 0) = 0$ at $x = 0.100$ m, we require

$$0 = (0.0800\text{ m})\sin\left(\frac{5}{2}\pi \times 0.1 + \phi\right)$$

$$\frac{5\pi}{2} \times 0.1 + \phi = 0$$

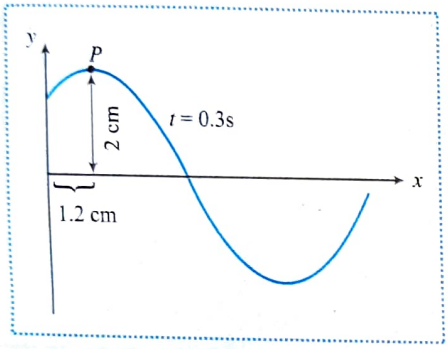
So we must have the phase constant $\phi = -\frac{\pi}{4}$ rad

Therefore, the wave function for all values of the variable x and t is

$$y = (0.0800\text{ m})\sin\left(\frac{5}{2}\pi x + 6.00\pi t - \frac{\pi}{4}\right)$$

ILLUSTRATION 6.24

Figure shows a snapshot of a sinusoidal travelling wave taken at $t = 0.3$ s. The wavelength is 7.5 cm and amplitude is 2 cm. If the crest P was at $x = 0$ at $t = 0$, write the equation of travelling wave.



Sol. Given, $A = 2$ cm, $\lambda = 7.5$ cm

$$k = \frac{2\pi}{\lambda} = \frac{4\pi}{15}\text{ cm}^{-1}$$

The wave has travelled a distance of 1.2 cm in 0.3 s. Hence, speed of the wave, $v = 1.2/0.3 = 4$ cm/s

Therefore, angular frequency $\omega = (v)\,(k) = \frac{16\pi}{15}$ rad/s

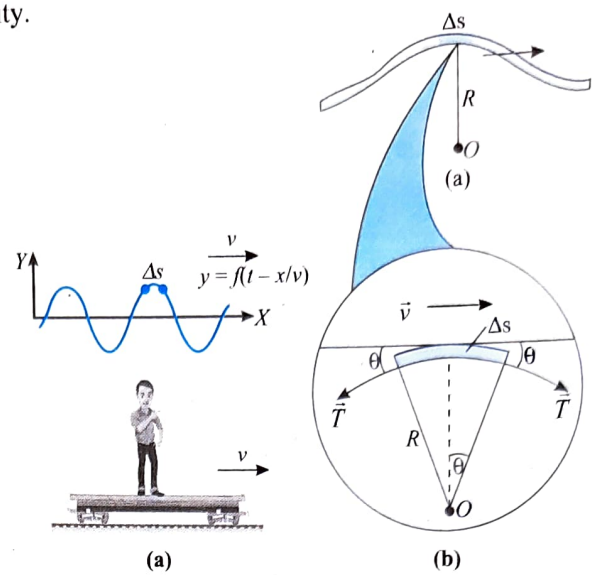
Since the wave is travelling along positive x-direction and crest (maximum displacement) is at $x = 0$ at $t = 0$, we can write the wave equation as,

$$y(x, t) = 2\cos(kx - \omega t) = 2\cos\left(\frac{4\pi}{15}x - \frac{16\pi}{15}t\right)$$

SPEED OF WAVES ON STRING

In this section, we will discuss how to find the speed of a transverse pulse travelling on taut string. The velocity of a wave travelling on a string depends on the elastic and the inertia properties of the string. When a part of the string gets disturbed, it exerts an extra force on the neighboring part because of the elastic property. The neighboring part responds to this force and the response depends on the inertia property. The elastic force in the string is measured by tension in the string is T and the inertia by its mass per unit length is μ.

Consider a pulse $y = f\left(t - \frac{x}{v}\right)$ is travelling on a taut string to the right with a uniform speed v measured relative to a stationary frame of reference. Let us choose a different inertial reference frame that moves along with the pulse with the same speed as the pulse so that the pulse is at rest within the frame. This change of reference frame is permitted because Newton's laws are valid in either a stationary frame, or one that moves with constant velocity.



Looking from this frame, the pattern of the string is at rest but the entire string is moving towards the negative x-direction with a speed v. If a crest is opposite to the observer at any instant, it will always remain opposite to him with the same shape while the string will pass through this crest in opposite direction like a snake.

Consider a small element of the string of length Δs at the highest point of a crest. Any small curve may be approximated by a circular arc. Suppose the small element forms an arc of radius R. The particles of the string in this element go in this circle with a speed v as the string slides through this part. The general situation is shown in Fig. (a) and the expanded view of the part near Δs is shown in Fig. (b).

We assume that the displacements are small so that the tension in the string does not appreciably change because of the disturbance. The element is pulled by the parts of the string to its right and to its left. Resultant force on this element is in the downward direction as shown in Fig. (b) and its magnitude is

$$F_r = T\sin\theta + T\sin\theta = 2T\sin\theta$$

As Δs is taken small, θ will be small and $\sin\theta \approx \frac{\Delta s/2}{R} = \frac{\Delta s}{2R}$

so that the resultant force on Δs is $F_r = 2T\left(\frac{\Delta s}{2R}\right) = \frac{T\Delta s}{R}$

If μ be the mass per unit length of the string, element has a mass $\Delta m = \mu\,\Delta s$. Its downward acceleration is

$$a = \frac{F_r}{\Delta m} = \frac{T\Delta s/R}{\mu\,\Delta s} = \frac{F}{\mu R}$$

But the element is moving in a circle of radius R with a constant speed v. Its acceleration is, therefore, $a = \frac{v^2}{R}$. The above equation becomes

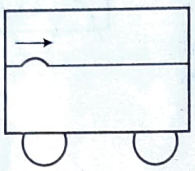
$$\frac{v^2}{R} = \frac{T}{\mu R}$$

or $v = \sqrt{\frac{T}{\mu}}$ ...(i)

The velocity of the wave on a string thus depends only on the tension T and the linear mass density μ. We have used the approximation that the tension T remains almost unchanged as the part of the string vibrates up and down. This approximation is valid only for small amplitudes because as the string vibrates, the lengths of its parts change during the course of vibration and hence, the tension changes.

ILLUSTRATION 6.25

A taut string having tension 100 N and linear mass density 0.25 kg/m is used inside a cart to generate a wave pulse starting at the left end, as shown. What should be the velocity of the cart so that pulse remains stationary w.r.t. ground.



Sol. The expression $v = \sqrt{\frac{T}{\mu}}$ gives the velocity of the wave in the string. Here the string is moving with cart.

Velocity of pulse $v = \sqrt{T/\mu} = 20\,\text{m/s}$

This velocity is w.r.t. car.

Velocity of the pulse can be given as

Now $\vec{v}_P = \vec{v}_{PC} + \vec{v}_{CG}$

$0 = 20i + \vec{v}_{CG}$

$\vec{v}_{CG} = -20i\ \text{m/s}$

Hence, the cart should move with velocity 20 m/s in the direction opposite to the pulse.

ILLUSTRATION 6.26

Transverse waves travel with a speed of 20.0 m/s in a string under a tension of 6.00 N. What tension is required for a wave speed of 30.0 m/s in the same string?

Sol. The two wave speeds can be written as:

$$v_1 = \sqrt{\frac{T_1}{\mu}} \quad \text{and} \quad v_2 = \sqrt{\frac{T_2}{\mu}}$$

Dividing the equations $\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$ so that $T_2 = \left(\frac{v_2}{v_1}\right)^2 T_1$

$$T_1 = \left(\frac{30.0}{20.0}\right)^2 (6.00\ \text{N}) = 13.5\ \text{N}$$

ILLUSTRATION 6.27

A rope of total mass m and length L is suspended vertically. Find the time taken by a transverse pulse travels the length of the rope.

Sol. The tension will increase as elevation x increases, because the upper part of the hanging rope must support the weight of the lower part. Then the wave speed increase with height.

We define $x = 0$ at the bottom of the rope and $x = L$ at the top of the rope. The tension in the rope at any point is the weight of the rope below that point. We can thus write the tension in the rope at each point x as $T = \mu x g$, where μ is the mass per unit length of the rope, which we assume uniform. The speed of the wave pulse at each point along the rope's length is, therefore,

$$v = \sqrt{\frac{T}{\mu}} \quad \text{or} \quad v = \sqrt{gx}$$

But at each point, the wave propagates at the rate of $v = \frac{dx}{dt}$

So we substitute for v and generate the differential equation:

$$\frac{dx}{dt} = \sqrt{gx} \quad \text{or} \quad dt = \frac{dx}{\sqrt{gx}}$$

Integrating both sides, $\Delta t = \frac{1}{\sqrt{g}} \int_0^L \frac{dx}{\sqrt{x}} = \frac{\left[2\sqrt{x}\right]_0^L}{\sqrt{g}} = 2\sqrt{\frac{L}{g}}$

What you have learned in mathematic class is another operation that can be done to both sides of an equation: Integrating from one physical point to another. Here the integral extends in time and space from the situation of the pulse being at the bottom on the string to the situation with the pulse reaching the top.

ILLUSTRATION 6.28

An aluminium wire is clamped at each end and under zero tension at room temperature. Reducing the temperature, which results in a decrease in the wire's equilibrium length, increases the tension in the wire. What strain ($\Delta L/L$) results in a transverse wave speed of 100 m/s? Take the cross-sectional area of the wire to be 5.00×10^{-6} m^2, the density to be 2.70×10^3 kg/m^3, and Young's modulus to be 7.00×10^{10} N/m^2.

Sol. We must review the relationship of strain to stress, and algebraically combine it with the equation for the speed of a string wave.

The expression for the elastic modulus $Y = \frac{F/A}{\Delta L/L}$

Becomes an equation for strain $\frac{\Delta L}{L} = \frac{F/A}{Y}$ ...(i)

We substitute into the equation for the wave speed

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{F}{\mu}}$$

where tension T means the same as stretching force F.

$$\mu = \frac{m}{L} = \frac{\rho(AL)}{L} = \rho A$$

We substitute and, solve,

$$v^2 = \frac{F}{\mu} = \frac{1}{\rho}\left(\frac{F}{A}\right) \quad \text{or} \quad \left(\frac{F}{A}\right) = \rho v^2 \qquad \text{...(ii)}$$

and substitute (ii) and (i) to obtain

$$\frac{\Delta L}{L} = \frac{\rho v^2}{Y} = \frac{(2.70 \times 10^3\ \text{kg/m}^3)(100\ \text{m/s})^2}{7.00 \times 10^{10}\ \text{N/m}^2} = 3.86 \times 10^{-4}$$

The Young's modulus equation describes stretching that does not really happen, being caused and cancelled out simultaneously by changes in tension and temperature. This wire could be used as a thermometer. Pluck it, pour liquid nitrogen onto it, pluck it again, and listen to the frequency go up.

ILLUSTRATION 6.29

A transverse wave of wavelength 50 cm is travelling towards +ve x-axis along a string whose linear density is 0.05 g/cm. The tension in the string is 450 N. At $t = 0$, the particle at $x = 0$ is passing through its mean position with an upward velocity. Form an equation describing the wave. The amplitude of the wave is 2.5 cm.

Sol. Let the wave be described by:

$$y(x, t) = A\sin(kx - \omega t + \phi_0) \quad \text{and} \quad A = 2.5 \text{ cm}$$

where $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{0.5} = 4\pi$

$$y(0, 0) = 0 \Rightarrow A\sin\phi_0 = 0$$

$$\phi_0 = 0, \pi \qquad \ldots(i)$$

we also have $\left(\dfrac{dy}{dt}\right)_{0,0} > 0 \Rightarrow -A\omega\cos\phi_0 > 0 \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get $\phi_0 = \pi$

Velocity of transverse wave in the string is given by

$$\sqrt{\frac{T}{\mu}} = \sqrt{\frac{450}{0.05\times10^{-1}}} = 300 \text{ m/s}$$

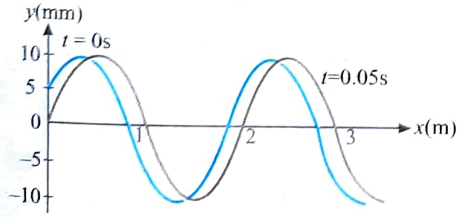
$$\omega = 2\pi f = \frac{2\pi c}{\lambda} = \frac{2\pi(300)}{0.5} = 1200\pi \text{ rad/s}$$

Using all the quantities, the equation is

$$y = 2.5\sin(4\pi x - 1200\pi t + \pi) \text{ cm}$$

ILLUSTRATION 6.30

Figure represents two snaps of a travelling wave on a string of mass per unit length $\mu = 0.25$ kg/m. The first snap is taken at $t = 0$ and the second is taken at $t = 0.05$ s. Determine **(a)** the speed of the wave, **(b)** the wavelength and frequency of the wave, **(c)** the maximum speed of the particle, **(d)** the tension in the string **(e)** the equation of the wave.



Sol. The wave is travelling along the positive x-axis can be given by equation.

$$y = A\sin[kx - \omega t + \phi]$$

at $x = 0$, $y = A\sin[-\omega t + \phi]$

also, at $t = 0$, $y = A\sin\phi = \dfrac{A}{2}$

$$\ldots(i)$$

or $\sin\phi = \dfrac{1}{2} \Rightarrow \phi = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \ldots$

As at $t = 0$ and $x = 0$,

$$\frac{dy}{dx} > 0 \text{ hence } \phi = \frac{\pi}{6}$$

and at $t = 0.05$ s

$$y = A\sin\left(\frac{-\omega}{20} + \phi\right) = 0$$

As at $x = 0$, $\dfrac{dy}{dx} > 0$

or $\dfrac{-\omega}{20} + \phi = 0, \pi, 2\pi, \ldots \qquad \ldots(ii)$

$$\phi - \frac{\omega}{20} = 0 \Rightarrow \omega = -\frac{10\pi}{3}$$

$$\therefore \quad f = \frac{\omega}{2\pi} = \frac{5}{3} \text{ Hz}$$

Velocity of the wave is $v = \lambda f = 2\left(\dfrac{0.5}{3}\right) = \dfrac{10}{3}$ m/s and maximum velocity of the particle is

$$v_{max} = \omega A = \left(\frac{10}{3}\pi\right)(10\times10^{-3}) = \frac{\pi}{30} \text{ m/s}$$

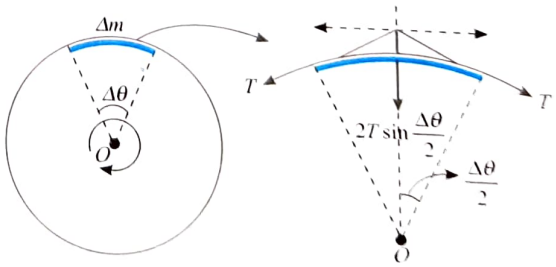
Tension in the string is $T = \mu v^2 = (0.25)\left(\dfrac{10}{3}\right)^2 = \dfrac{25}{3}$ N

The equation of the wave is $y = 10\sin\left[\pi x - \dfrac{10}{3}\pi t + \dfrac{\pi}{6}\right]$

ILLUSTRATION 6.31

A circular loop of string rotates about its axis on a frictionless horizontal plane at a uniform rate so that the tangential speed of any particle of the string is v. If a small transverse disturbance is produced at a point of the loop, with what speed (relative to the string) will this disturbance travel on the string?

Sol. Tension in loop will be developed due to rotation of the loop. Let us consider an elemental arc on the rotating loop and draw the tension force (T) on it as shown in figure. Resolving the forces along the radial direction,



$$T\cos\left(90° - \frac{\Delta\theta}{2}\right) + T\cos\left(90° - \frac{\Delta\theta}{2}\right) = (\Delta m)\left(\frac{v^2}{R}\right)$$

or $2T\sin\dfrac{\Delta\theta}{2} = (\Delta m)\left(\dfrac{v^2}{R}\right)$

The length of the elemental arc is $R\Delta\theta$. As the total mass of the string is m, the mass of this arc will be

$$\Delta m = \frac{m}{2\pi R} R\Delta\theta = \frac{m\Delta\theta}{2\pi}$$

Putting Δm in Eq. (i), $2T\sin\frac{\Delta\theta}{2} = \frac{m}{2\pi}\Delta\theta\left(\frac{v^2}{R}\right)$

or $$T = \frac{mv^2}{2\pi R}\frac{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)}$$

As $\Delta\theta$ is very small, $\frac{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)} = 1$ and $T = \frac{mv^2}{2\pi R}$

Mass per unit length of the string is $\mu = \left(\frac{m}{2\pi R}\right)$

$\therefore$ The speed of the disturbance $= \sqrt{\frac{T}{\mu}} = \sqrt{\frac{mv^2/2\pi R}{m/2\pi R}} = v$

CONCEPT APPLICATION EXERCISE 6.4

1. The equation of transverse wave travelling on a rope is given by

$y = 5\sin(4.0t - 0.02x)$

where y and x are in centimetres and time t in seconds

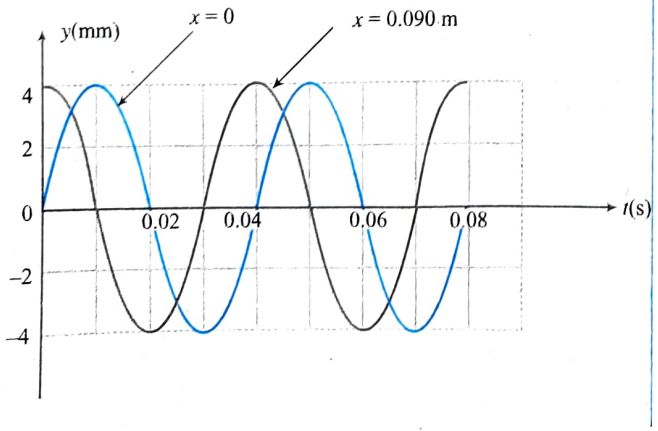
(a) Calculate the amplitude, frequency, velocity, and wavelength.

(b) Calculate the maximum transverse speed of a particle in the rope.

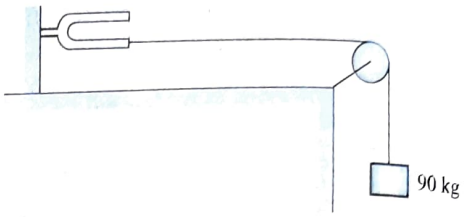
2. The following equation gives the displacement y at time t for a particle at a distance x:

$y = 0.01\sin 500\,\pi(t - x/30)$ where all are in SI units. Find **(a)** the wavelength, **(b)** the speed of the wave, **(c)** the velocity amplitude of the particles of the medium and **(d)** the acceleration amplitude of the particles of the medium.

3. A sinusoidal wave is propagating along a stretched string that lies along the x-axis. The displacement of the string as a function of time is graphed in figure for particles at $x = 0$ and at $x = 0.0900$ m. **(a)** What is the amplitude of the wave? **(b)** What is the period of the wave? **(c)** You are told that the two points $x = 0$ and $x = 0.0900$ m are within one wavelength of each other. If the wave is moving in the $+x$-direction, determine the wavelength and the wave speed.

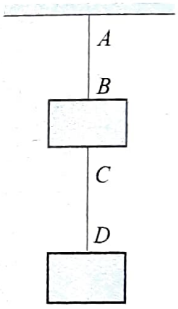


4. One end of a long string of linear mass density 10^{-2} kg m^{-1} is connected to an electrically driven tuning fork of frequency 150 Hz. The other end passes over a pulley and is tied to a pan containing a mass of 90 kg. The pulley end absorbs all the incoming energy so that reflected waves from this end have negligible amplitude. At $t = 0$, the left end (fork end) of the string is at $x = 0$ has a transverse displacement of 2.5 cm and is moving along positive y-direction. The amplitude of the wave is 5 cm. Write down the transverse displacement y (in centimetres) as function of x (in metres) and t (in seconds) that describes the wave on the string.

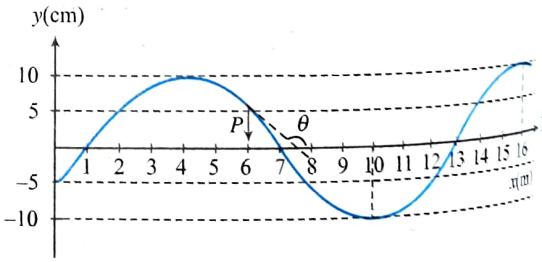


5. A 4.0 kg block is suspended from the ceiling of an elevator through a string having a linear mass density of 19.2×10^{-3} kg/m. Find the speed (with respect to the string) with which a wave pulse can proceed on the string if the elevator accelerate up at the rate of 2.0 m/s^2.

6. Two blocks each having a mass of 3.2 kg are connected by a wire CD and the system is suspended from the ceiling by another wire AB. The linear mass density of the wire AB is 10 g/m and that of CD is 8 g/m. Find the speed of a transverse wave pulse produced in AB and CD.



7. Figure shows the position of a medium particles at $t = 0$, supporting a simple harmonic wave travelling either along or opposite to the positive x-axis.



(a) Write down the equation of the curve.

(b) Find the angle θ made by the tangent at point P with the x-axis.

(c) If the particle at P has a velocity v_p m/s, in the negative y-direction, as shown in figure, then determine the speed and direction of the wave.

(d) Find the frequency of the waves.

(e) Find the displacement equation of the particle at the origin as a function of time.

(f) Find the displacement equation of the wave.

8. The equation of a progressive wave travelling along a string is given by
$y = 10 \sin\pi(0.01x - 2.00t)$
where x and y are in centimetres and t in seconds. Find the **(a)** velocity of a particle at $x = 2$ m and $t = 5/6$ s. **(b)** acceleration of a particle at $x = 1$ m and $t = 1/4$ s. Also, find the velocity amplitude and acceleration amplitude for the wave.

9. Write down the wave equation for a plane progressive wave of amplitude 5 cm, wavelength 2 m and speed 330 ms^{-1}. If the wave is moving in positive x-direction and at $t = 0$ and $x = 0$: $y = 0$ and $dy/dt > 0$.

10. A simple harmonic wave travelling along the positive x-axis having amplitude of 7 cm, velocity 80 cm s^{-1} and frequency 50 vibration per second. The wave travelling along the positive x-axis and at $t = 0$ and $x = 0$; $y = 0$ and $dy/dt > 0$. Calculate the displacement, velocity and acceleration of a medium particle at a position $x = 399$ cm at the instant $t = 5$ s.

ANSWERS

1. (a) 5cm, $\frac{2}{\pi}s^{-1}$, 200 cm/s and 100 π cm (b) 20 cm/s

2. (a) 0.12 m (b) 30 m/s (c) 15.7 m/s (d) 2.47×10^4 m/s^2

3. (a) 4 mm (b) 0.04 s (c) 0.12 m, 3 m/s

4. $y = 5\sin\left\{\pi(300t - x) + \frac{\pi}{6}\right\}$ **5.** 50 m/s

6. 80 m/s, 63.24 m/s

7. (a) $y = 0.10(\text{m})\sin\left(\frac{\pi}{6}x - \frac{\pi}{6}\right)$ (b) $\tan^{-1}\left(-\frac{\pi\sqrt{3}}{120}\right)$
(c) $\frac{-40\sqrt{3}}{\pi}v_P$ along negative x-axis. (d) $\frac{10\sqrt{3}}{3\pi}v_P s^{-1}$
(e) $y = 10(\text{cm})\sin\left[\frac{20\sqrt{3}}{3}v_P t - \frac{\pi}{6}\right]$
(f) $y = 0.1(\text{m})\sin\left[\frac{\pi}{6}x(\text{m}) + \frac{20\sqrt{3}}{3}v_P t - \frac{\pi}{6}\right]$

8. (a) $-10.0\,\pi$ cm/s, velocity amplitude $= 20.0\,\pi$ cm/s
(b) $-40.0\,\pi^2$ cm/s^2, acceleration amplitude $= 40.0\,\pi^2$ cm/s^2

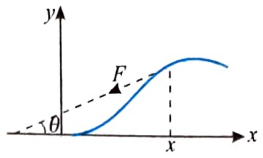
9. $y = 5\text{cm}\sin 2\pi\left[165t - \frac{x}{200}\right]$

10. $\frac{-7\sqrt{2}}{2}$cm, $-350\pi\sqrt{2}\,\text{cms}^{-1}$, $35000\pi^2\sqrt{2}\,\text{cms}^{-2}$

POWER TRANSMITTED ALONG THE STRING BY A SINE WAVE

Consider a sinusoidal wave travelling on a string. The source of the energy is some external agent at the left end of the string, which does work in producing the oscillations. Let us consider a harmonic wave travelling along a stretched string in positive x-direction. The equation for the displacement in y-direction is

$$y = A\sin\omega\left(t - \frac{x}{v}\right) \quad \text{... (i)}$$



Now consider a portion of the string at a time t to the right of position x as shown in figure. The string on the left of the point x exerts a force F on this part. The direction of this force is along the tangent to the string at position x. The component of the force along the Y-axis is

$$F_y = -F\sin\theta \approx -F\tan\theta = -F\frac{\partial y}{\partial x}$$

The power delivered by the force F to the string on the right of position x is, therefore,

$$P = \left(-F\frac{\partial y}{\partial x}\right)\frac{\partial y}{\partial t}$$

By (i), it is $P = -F\left[\left(-\frac{\omega}{v}\right)A\cos\omega\left(t - \frac{x}{v}\right)\right]\left[\omega A\cos\omega\left(t - \frac{x}{v}\right)\right]$

$$\Rightarrow \quad P = \frac{\omega^2 A^2 F}{v}\cos^2\omega\left(t - \frac{x}{v}\right)$$

This is the rate at which energy is being transmitted from left to right across the point at x. The $\cos^2$ term oscillates between 0 and 1 during a cycle and its average value is 1/2. The average power transmitted across any point is, therefore,

$$P_{av} = \frac{1}{2}\frac{\omega^2 A^2 F}{v} \quad \text{...(ii)}$$

Here F is nothing but tension in string. The wave velocity depends upon tension T and the linear mass density μ of the string, we define wave velocity as $v = \sqrt{\frac{T}{\mu}}$

Hence, $T = F = v^2\mu$ ...(iii)

From (ii) and (iii) we get

$$P_{av} = \frac{1}{2}\frac{(2\pi f)^2 A^2 (v^2\mu)}{v} = 2\pi^2\mu v A^2 f^2 \quad \text{...(iv)}$$

Hence the power transmitted along the string is proportional to the square of the amplitude and square of the frequency of the wave.

POWER OF A WAVE

Equation (iv) shows that the rate of energy transfer by a sinusoidal wave on a string is proportional to (a) the square of the frequency, (b) the square of the amplitude and (c) the wave speed. In fact, the rate of energy transfer in any sinusoidal wave is proportional to the square of the angular frequency and to the square of the amplitude.

INTENSITY OF THE WAVE

Flow of energy per unit area of cross section of the string in the unit time is known as the intensity of the wave. Thus,

$$I = \frac{\text{power}}{\text{area of cross section}} = \frac{(1/2)\mu\omega^2 A^2 v}{s}$$

$$= \frac{(1/2)\rho s\omega^2 A^2 v}{s} = \frac{1}{2}\rho\omega^2 A^2 v$$

This is the average intensity transmitted through the string.

ILLUSTRATION 6.32

Power supplied to a vibrating string: A string with linear mass density $\lambda = 5.00 \times 10^{-2}$ kg/m is under a tension of 80.0 N. How much power must be supplied to the string to generate sinusoidal waves at a frequency of 60.0 Hz and an amplitude of 6.00 cm?

Sol. The wave speed on the string is

$$v = \sqrt{\frac{T}{\mu}} = \left(\frac{80.0\,\text{N}}{5.00\times10^{-2}\,\text{kg/m}}\right)^{1/2} = 40.0\ \text{m/s}$$

Because $f = 60$ Hz, the angular frequency ω of the sinusoidal waves on the string has the value

$$\omega = 2\pi f = 2\pi(60.0\ \text{Hz}) = 377\ \text{s}^{-1}$$

Using these values in Eq. (iii) for the power, with amplitude $A = 6.00 \times 10^{-2}$ m, gives

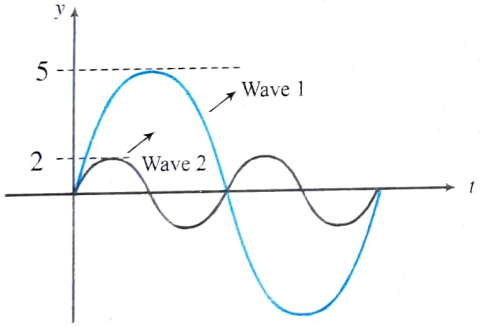
$$p = \frac{1}{2}\mu\omega^2 A^2 v$$

$$= \frac{1}{2}(5.00\times10^{-2}\text{kg/m})(377\text{s}^{-1})^2\times(6.00\times10^{-2}\text{m})^2(40.0\text{m/s})$$

$$= 512\ \text{W}$$

ILLUSTRATION 6.33

Two waves in the same medium are represented by y–t curves in the figure. Find the ratio of their average intensities?



Sol. $\frac{I_1}{I_2} = \frac{\omega_1^2 A_1^2}{\omega_2^2 A_2^2} = \frac{f_1^2 \times A_1^2}{f_2^2 \times A_2^2} = \frac{1\times25}{4\times4} = \frac{25}{16}$

ILLUSTRATION 6.34

Sinusoidal waves 5.00 cm in amplitude are to be transmitted along a string that has a linear mass density of 4.00×10^{-2} kg/m. The source can deliver a maximum power of 300 W and the string is under a tension of 100 N. What is the highest frequency at which the source can operate?

Sol. Turning up the source, frequency will increase the power carried by the wave. Perhaps on the order of a hundred hertz will be the frequency at which the energy per second is 300 J/s.

We will use the expression for power carried by a wave on a string.

The wave speed is $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{100\ \text{N}}{4.00\times10^{-2}\ \text{kg/m}}} = 50.0\ \text{m/s}$

From $P = \frac{1}{2}\mu\omega^2 A^2 v$, we have

$$\omega^2 = \frac{2P}{\mu A^2 v} = \frac{2(3.00\ \text{N m/s})}{(4.00\times10^{-2}\ \text{kg/m})(5.00\times10^{-2}\ \text{m})^2(50.0\ \text{m/s})}$$

Computing, $\omega = 346.4$ rad/s and $f = \frac{\omega}{2\pi} = 55.1$ Hz

This string wave would softly broadcast sound into the surrounding air, at the frequency of the second-lowest note called A on piano. If we tried to turn the source to a higher frequency, it might just vibrate with smaller amplitude. The power is generally proportional to the squares of both the frequency and the amplitude.

ILLUSTRATION 6.35

A sinusoidal wave on a string is described by the wave function

$$y = (0.15\ \text{m})\sin(0.80x - 50t)$$

where x and y are in metres and t is in seconds. The mass per unit length of this string is 12.0 g/m. Determine **(a)** the speed of the wave, **(b)** the wavelength, **(c)** the frequency and **(d)** the power transmitted to the wave.

Sol. Comparing the given wave function,

$$y = (0.15\ \text{m})\sin(0.80x - 50t)$$

with the general wave function, $y = A\sin(kx - \omega t)$

we have $k = 0.80$ rad/m and $\omega = 50$ rad/s and $A = 0.15$ m

(a) The wave speed is then

$$v = f\lambda = \frac{\omega}{k} = \frac{50.0\ \text{rad/s}}{0.80\ \text{rad/m}} = 62.5\ \text{m/s}$$

(b) The wavelength is $\lambda = \frac{2\pi}{k} = \frac{2\pi\ \text{rad}}{0.80\ \text{rad/m}} = 7.85\ \text{m}$

(c) The frequency is $f = \frac{\omega}{2\pi} = \frac{50\ \text{rad/s}}{2\pi\ \text{rad}} = 7.96\ \text{Hz}$

(d) The wave carries power

$$P = \frac{1}{2}\mu\omega^2 A^2 v$$

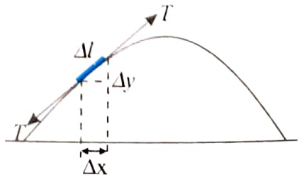
$$= \frac{1}{2}(0.0120\ \text{kg/m})(50.0\ \text{s}^{-1})^2(0.150\ \text{m})^2(62.5\ \text{m/s})$$

$$= 21.1\ \text{W}$$

ILLUSTRATION 6.36

A sinusoidal transverse wave having wave equation is $y = a\sin(kx - \omega t)$ is travelling on a stretched long string. The linear mass density (mass per unit length of the string) is μ. Considering the amplitude of the wave small, take a small element of length Δx on the string at $x = 0$, calculate the elastic potential energy stored in the element at time $t = 0$. Also find the kinetic energy of the element at $t = 0$.

Sol. Let us take an element of small length Δx. At some instant it is stretched to length Δl.



Extension of this element $\delta l = \Delta l - \Delta x = (\Delta x^2 + \Delta y^2)^{\frac{1}{2}} - \Delta x$

$$\delta l = \Delta x\left[1+\left(\frac{\Delta y}{\Delta x}\right)^2\right]^{1/2} - \Delta x \qquad \ldots(i)$$

The amplitude of the wave small, hence $\frac{\Delta y}{\Delta x}$ is small, we can write Eq. (i) as

$$\delta l = \Delta x\left[1+\frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2\right] - \Delta x$$

Extension $\delta l \simeq \frac{1}{2}\left(\frac{\Delta y}{\Delta x}\right)^2 \Delta x = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \Delta x$

Elastic PE = work done by tension in stretching = Tension × Extension

$$\Delta U = T\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \Delta x = \frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2 \Delta x \quad \left[\because v = \sqrt{\frac{T}{\mu}}\right]$$

$$\Delta U = \frac{1}{2}\mu\frac{\omega^2}{k^2}a^2k^2\cos^2(kx-\omega t)\Delta x$$

At $t = 0$, $\Delta U = \frac{1}{2}\mu\omega^2a^2\Delta x\cos^2(\omega t)$

At $t = 0$, $\Delta U = \frac{1}{2}\mu\omega^2a^2\Delta x$

And the kinetic energy,

$$\Delta K = \frac{1}{2}(\mu\Delta x)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu\Delta x a^2\omega^2\cos^2(kx-\omega t)$$

At $x = 0$; $t = 0$ $\Delta K = \frac{1}{2}\mu a^2\omega^2\Delta x$

ILLUSTRATION 6.37

A transverse harmonic wave is propagating along a taut string. Tension in the string is 50 N and its linear mass density is 0.02 kg m^{-1}. The string is driven by a 80 Hz oscillator tied to one end oscillating with an amplitude of 2 mm. The other end of the string is terminated so that all the wave energy is absorbed and there is no reflection

(a) Calculate the power of the oscillator.

(b) The tension in the string is quadrupled. What is new amplitude of the wave if the power of the oscillator remains same?

(c) Calculate the average energy of the wave on a 1.0 m long segment of the string.

Sol.

(a) Wave speed $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{50}{0.02}} = \sqrt{2500} = 50$ ms^{-1}

Power of oscillator = average power transmitted through a cross section, $P_{av} = 2\pi^2 f^2 a^2 \mu v$

$$P_{av} = 2\pi^2 f^2 a^2 \mu v = 2\pi^2 \times (80)^2 \times (2\times10^{-3})^2 \times 0.02 \times 50$$
$$= 0.512 \text{ W}$$

(b) As $v \propto \sqrt{T}$, it means if tension is made 4 times the wave speed becomes double.

And $P_{av} \propto a^2 v$, hence to keep the power same amplitude will become $\frac{1}{\sqrt{2}}$ times, i.e., $\sqrt{2}$ mm.

(c) Average energy on a segment of length L,

$E_{av} = P_{av} \times$ (time required for the wave to travel a distance L)

$$\Rightarrow E_{av} = P_{av}\frac{L}{v} = 0.512 \times \frac{1.0}{50} = 0.01024 \text{ J} = 10.24 \text{ mJ}$$

CONCEPT APPLICATION EXERCISE 6.5

1. A wave travels out in all directions from a point source. Justify the expression $y = (a_0/r)\sin k(r - vt)$, at a distance r from the source. Find the intensity of the wave.
2. For plane waves in the air of frequency 1000 Hz and the displacement amplitude 2×10^{-8} m, deduce **(a)** the velocity amplitude, and **(b)** the intensity. (Take $\rho = 1.3$ kg/m^3, $c = 340$ m/s)
3. A piano wire with mass 3.00 g and length 80.0 cm is stretched with a tension of 25.0 N. A wave with frequency 120.0 Hz and amplitude 1.6 mm travels along the wire. **(a)** Calculate the average power carried by the wave. **(b)** What happens to the average power if the wave amplitude is halved?
4. A steel wire has a mass of 50 g/m and is under tension 450 N.
   (a) Find the maximum average power that can be carried by the transverse wave in the wire if the amplitude is not to exceed 20% of the wavelength.
   (b) The change in maximum average power if the mass per unit length of the wire is doubled. (Use $\pi^2 \cong 10$)
5. Two travelling transverse waves each of frequency f amplitude A and phase difference ϕ superimpose to form another travelling wave. The resulting wave travels in a string of tension T and linear mass density μ. Find the average power of the wave in the string.

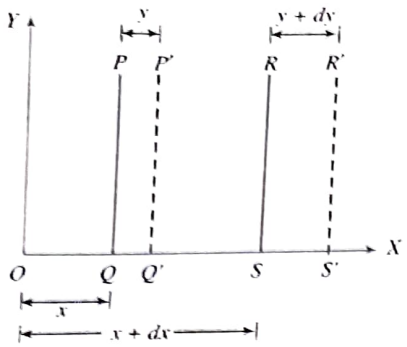
ANSWERS

1. $\frac{1}{2}\rho a_0^2k^2v^3/r^2$ 2. (a) 1.3×10^{-4} m/s (b) 3.7×10^{-6} W/m^2
3. (a) 0.223 W (b) 0.056 W
4. (a) $1080\sqrt{10}$ W (b)) Power becomes $\sqrt{2}$ times
5. $4\pi^2 f^2(1+\cos\theta)A^2\sqrt{T\mu}$

INTERPRETATION OF dy/dx IN LONGITUDINAL WAVES AND TRANSVERSE WAVES

Consider a longitudinal wave travelling along the positive x-axis. Let us choose two layers of medium PQ and RS with planes normal to the direction of wave propagation, at distance x and $x + dx$ from the origin O. After the wave passes through these sections, the layers get displaced from their mean positions, say by y and $y + dy$ respectively. (as shown in figure) if a be the

cross-sectional area of the layer, then, the initial volume of the cylinder $PQRS = a\,(dx)$



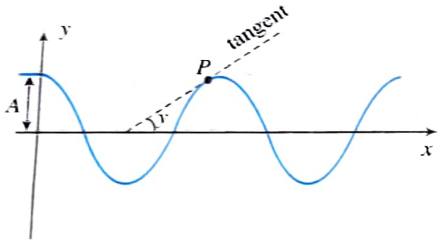
and final volume = volume of cylinder $P'Q'R'S'$

$$= a\,[x + dx + y + dy) - (x + y)] = a\,(dx + dy)$$

$\therefore$ Change in volume $= a\,dy$

$\therefore$ Volume strain $= \dfrac{\text{Change in volume}}{\text{Original volume}} = \dfrac{a\,dy}{a\,dx} = \dfrac{dy}{dx}$

Hence, in case of longitudinal waves dy/dx represents the volume strain.



In case of transverse waves, the term dy/dx has its usual geometrical meaning. If a 'snapshot' be taken, of the particles of a medium at any instant, in which a simple harmonic wave is travelling, then the shape of the waveform will be a sine curve (as shown in the figure) dy/dx, at the instant for any particle say P) represents the slope of the curve at that point. Thus, if ψ is be the angle made by the geometrical tangent drawn to the curve at that point, with the positive direction of the x-axis, then $dy/dx = \tan\psi$.

Solved Examples

EXAMPLE 6.1

A harmonic wave is travelling in a stationary medium whose equation is given by $y = A\sin(\omega t - kx)$. Find the equation of this wave w.r.t. a frame which is moving along –ve x-axis with a constant speed v w.r.t. stationary medium. Also, find speed of wave in moving frame.

Sol. Let in time t, wave gets displaced by x w.r.t. stationary medium, the displacement of wave w.r.t. moving frame is $x' = x + vt$.

So, equation of wave in moving frame is

$Y = A\sin[\omega t - k(x' - vt)]$ $\quad [\because\ x = x' - vt]$

$Y = A\sin[(\omega + kv)\,t - kx']$

Speed of wave in moving frame $V' = \dfrac{dx'}{dt} = \dfrac{\omega + kv}{k} = \left(\dfrac{\omega}{k} + v\right)$

EXAMPLE 6.2

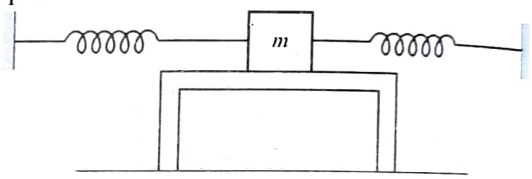
One end of each of two identical springs, each of force constant 0.5 N/m are attached on the opposite sides of wooden block of mass 0.01 kg. The other ends of the springs are connected to separate rigid supports such that the springs are unstretched and are collinear in a horizontal plane. To the wooden piece is fixed a pointer which touches a vertically moving plane paper. The wooden piece kept on a smooth horizontal table is now displaced by 0.02 m along the line of springs and released. If the speed of the paper, perpendicular to the springs' length, is 0.1 m/s, find the equation of the path traced by the pointer on the paper and the distance between two consecutive maxima on this path.

Sol. The effective force constant of the spring system is $2k$ (since they constitute a parallel combination). The angular frequency of simple harmonic oscillation is

$$\omega = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2\times 0.5}{0.01}} = 10\ \text{rad/s}$$

The amplitude $A = 0.02$ m

The speed of the paper may be assumed as the speed of wave-propagators and the curve traced on the paper can be represented by the equation



$$y = A\sin(\omega t - kx) \quad \text{...(i)}$$

The wavelength $\lambda = \dfrac{v}{f} = \dfrac{v}{\omega/2\pi} = \dfrac{2\pi v}{\omega} = \dfrac{2\pi\times 0.1}{10} = \dfrac{\pi}{50}$ m ...(ii)

$\therefore$ $\quad k = \dfrac{2\pi}{\lambda} = \dfrac{\omega}{v} = \dfrac{10}{0.1} = 100$

Substituting the value in (i), we get the required equation of the path

$$y = A\sin kx = 0.02\sin(100x)$$

EXAMPLE 6.3

A wave pulse starts propagating in the +x-direction along a non-uniform wire of length 10 m with mass per unit length given by $\mu = \mu_0 + ax$ and under a tension of 100 N. Find the time taken by a pulse to travel from the lighter end ($x = 0$) to the heavier end. ($\mu_0 = 10^{-2}$ kg/m and $a = 9\times 10^{-3}$ kg/m^2).

Sol. The speed of the wave pulse

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{\mu_0 + ax}} = \frac{dx}{dt}$$

$\therefore$ $\quad t = \displaystyle\int_0^t dt = \int_0^L \sqrt{\frac{\mu_0 + ax}{T}}\,dx$

$$= \frac{2}{3}\frac{1}{\alpha\sqrt{T}}[(\mu_0 + aL)^{3/2} - (\mu_0)^{3/2}]$$

Substituting the values, we get

$$t=\frac{2}{3}\times\frac{1}{9\times10^{-3}\times\sqrt{100}}[(10^{-2}+9\times10^{-3}\times10^{-2})^{3/2}-(10^{-2})^{3/2}]$$

$$=\frac{2\times100}{27}[(10)^{-3/2}-(10)^{-3}]=0.227\text{ s}$$

EXAMPLE 6.4

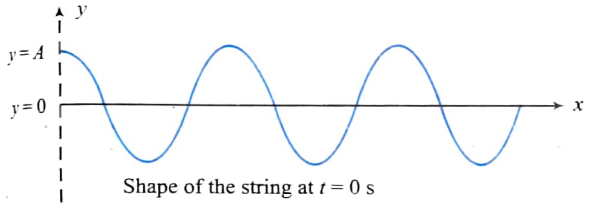
A 100 Hz sinusoidal wave is travelling in the positive x-direction along a string with a linear mass density of 3.5×10^{-3} kg/m and a tension of 35 N. At time $t=0$, the point $x=0$ has maximum displacement in the positive y-direction. Next, when this point has zero displacement the slope of the string is $\pi/20$. Find **(a)** amplitude **(b)** wave speed (iii) the expression which represents the displacement of string as a function of x (in metres) and t (in seconds).

Sol. The speed of wave in string is

$$V=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{35}{3.5\times10^{-3}}}=100\text{ m/s}$$

$$\lambda=\frac{v}{f}=\frac{100}{100}=1\text{ m}$$

The shape of the string at $t=0$ s is as shown in figure.



Shape of the string at $t=0$ s

Let us assume that $y=A\sin(\omega t-kx+\theta)$

putting $t=0$, $x=0$ and $y=+A$

gives $\theta=\pi/2$

also $\omega=2\pi f=2\pi\times100=200\pi$ rad/s

$$k=\frac{2\pi}{\lambda}=\frac{2\pi}{1}=2\pi/\text{m}$$

Hence $y=A\sin\left(200\pi t-2\pi x+\frac{\pi}{2}\right)$

It is given that $\frac{\partial y}{\partial x}=\frac{\pi}{20}$ at $t=\frac{T}{4}=\frac{1}{400}$ s $\left[\because T=\frac{1}{f}=\frac{1}{100}\text{s}\right]$

$$\left.\frac{\partial y}{\partial x}\right]_{t=\frac{1}{400}\text{s}}=A2\pi\cos\left(200t-2\pi x+\frac{\pi}{2}\right)=-2A\pi=\frac{\pi}{20}$$

$$A=\frac{1}{40}=0.025$$

(i) Amplitude $A=0.25$ m

(ii) Wave speed = 100 m/s

(iii) Wave equation is $y=0.025\sin\left(200\pi t-2\pi x+\frac{\pi}{2}\right)$

EXAMPLE 6.5

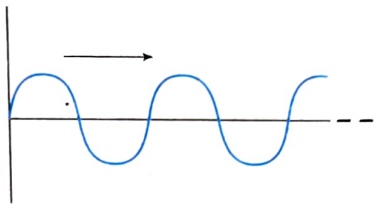
A harmonic wave has been set up on a very long string which travels along the length of the string. The wave has a frequency of 50 Hz, amplitude 1 cm and wavelength 0.5 m. Find

(a) the time taken by the wave to travel a distance of 8 m along the length of string

(b) the time taken by a point on the string to travel a distance of 8 m, once the wave has reached the point and sets it into motion

(c) also, consider the above case when the amplitude gets doubled

Sol. The wave is travelling along the length of string, while points on the string are moving perpendicular to the length of the string.



In one time period (cycle), the wave moves forward by one wavelength while the particle on string travels a distance of 4 times the amplitude. Here, time period.

$$T=\frac{1}{f}=\frac{1}{50}\text{ s}=0.02\text{ s}$$

And wave speed, $v=f\lambda=50\times0.5=25$ m/s

(a) Time taken by wave to travel a distance of 8 m, $t_1=8/25$ s

(b) Time taken by a point on string to travel a distance of 8 m.

$$t_2=\frac{8}{4A}\times T=\frac{8}{4\times10^{-2}}\times0.02=4\text{ s}$$

(c) When the amplitude gets doubled, the answer (c) for (a). does not change.

For (b), $t_2'=\frac{8}{4A'}\times T=\frac{8}{4\times2A}\times0.02=2$ s

EXAMPLE 6.6

A transverse harmonic wave of amplitude 0.01 m is generated at one end ($x=0$) of a long horizontal string by a tuning fork of frequency 500 Hz. At a given instant of time the displacement of the particle at $x=0.1$ m is –0.005 m and that of the particle at $x=0.2$ m is +0.005 m. Calculate the wavelength and the wave velocity. Obtain the equation of the wave assuming that the wave is travelling along the + x-direction and that the end $x=0$ is at the equilibrium position at $t=0$.

Sol. Since the wave is travelling along + x-direction and the displacement of the end $x=0$ is at time $t=0$, the general equation of this wave is

$$y(x,t)=A\sin\left\{\frac{2\pi}{\lambda}(vt-x_1)\right\}\qquad\text{...(i)}$$

where $A=0.01$ m

when $x=0.1$ m, $y=-0.005$ m

Putting all values, we get $-0.005=0.01\sin\left\{\frac{2\pi}{\lambda}(vt-x_1)\right\}$

where $x_1=0.1$m

or, $\sin\left\{\frac{2\pi}{\lambda}(vt-x_1)\right\}=-\frac{1}{2}$

$\therefore$ Phase $\phi_1 = \dfrac{2\pi}{\lambda}(vt - x_1) = \dfrac{7\pi}{6}$ ...(ii)

when $x = 0.2$ m, $y = +0.005$ m. Therefore, we have

$$+0.005 = 0.01\sin\left\{\frac{2\pi}{\lambda}(vt - x_2)\right\}$$

where $x_2 = 0.2$ m

$$\phi_2 = \frac{2\pi}{\lambda}(vt - x_2) = \frac{\pi}{6} \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), $\Delta\theta = \theta_1 - \theta_2 = \pi$

Now, $\Delta\phi = \dfrac{2\pi}{\lambda}\Delta x$

Thus, $\pi = -\dfrac{2\pi}{\lambda}(x_1 - x_2) = \dfrac{-2\pi}{\lambda}(0.1 - 0.2)$

or $\lambda = 0.2$ m

Now, frequency f of the wave = frequency of the tuning for $k = 500$ Hz.

Hence, wave velocity $v = f\lambda = 500 \times 0.2 = 100$ m/s

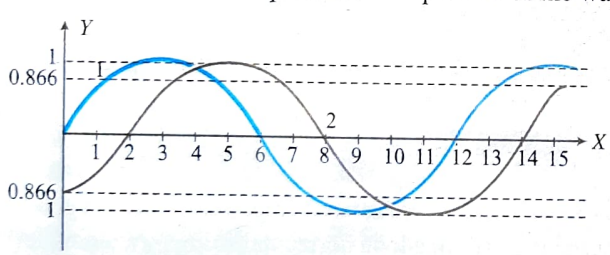
Substituting for A, λ and v in Eq. (i) we get

$$y(x, t) = 0.01\sin(10\pi(100t - x))$$

This is the equation of the wave where y and x are in metres and t in seconds.

EXAMPLE 6.7

Figure shows two snapshots of medium particles within a time interval of 1/60 s. Find the possible time periods of the wave



Sol. The wavelength of the wave

$\lambda = (12 - 0) = (14 - 2) = 12$ units

Therefore, in time 1/60 s, the distance moved by the wave is

$$d = 2 + n(\lambda) = \frac{\lambda}{6} + n\lambda;\ n \in J$$

Velocity of wave: $v = \dfrac{d}{(1/60)} = \left(\dfrac{\lambda}{6} + n\lambda\right)60$

$$T = \frac{\lambda}{v} = \frac{\lambda}{\left(\frac{\lambda}{6} + n\lambda\right)60} = \frac{1}{10(6n+1)}$$

EXAMPLE 6.8

A sinusoidal wave travelling in the positive direction on a stretched string has amplitude 2.0 cm, wavelength 1.0 m and wave velocity 5.0 m/s. At $x = 0$ and $t = 0$, it is given that $y = 0$ and $\dfrac{\partial y}{\partial t} < 0$. Find the wave function $y(x, t)$.

Sol. Let the equation of the wave moving towards positive x-direction, $y(x,t) = A\sin(kx - \omega t + \phi)$

Given: The amplitude given is $A = 2.0$ cm $= 0.02$ m and the wavelength $\lambda = 1.0$ m

Hence the wave number, $k = \dfrac{2\pi}{\lambda} = 2\pi$ m^{-1}

And angular frequency, $\omega = vk = 10\pi$ rad/s

$$y(x,t) = (0.02)\sin[2\pi(x - 5.0t) + \phi] \quad \text{...(i)}$$

and $$\frac{\partial y}{\partial t} = (0.02)\times(-10\pi)\cos[2\pi(x - 5.0t) + \phi] \quad \text{.....(ii)}$$

It is also given, for $x = 0$, $t = 0$, $y = 0$ and $\dfrac{\partial y}{\partial t} < 0$

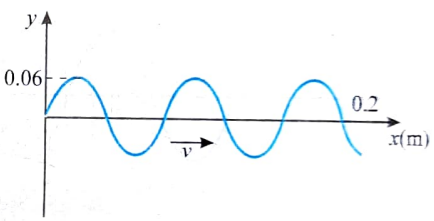
i.e., $0.02\sin\phi = 0$ (as $y = 0$) and $-0.2\pi\cos\phi < 0$

From these conditions, we may conclude that $\phi = 2n\pi$ where $n = 0, 1, 2, 3, ...$

Therefore, $y(x,t) = (0.02\,m)\sin[(2\pi m^{-1})x - (10\pi s^{-1})t]$m

EXAMPLE 6.9

For the wave shown in figure, write the equation of this wave if its position is shown at $t = 0$. Speed of wave is $v = 300$ m/s.



Sol. From given figure the amplitude, $A = 0.06$ m

And, $\dfrac{5}{2}\lambda = 0.2$ m $\therefore$ $\lambda = 0.08$ m

The frequency of wave, $f = \dfrac{v}{\lambda} = \dfrac{300}{0.08} = 3750$ Hz

Wave number, $k = \dfrac{2\pi}{\lambda} = 78.5$ m^{-1} and angular frequency,

$\omega = 2\pi f = 23562$ rad/s

From given figure, at $t = 0$, $x = 0$, and $\dfrac{dy}{dx}$ = positive

The given curve is a sine curve and wave travelling in positive x-direction. Hence, the equation should have the form,

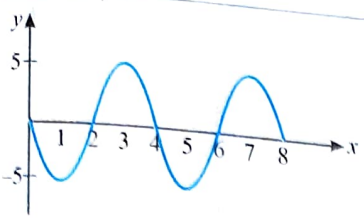
$$y(x,t) = A\sin(kx - \omega t)$$

Substituting the values, we have

$$y = (0.06\,m)\sin[(78.5\,m^{-1})x - (23562\,s^{-1})t]m$$

EXAMPLE 6.10

Figure shows the shape of a progressive wave at time $t = 0$. After a time of $t = 1/80$ sec, the particle at the origin has its maximum negative displacement. If the wave speed be 80 m/s, then what will be the equation of the progressive wave?



Sol. The wavelength $\lambda = 4$ m

Time period, $T = \frac{\lambda}{v} = \frac{4}{80} = \frac{1}{20}$ s

Since after $t = \frac{1}{80}$ s $= \frac{T}{4}$

In time $T/4$ the entire wave pattern would have shifted by a distance of $\lambda/4$. It is given that the particle at the origin is at its maximum negative, it means the entire wave pattern is moving to the left.

The wave moving in the negative x- axis direction, given as

$$y = A \sin\left[2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) + \phi\right]$$

We have initially $y = 0$ for $t = 0$, and $x = 0$

$\phi = 0, \pi, 2\pi$ etc

Also, $y = -A$; when $t = T/4$ and $x = 0$ ...(ii)

$$-A = A \sin\left[2\pi\left(\frac{1}{4}\right) + \phi\right] \Rightarrow \frac{\pi}{2} + \phi = \frac{3\pi}{2}, \frac{7\pi}{2}, \text{ etc}$$

Hence, $\phi = \pi, 3\pi, 2\pi$ etc.

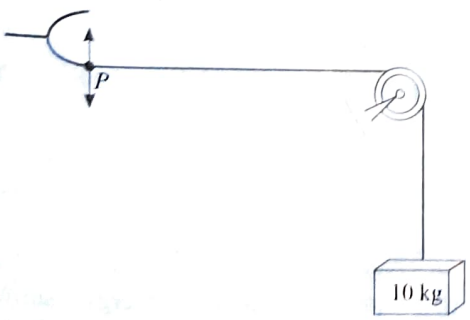
For Eqs. (i) and (ii), $\phi = \pi$ ....(iii)

Hence the equation of the progressive wave is,

$$y = 5 \sin\left[2\pi\left(\frac{t}{(1/20)} + \frac{x}{(4)} + \frac{1}{2}\right)\right]$$

EXAMPLE 6.11

A horizontal long taut string having linear mass density 0.1 kg/m have tension of 100 N. A tunning fork of frequency 100 Hz is connected to one end of the string. Find the: **(a)** wavelength of the transverse waves produced by the tunning fork in the string **(b)** wave equation $y = f(x,t)$ if the amplitude of vibration is 0.02 m, assuming that the particle of the string at a distance $\frac{1}{\sqrt{10}}$ m from the tunning fork at $t = 4$ s moves up with a speed $v_p = 2$ m/s.



Sol.

(a) The wave velocity in taut string is given by,

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{100}{0.1}} = \sqrt{1000} \text{ m/s} = 10\sqrt{10} \text{ m/s}$$

The wave length of wave in string,

$$\lambda = \frac{v}{f} = \frac{10\sqrt{10}}{100} = \frac{1}{\sqrt{10}} \text{m}$$

The angular wave number , $k = \frac{2\pi}{\lambda} = 2\sqrt{10}\,\pi$

The angular frequency, $\omega = 2\pi f = 2\pi \times 100 = 200\pi$

Let the equation of the wave is, $y = A\sin(\omega t - kx + \phi_0)$

$$y = 0.02 \sin(200\pi t - 2\sqrt{10}\pi x + \phi_0)$$

Particle velocity,

$$\frac{\partial y}{\partial t} = (200\pi)0.02 \cos(200\pi t - 2\sqrt{10}\pi x + \phi_0)$$

$$v_p = 4\pi \cos(200t - 2\sqrt{10}\pi x + \phi_0) \quad \ldots(i)$$

At $t = 4$ s, at $x = \frac{1}{\sqrt{10}}$ m the point moves up with a speed $v_p = 2\pi$ m/s

Then, substituting $v_p = +2\pi, x = \frac{1}{\sqrt{10}}$ m, $t = 4$ s in Eq. (i)

$$+2\pi = 4\pi \cos\left[200\pi \times 4 - 2\sqrt{10}\pi\left(\frac{1}{\sqrt{10}}\right) + \phi_0\right]$$

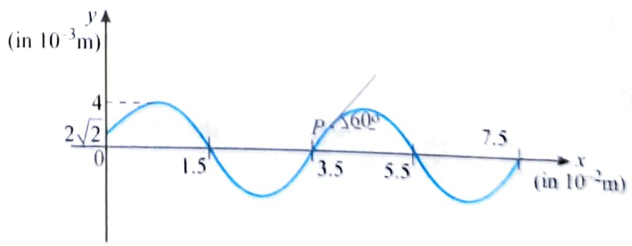
or $$\frac{1}{2} = \cos(798\pi + \phi) = \cos\phi_0 \Rightarrow \phi_0 = \frac{\pi}{3}$$

Hence, the equation of the wave should be

$$y = 0.02 \sin\left(200\pi t - 2\sqrt{10}x + \frac{\pi}{3}\right)$$

EXAMPLE 6.12

The figure shows a snap photograph of a vibrating string at $t = 0$. The particle P is observed moving up with velocity $20\sqrt{3}$ cm/s . The tangent at P makes an angle 60° with x-axis. Find



(a) the wave velocity. **(b)** the equation of wave.

Sol.

(a) At point P: Slope of tangent, $\frac{dy}{dx} = \tan 60° = \sqrt{3}$

Particle velocity, $v_p = -v\frac{dy}{dx} \Rightarrow 20\sqrt{3} = -v\sqrt{3}$

$\Rightarrow \quad v = -20\,\text{cm/s} = -\frac{1}{5}\,\text{m/s}$

Hence the move in travelling in negative x direction with velocity 20 cm/s.

(b) From given graph, amplitude $A = 4\times10^{-3}\,\text{m}$

And wave length $\lambda = (5.5-1.5) = 4\times10^{-2}\,\text{m}$

$$\text{Wave number } k = \frac{2\pi}{\lambda} = \frac{2\pi}{4\times10^{-2}} = 50\pi\ \frac{1}{m}$$

and $\quad \text{Angular frequency } \omega = kv = 50\pi\times\frac{1}{5} = 10\pi\,\text{s}^{-1}$

Hence equation can be written as, $y = A\sin(\omega t + kx + \phi)$

$\Rightarrow \quad y = (4\times10^{-3})\sin(10\pi t + 50\pi x + \phi) \qquad \ldots(i)$

When $y = 2\sqrt{2}\times10^{-3} \Rightarrow 2\sqrt{2}\times10^{-3} = 4\times10^{-3}\sin(\phi)$

$\Rightarrow \quad \sin\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}, \frac{3\pi}{4}$

From graph we can observe, at $t = 0$, $x = 0$, $\frac{dy}{dx} < 0$

hence, $\phi = \frac{\pi}{4}$

It means the required equation should be,

$$y = (4\times10^{-3})\sin\left(10\pi t + 50\pi x + \frac{\pi}{4}\right)$$

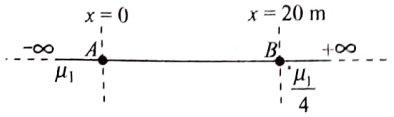
EXAMPLE 6.13

A non-uniform wire under constant tension has a variable linear mass density. Mass density is uniform for $-\infty \le x \le 0$. In this region, a transverse wave has the form $y = 0.003\cos(25x - 50t)$ where x and y are in meter and t in second. From $x = 0$ to $x = 20$ m the linear mass density decreases gradually from μ_1 to $\mu_1/4$. For $20\ m \le x \le \infty$, the linear mass density is $\mu = \frac{\mu_1}{4}$. Assume that variation in linear mass density is gradual so that an incident wave is transmitted without reflection. **(a)** Find the wave velocity for large values of x. **(b)** Find the amplitude of the wave for large values of x. **(c)** Give the wave function for $20\,m \le x \le \infty$.

Sol.

(a) For the region $-\infty \le x \le 0$ the wave function, $y = 0.003\cos(25x - 50t)$

which gives $k = \frac{2\pi}{\lambda} = 25\ \text{m}^{-1}$ and $\omega = 2\pi f = 50\ \text{s}^{-1}$



$$\text{Wave velocity } v = f\lambda = \frac{50}{2\pi}\times\frac{2\pi}{25} = 2\ \text{m/s}$$

If T is the tension in string, velocity of wave at A,

$$v_A = \sqrt{\frac{T}{\mu_1}} = 2\ \text{m/s}$$

Wave velocity at B, $v_B = \sqrt{\frac{T}{\mu_1/4}} = 2\sqrt{\frac{T}{\mu_1}} = 4\ \text{m/s}$

(b) Since density in the region $20\,\text{m} \le x \le \infty$ is constant the velocity in the entire region will be constant and equal to 4 m/s. As the wave is transmitted without reflection at junction at points A and B, we can assume the intensity to remain constant.

$$I = \frac{1}{2}\mu_1 v_1\omega^2 A_{in}^2 = \frac{1}{2}\left(\frac{\mu_1}{4}\right)v_2\omega^2 A_t^2$$

$$A_t^2 = 4A_{in}^2\frac{v_1}{v_2} \Rightarrow A_t^2 = 2A_{in}^2$$

Hence, $A_t = \sqrt{2}\,(0.003) = 0.0042\,\text{m}$

(c) For $20\,\text{m} \le x \le \infty$, frequency will remain unchanged hence ω is unchanged therefore k will be halved so that speed becomes twice.

Also, A = 0.0042 m and wave number, $k = \frac{2\ \text{s}}{2} = 12.5\ \text{m}^{-1}$ and angular frequency, ω = 50 rad/s

Therefore, the wave function for

$$y(x,t) = 0.0042\cos(12.5x - 50t)$$

EXAMPLE 6.14

A sinusoidal wave travels along a taut string of linear mass density 0.1 g/cm. The particles oscillate along y-direction and the disturbance moves in the positive x-direction. The amplitude and frequency of oscillation are 2 mm and 50 Hz respectively. The minimum distance between two particles oscillating in the same phase is 4.0 m.

(a) Find the tension in the string.

(b) Find the amount of energy transferred through any point of the string in one second.

(c) If it is observed that the particle at x = 2 m is at y = 1 mm at t = 2 s, and its velocity is in positive y-direction, then write the equation of this travelling wave.

Sol.

(a) The minimum distance between two particles oscillating in the same phase is 4.0 m, hence the wavelength of the wave should be 4.0 m.

Velocity of the wave, $\therefore\ v = f' = 200$ m/s

As $v = \sqrt{\frac{T}{\mu}} \Rightarrow T = \propto v^2 = (0.1)\times(200)^2 = 400$ N

(b) Number of waves that will cross a point is 1 second is 50, therefore power transmitted can be written as average power (If a fractional number of waves were crossing a point in a given interval then we cannot multiply average power with time to get the total energy transmitted)

Average power transmitted through string,

$$P_{av} = 2\pi^2 f^2 A^2 \mu v$$

$$\Rightarrow P_{av} = 2\times\pi^2\times(50)^2\times(2\times10^{-3})^2\times(0.01)\times200 = \frac{\pi^2}{25}\text{J}$$

Hence the amount of energy transferred through any point of the string in one second,

$$E = P_{av}\times t = \frac{\pi^2}{25}\times1 = \frac{\pi^2}{25}J$$

(b) The equation of wave is $y = A\sin(kx - \omega t + \phi_0)$

The wave number, $K = \frac{2\pi}{\lambda} = \frac{\pi}{2}m^{-1}$, angular frequency, $\omega = 2\pi f = 100\pi\ s^{-1}$ and amplitude $A = 2$ mm

At $x = 2$ m and $t = 2$ s; $y = 1$ mm

$\therefore\ 1 = 2\sin(\pi - 200\pi + \phi_0)$ on solving we get, $\phi_0 = -\frac{\pi}{6}$

Hence equation of wave $y = 2\sin\left(\frac{\pi x}{2} - 100\pi t - \frac{\pi}{6}\right)$

Exercises

Single Correct Answer Type

1. The equation of a wave travelling on a string is

$$y = 4\sin\frac{\pi}{2}\left(8t - \frac{x}{8}\right)$$

if x and y are in centimetres, then velocity of wave is

(1) 64 cm/s in –ve x-direction

(2) 32 cm/s in –ve x-direction

(3) 32 cm/s in +ve x-direction

(4) 64 cm/s in +ve x-direction

2. The amplitude of a wave disturbance propagating in the positive x-direction is given by

$$y = \frac{1}{1+x^2} \text{ at } t = 0 \text{ and } y = \frac{1}{[1+(x-1)^2]} \text{ at } t = 2\text{ s}$$

The wave speed is

(1) 1 m/s (2) 1.5 m/s

(3) 0.5 m/s (4) 2 m/s

3. A sinusoidal wave is generated by moving the end of a string up and down, periodically. The generator must apply the energy at maximum rate when the end of the string attached to generator has ..... X..... and least power when the end of the string attached to generator has ..... Y..... . The most suitable option which correctly fills blanks X and Y, is

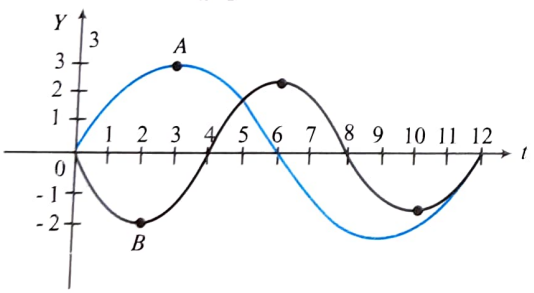
(1) maximum displacement, least acceleration

(2) maximum displacement, maximum acceleration

(3) least displacement, maximum acceleration

(4) least displacement, least acceleration

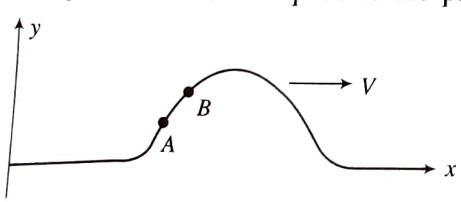
4. The displacement *vs* time graph for two waves A and B which travel along the same string are shown in the figure. Their intensity ratio I_A/I_B is



(1) $\frac{9}{4}$ (2) 1

(3) $\frac{81}{16}$ (4) $\frac{3}{2}$

5. A wave pulse is generated in a string that lies along x-axis. At the points A and B, as shown in figure, if R_A and R_B are ratios of magnitudes of wave speed to the particle speed, then



(1) $R_A > R_B$

(2) $R_B > R_A$

(3) $R_B = R_A$

(4) Information is not sufficient

6. The speed of a wave in a certain medium is 960 m/s. If 3600 waves pass over a certain point of the medium in 1 min, the wavelength is

(1) 2 m (2) 4 m

(3) 8 m (4) 16 m

7. A transverse wave is described by the equation

$$y = y_0 \sin 2\pi\left(ft - \frac{x}{\lambda}\right)$$

The maximum particle velocity is four times the wave velocity if

(1) $\lambda = \frac{\pi y_0}{4}$ (2) $\lambda = \frac{\pi y_0}{2}$

(3) $\lambda = \pi y_0$ (4) $\lambda = 2\pi y_0$

8. A simple harmonic progressive wave is represented by the equation $y = 8 \sin 2\pi\,(0.1x - 2t)$ where x and y are in centimetres and t is in seconds. At any instant the phase difference between two particles separated by 2.0 cm along the x-direction is

(1) 18° (2) 36°

(3) 54° (4) 72°

9. The equation of a transverse wave travelling on a rope is given by $y = 10 \sin \pi(0.01x - 2.00t)$ where y and x are in centimetres and t in seconds. The maximum transverse speed of a particle in the rope is about

(1) 63 cm/s (2) 75 cm/s

(3) 100 cm/s (4) 121 cm/s

10. A simple harmonic wave is represented by the relation

$$y(x,t) = a_0 \sin 2\pi\left(\nu t - \frac{x}{\lambda}\right)$$

If the maximum particle velocity is three times the wave velocity, the wavelength λ of the wave is

(1) $\pi a_0/3$ (2) $2\pi a_0/3$

(3) πa_0 (4) $\pi\, a_0/2$

11. The equation of a travelling wave is

$$y = 60\cos(1800t - 6x)$$

where y is in microns, t in seconds and x in metres. The ratio of maximum particle velocity to velocity of wave propagation is

(1) 3.6 (2) 3.6×10^{-6}

(3) 36×10^{-11} (4) 3.6×10^{-4}

12. The equation of a progressive wave is

$$y = 0.02\sin 2\pi\left[\frac{t}{0.01} - \frac{x}{0.30}\right]$$

here x and y are in metres and t is in seconds. The velocity of propagation of the wave is

(1) 300 m/s (2) 30 m/s

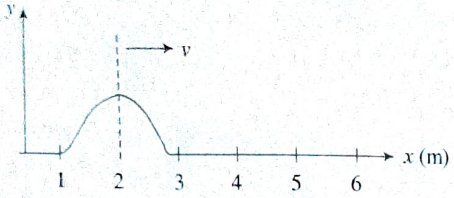
(3) 400 m/s (4) 40 m/s

13. In a medium in which a transverse progressive wave is travelling, the phase difference between two points with a separation of 1.25 cm is $(\pi/4)$. If the frequency of wave is 1000 Hz. Its velocity will be

(1) 10^4 m/s (2) 125 m/s

(3) 100 m/s (4) 10 m/s

14. Wave pulse on a string shown in figure is moving to the right without changing shape. Consider two particles at positions $x_1 = 1.5$ m and $x_2 = 2.5$ m. Their transverse velocities at the moment shown in figure are along directions



(1) positive y-axis and positive y-axis, respectively

(2) negative y-axis and positive y-axis, respectively

(3) positive y-axis and negative y-axis, respectively

(4) negative y-axis and negative y-axis, respectively

15. Small amplitude progressive wave in a stretched string has a speed of 100 cm/s, and frequency 100 Hz. The phase difference between two points 2.75 cm apart on the string, in radians, is

(1) 0 (2) $11\pi/2$

(3) $\pi/4$ (4) $3\pi/8$

16. The linear density of a vibrating string is 10^{-4} kg/m. A transverse wave is propagating on the string, which is described by the equation $y = 0.02 \sin (x + 30\ t)$, where x and y are in metres and time t in seconds. Then tension in the string is

(1) 0.09 N (2) 0.36 N

(3) 0.9 N (4) 3.6 N

17. The amplitude of a wave represented by displacement equation

$$y = \frac{1}{\sqrt{a}} \sin \omega t \pm \frac{1}{\sqrt{b}} \cos \omega t$$

will be

(1) $\dfrac{a+b}{ab}$ (2) $\dfrac{\sqrt{a}+\sqrt{b}}{ab}$

(3) $\dfrac{\sqrt{a} \pm \sqrt{b}}{ab}$ (4) $\sqrt{\dfrac{a+b}{ab}}$

18. A plane sound wave is travelling in a medium. In reference to a frame A, its equation is $y = a \cos (\omega t - kx)$. Which reference to frame B, moving with a constant velocity v in the direction of propagation of the wave, equation of the wave will be

(1) $y = a\cos[(\omega t + kv)t - kx]$

(2) $y = -a\cos[(\omega t - kv)t - kx]$

(3) $y = a\cos[(\omega t - kv)t - kx]$

(4) $y = a\cos[(\omega t + kv)t + kx]$

19. Two blocks of masses 40 kg and 20 kg are connected by a wire that has a linear mass density of 1 g/m. These blocks are being pulled across horizontal frictionless floor by horizontal force F that is applied to 20 kg block. A transverse wave travels on the wave between the blocks with a speed of 400 m/s (relative to the wire). The mass of the wire is negligible compared to the mass of the blocks. The magnitude of F is

(1) 160 N (2) 240 N

(3) 320 N (4) 400 N

20. At $t = 0$, the shape of a travelling pulse is given by

$$y(x,0) = \frac{4\times10^{-3}}{8-(x)^2}$$

where x and y are in metres. The wave function for the travelling pulse if the velocity of propagation is 5 m/s in the x direction is given by

(1) $y(x,t) = \dfrac{4\times10^{-3}}{8-(x^2-5t)}$ (2) $y(x,t) = \dfrac{4\times10^{-3}}{8-(x-5t)^2}$

(3) $y(x,t) = \dfrac{4\times10^{-3}}{8-(x+5t)^2}$ (4) $y(x,t) = \dfrac{4\times10^{-3}}{8-(x^2+5t)}$

21. Two particles of medium disturbed by the wave propagation are at $x_1 = 0$ and $x_2 = 1$ cm. The respective displacements (in cm) of the particles can be given by the equations:

$y_1 = 2 \sin 3\pi t$, $y_2 = 2 \sin (3\pi t - \pi/8)$

The wave velocity is

(1) 16 cm/s (2) 24 cm/s

(3) 12 cm/s (4) 8 cm/s

22. At $t = 0$, a transverse wave pulse travelling in the positive x direction with a speed of 2 m/s in a wire is described by the function $y = 6/x^2$ given that $x \neq 0$. Transverse velocity of a particle at $x = 2$ m and $t = 2$ s is

(1) 3 m/s (2) –3 m/s

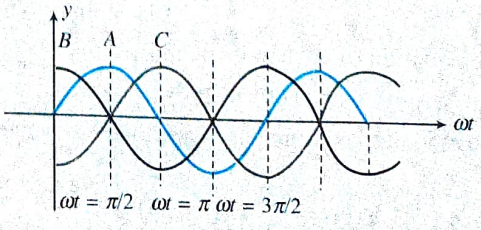
(3) 8 m/s (4) –8 m/s

23. Sinusoidal waves 5.00 cm in amplitude are to be transmitted along a string having a linear mass density equal to 4.00×10^{-2} kg/m. If the source can deliver a maximum power of 90 W and the string is under a tension of 100 N, then the highest frequency at which the source can operate it, is (take $\pi^2 = 10$)

(1) 45.3 Hz (2) 50 Hz

(3) 30 Hz (4) 62.3 Hz

24. The figure shows three progressive waves A, B and C. What can be concluded from the figure that with respect to wave A?



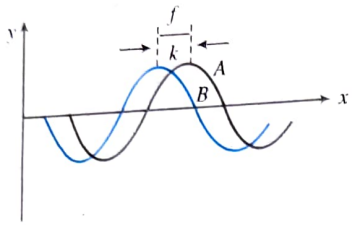
(1) The wave C is ahead by a phase angle of $\pi/2$ and the wave B lags behind by a phase angle $\pi/2$.

(2) The wave C is lag behind by a phase angle of $\pi/2$ and the wave B is ahead by a phase angle $\pi/2$.

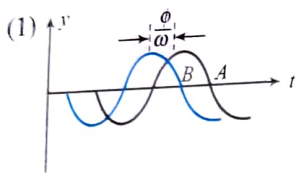
(3) The wave C is ahead by a phase angle of π and the wave B lags behind by a phase angle π.

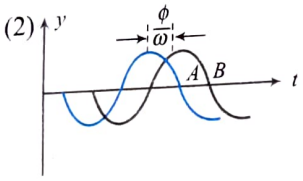
(4) The wave C lags behind by a phase angle of π and the wave B is ahead by a phase angle π.

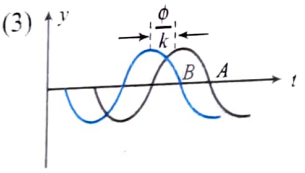
25. Adjoining figure shows the snapshot of two waves A and B at any time t. The equation for A is $y = A \sin(kx - \omega t - \phi)$, and for B it is $y = A \sin(kx - \omega t)$. It is clearly shown in the figure that wave A is ahead of B by a distance ϕ/k.

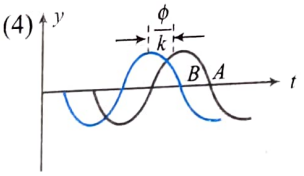


The motion of a single point in time, i.e., y versus t for two waves is best represented by

(1) 

(2) 

(3) 

(4) 

26. At $t = 0$, a transverse wave pulse in a wire is described by the function $y = 6/(x^2 - 3)$ where x and y are in metres. The function $y(x, t)$ that describes this wave equation if it is travelling in the positive x direction with a speed of 4.5 m/s is

(1) $y = \dfrac{6}{(x+4.5t)^2 - 3}$ (2) $y = \dfrac{6}{(x-4.5t)^2 + 3}$

(3) $y = \dfrac{6}{(x+4.5t)^2 + 3}$ (4) $y = \dfrac{6}{(x-4.5t)^2 - 3}$

27. The amplitude of a wave disturbance propagating along positive X-axis is given by $y = 1/(1 + x^2)$ at $t = 0$ and $y = 1/[1 + (x - 2)^2]$ at $t = 4$ s where x and y are in metre. The shape of wave disturbance does not change with time. The velocity of the wave is

(1) 0.5 m/s (2) 1 m/s

(3) 2 m/s (4) 4 m/s

28. The equation of a wave is given by

$$y = 0.5 \sin(100\,t + 25x)$$

The ratio of maximum particle velocity to wave velocity is:

(1) 12.5 (2) 25

(3) 4 (4) 1/8

29. A transverse wave is travelling in a string. Study followin statements.

(i) Equation of the wave is equal to the shape of the strin at an instant t.

(ii) Equation of the wave is general equation for displacemer of a particle of the string

(iii) Equation of the wave must be sinusoidal equation

(iv) Equation of the wave is an equation for displacement c the particle at one end only.

Correct statements are

(1) (i) and (ii) (2) (ii) and (iii)

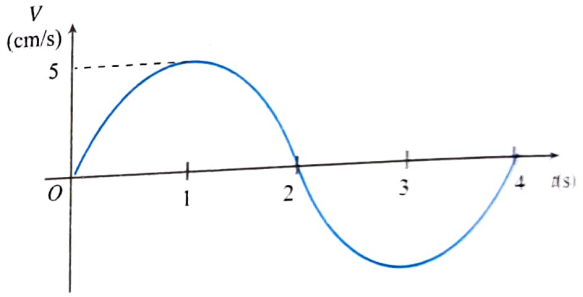
(3) (i) and (iii) (4) (ii) and (iv)

30. If the maximum speed of a particle on a travelling wave v_0, then find the speed of a particle when the displacement half of the maximum value.

(1) $\dfrac{v_0}{2}$ (2) $\dfrac{\sqrt{3}\,v_0}{4}$

(3) $\dfrac{\sqrt{3}\,v_0}{2}$ (4) v_0

31. A certain transverse sinusoidal wave of wavelength 20 cm moving in the positive x direction. The transverse velocity of the particle at $x = 0$ as a function of time is shown. The amplitude of the motion is:



(1) $\dfrac{5}{\pi}$ cm (2) $\dfrac{\pi}{2}$ cm

(3) $\dfrac{10}{\pi}$ cm (4) 2π cm

32. A point source of sound is placed in a non-absorbing medium. Two points A and B are at the distances of 1 m and 2 m, respectively, from the source. The ratio of amplitudes of waves at A to B is

(1) 1:1 (2) 1:4

(3) 1:2 (4) 2:1

33. Two canoes are 10 m apart on a lake. Each bobs up and down with a period of 4.0 s. When one canoe is at its highest point, the other canoe is at its lowest point. Both canoes are always within a single cycle of the waves. Determine the speed of the wave.

(1) 2.5 m/s (2) 5 m/s

(3) 40 m/s (4) 4 m/s

34. The mathematical form of three travelling waves are given by

$Y_1 = (2\text{ cm}) \sin(3x - 6t)$

$Y_2 = (3\text{ cm}) \sin(4x - 12t)$

And $Y_3 = (4\text{ cm})\sin(5x - 11t)$ of these waves,

(1) Wave 1 has greatest wave speed and greatest maximum transverse string speed

(2) Wave 2 has greatest wave speed and wave 1 has greatest maximum transverse string speed

(3) Wave 3 has greatest wave speed and wave 1 has greatest maximum transverse string speed

(4) Wave 2 has greatest wave speed and wave 3 has greatest maximum transverse string speed.

35. A harmonic wave has been set up on a very long string which travels along the length of string. The wave has frequency of 50 Hz, amplitude 1 cm and wavelength 0.5 m. For the above described wave.

Statement I: Time taken by wave to travel a distance of 8 m along the length of string is 0.32 s.

Statement II: Time taken by a point on the string to travel a distance of 8 m, once the wave has reached at that point and sets it into motion is 0.32 s.

(1) Both the statements are correct.

(2) Statement I is correct but Statement II is incorrect.

(3) Statement I is incorrect but Statement II is correct.

(4) Both the statements are incorrect.

36. A wave is represented by the equation

$$y = y_0 \sin[10\pi x - 15\pi t + (\pi/3)]$$

where x is in metres and t in seconds. The equation represents a travelling wave:

(1) in the positive direction with a velocity 1.5 m/s and wavelength 0.2 m.

(2) in the negative direction with a velocity 1.5 m/s and wavelength 0.2 m.

(3) in the positive direction with a velocity 2 m/s and wavelength 0.2 m.

(4) in the negative direction with a velocity 2 m/s and wavelength 1.5 m.

37. A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar that moves the end up and down through a distance by 2.0 cm. The motion of bar is continuous and is repeated regularly 125 times per second. If the distance between adjacent wave crests is observed to be 15.6 cm and the wave is moving along positive x-direction, and at $t = 0$ the element of the string at $x = 0$ is at mean position $y = 0$ and is moving downward, the equation of the wave is best described by

(1) $y = (1\text{ cm})\sin[(40.3\text{ rad/m})x - (786\text{ rad/s})t]$

(2) $y = (2\text{ cm})\sin[(40.3\text{ rad/m})x - (786\text{ rad/s})t]$

(3) $y = (1\text{ cm})\cos[(40.3\text{ rad/m})x - (786\text{ rad/s})t]$

(4) $y = (2\text{ cm})\cos[(40.3\text{ rad/m})x - (786\text{ rad/s})t]$

38. A transverse wave on a string has an amplitude of 0.2 m and a frequency of 175 Hz. Consider a particle of the string at $x = 0$. It begins with a displacement $y = 0$, at $t = 0$, according to equation $y = 0.2\sin(kx \pm \omega t)$. How much time passes between the first two instant when this particle has a displacement of $y = 0.1$ m?

(1) 1.9 ms (2) 3.9 ms

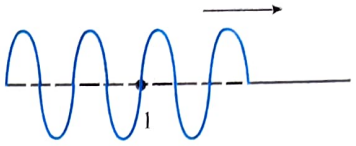
(3) 2.4 ms (4) 0.5 ms

39. A water surfer is moving at a speed of 15 m/s. When he is surfing in the direction of wave, he swings upwards every 0.8 s because of wave crests. While surfing in opposite direction to that of wave motion, he swings upwards every 0.6 s. Determine the wavelength of transverse component of the water wave.

(1) 15 m (2) 10.3 m

(3) 21.6 m (4) Information insufficient

40. A transverse wave on a string travelling along +ve x-axis has been shown in the figure below:



The mathematical form of the shown wave is

$$y = (3.0\text{ cm})\sin\left[2\pi \times 0.1\,t - \frac{2\pi}{100}x\right]$$

where t is in seconds and x is in centimetres. Find the total distance travelled by the particle at (1) in 10 min 15 s, measured from the instant shown in the figure and direction of its motion at the end of this time.

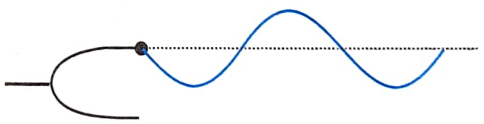
(1) 6 cm, in upward direction

(2) 6 cm, in downward direction

(3) 738 cm, in upward direction

(4) 732 cm, in upward direction

41. The prong of an electrically operated tuning fork is connected to a long string of $\mu = 1$ kg/m and tension 25 N. The maximum velocity of the prong is 1 cm/s, then the average power needed to drive the prong is:



(1) 5×10^{-4} W (2) 2.5×10^{-4} W

(3) 1×10^{-4} W (4) 10^{-3} W

42. A string of length $2L$, obeying Hooke's law, is stretched so that its extension is L. The speed of the transverse wave travelling on the string is v. If the string is further stretched so that the extension in the string becomes $4L$. The speed of transverse wave travelling on the string will be

(1) $\frac{1}{\sqrt{2}}v$ (2) $\sqrt{2}\,v$

(3) $\frac{1}{2}v$ (4) $2v$

43. Consider a wave represented by $y = a\cos^2(\omega t - kx)$ where symbols have their usual meanings. This wave has

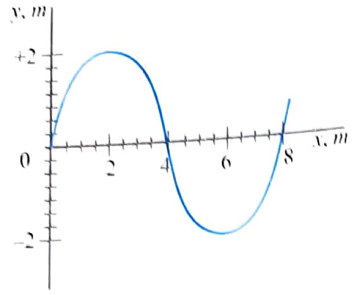
(1) an amplitude a, frequency ω, and wavelength λ.

(2) an amplitude a, frequency 2ω, and wavelength 2λ.

(3) an amplitude $a/2$, frequency 2ω, and wavelength $\lambda/2$.

(4) am amplitude $a/2$, frequency 2ω, and wavelength λ.

44. The graph shows a wave at $t = 0$ travelling to the right with a velocity of 4 m/s. The equation that best represents the wave is



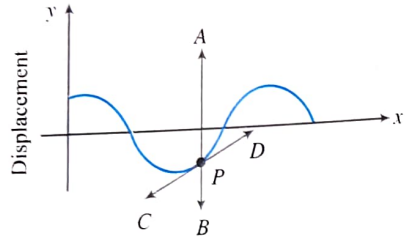
(1) $y(x, t) = 2\sin(\pi x/4 - \pi t)$ metres

(2) $y(x, t) = 2\sin(16\pi x - 8\pi t)$ metres

(3) $y(x, t) = 2\sin(\pi x/4 + \pi t)$ metres

(4) $y(x, t) = 4\sin(\pi x/4 - \pi t)$ metres

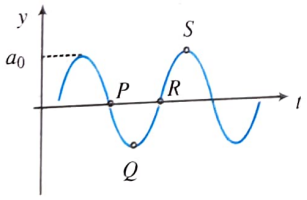
45. The figure below is a representation of a simple harmonic progressive wave, progressing in the negative X-axis, at a given instant. The direction of the velocity of the particle at stage P in the figure is best represented by the arrow.



(1) $\vec{PA}$ (2) $\vec{PB}$

(3) $\vec{PC}$ (4) $\vec{PD}$

46. The wave motion has the function $y = a_0\sin(\omega t - kx)$. The graph in the figure shows how the displacement y at a fixed point varies with time t. Which one of the labelled points shows a displacement equal to that at the position $x = \pi/2k$ at time $t = 0$?



(1) P (2) Q

(3) R (4) S

47. A sinusoidal wave travelling in the positive direction on stretched string has amplitude 20 cm, wavelength 1.0 m and wave velocity 5.0 m/s. At $x = 0$ and $t = 0$ it is given that $y = 0$ and $\partial y/\partial t < 0$. Find the wave function $y(x, t)$.

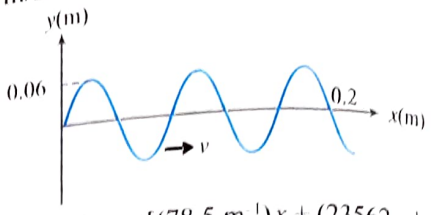
(1) $y(x,t) = (0.02\text{ m})\sin\left[(2\pi\text{ m}^{-1})x + (10\pi\text{ s}^{-1})t\right]$ m

(2) $y(x,t) = (0.02\text{ m})\cos[(10\pi\text{ s}^{-1})t + (2\pi\text{ m}^{-1})x]$ m

(3) $y(x,t) = (0.02\text{ m})\sin\left[(2\pi\text{ m}^{-1})x - (10\pi\text{ s}^{-1})t\right]$ m

(4) $y(x,t) = (0.02\text{ m})\sin\left[(\pi\text{ m}^{-1})x + (5\pi\text{ s}^{-1})t\right]$ m

48. For the wave shown in figure, write the equation of the wave if its position is shown at $t = 0$. Speed of wave is $v = 300$ m/s.



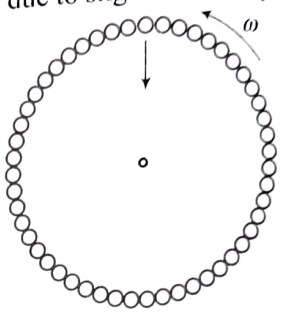
(1) $y = (0.06\text{ m})\cos[(78.5\text{ m}^{-1})x + (23562\text{ s}^{-1})t]$ m

(2) $y = (0.06\text{ m})\sin[(78.5\text{ m}^{-1})x - (23562\text{ s}^{-1})t]$ m

(3) $y = (0.06\text{ m})\sin[(78.5\text{ m}^{-1})x + (23562\text{ s}^{-1})t]$ m

(4) $y = (0.06\text{ m})\cos[(78.5\text{ m}^{-1})x - (28562\text{ s}^{-1})t]$ m

49. A circular loop of rope of length L rotates with uniform angular velocity ω about an axis through its centre on a horizontal smooth platform. Velocity of pulse (with respect to rope) produced due to slight radial displacement is given by



(1) ωL (2) $\dfrac{\omega L}{2\pi}$

(3) $\dfrac{\omega L}{\pi}$ (4) $\dfrac{\omega L}{4\pi^2}$

50. In question (49), if the motion of the pulse and rotation of the loop, both are in same direction then the velocity of the pulse w.r.t. to ground will be:

(1) ωL (2) $\dfrac{\omega L}{2\pi}$

(3) $\dfrac{\omega L}{\pi}$ (4) $\dfrac{\omega L}{4\pi^2}$

51. In question (49), if both are in opposite direction, then the velocity of the pulse w.r.t. to ground will be:

(1) ωL (2) $\dfrac{\omega L}{2\pi}$

(3) $\dfrac{\omega L}{\pi}$ (4) 0

Multiple Correct Answers Type

1. Mark out the correct statement(s) w.r.t. wave speed and particle velocity for a transverse travelling mechanical wave on a string.

(1) The wave speed is same for the entire wave, while particle velocity is different for different points at a particular instant.

(2) Wave speed depends upon property of the medium but not on the wave properties.

(3) Wave speed depends upon both the properties of the medium and on the properties of wave.

(4) Particle velocity depends upon properties of the wave and not on medium properties.

2. A wave moves at a constant speed along a stretched string. Mark the incorrect statement out of the following:

(1) Particle speed is constant and equal to the wave speed.

(2) Particle speed is independent of amplitude of the periodic motion of the source.

(3) Particle speed is independent of frequency of periodic motion of the source.

(4) Particle speed is dependent on tension and linear mass density the string.

3. A harmonic wave is travelling along +ve x-axis, on a stretched string. If wavelength of the wave gets doubled, then

(1) Frequency of wave may change

(2) Wave speed may change

(3) Both frequency and speed of wave may change

(4) Only frequency will change

4. As a wave propagates

(1) the wave intensity remains constant for a plane wave

(2) the wave intensity decreases as the inverse square of the distance from the source for a spherical wave

(3) the wave intensity decreases as the inverse of the distance from a line source

(4) total power of the spherical wave over the spherical surface centred at the source remains constant at all the times

5. Two particles P and Q have a phase difference of π when a sine wave passes through the region:

(1) P oscillates at half the frequency of Q.

(2) P and Q move in opposite directions.

(3) P and Q must be separated by half of the wavelength.

(4) The displacements of P and Q have equal magnitudes.

6. $y(x,t)=0.8/[(4x+5t)^2+5]$ represents a moving pulse, where x and y are in metres and t is in seconds, then

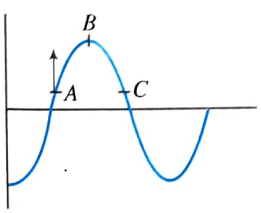
(1) pulse is moving in $+x$ direction

(2) in 2 s it will travel a distance of 2.5 m

(3) its maximum displacement is 0.16 m

(4) it is a symmetric pulse

7. A wave is travelling along a string. At an instant shape of the string is as shown in the enclosed figure. At this instant, point A is moving upwards. Which of the following statements are correct?



(1) The wave is travelling to the right

(2) Displacement amplitude of the wave is equal to the displacement of B at this instant

(3) At this instant velocity of C is also upwards

(4) Phase difference between A and C may be equal to $\pi/2$.

8. A wave equation which gives the displacement along y-direction is given by

$$y = 10^{-4} \sin(60t + 2x)$$

where x and y are in metres and t is time in seconds. This represents a wave

(1) travelling with a velocity of 30 m/s in the negative x-direction

(2) of wavelength π metres

(3) of frequency $30/\pi$ Hertz

(4) of amplitude 10^{-4} m travelling along the negative x-direction.

9. Consider a wave represented by $y = \cos(500t - 70x)$ where y is in millimetres, x in meters and t in second. Which of following are true?

(1) The wave is a standing wave.

(2) The speed of the wave is 50/7 m/s

(3) The frequency of oscillations is $500 \times 2\pi$ Hz.

(4) Two nearest points in the same phase have separation $20\pi/7$ cm.

10. A simple harmonic progressive wave in a gas has a particle displacement of $y = a$ at time $t = T/4$ at the origin of the wave and a particle velocity of $\dot{y} = v$ at the same instant but at a distance $x = \lambda/4$ from the origin where T and λ are the periodic time and wavelength of the wave respectively. Then for this wave,

(1) the amplitude A of the wave is $A = 2a$

(2) the amplitude A of the wave is $A = a$

(3) The equation of the wave can be represented by

$$y = a \sin \frac{v}{a}\left[t - \frac{x}{v}\right]$$

(4) The equation of the wave can be represented by

$$y = 2a \cos \frac{v}{a}\left[t - \frac{x}{v}\right]$$

11. Equation of a wave travelling in a medium is: $y = a \sin(bt - cx)$. Which of the following are correct?

(1) Ratio of the displacement amplitude, with which the particles of the medium oscillate, to the wavelength is equal to $ac/2\pi$.

(2) Ratio of the velocity oscillation amplitude of medium particles to the wave propagation velocity is equal to ac.

(3) Oscillation amplitude of relative deformation of the medium is directly proportional to velocity oscillation amplitude of medium particles.

(4) None of the above

12. A transverse wave travelling on a taut string is represented by:

$$y = 0.01 \sin 2\pi(10t - x)$$

y and x are in meters and t in seconds. Then,

(1) The speed of the wave is 10 m/s.

(2) Closest points on the string which differ in phase by 60° are (1/6) m apart.

(3) Maximum particle velocity is $\pi/4$ m/s.

(4) The phase of a certain point on the string changes by 120° in (1/20) seconds.

13. The equation to a transverse wave travelling in a rope is given by

$$y = A\cos\frac{\pi}{2}[kx - \omega t - \alpha]$$

where $A = 0.6$ m, $k = 0.005$ cm^{-1}, $\omega = 8.0$ s^{-1} and α is a non-vanishing constant. Then for this wave,

(1) the wavelength of the wave is $\lambda = 8$ m

(2) the maximum velocity v_m of a particle of the rope will be, $v_m = 7.53$ m/s.

(3) the equation of a wave which, when superposed with the given wave can produce standing waves in the rope is

$$y = A\cos\frac{\pi}{2}(kx - \omega t + \alpha)$$

(4) the equation of a wave which, when superposed with the given wave can produce standing waves in the rope is

$$y = A\cos\frac{\pi}{2}(kx + \omega t - \alpha)$$

14. The equation of a wave is $y = 4\sin\left[\frac{\pi}{2}\left(2t + \frac{1}{8}x\right)\right]$

where y and x are in centimetres and t is in seconds.

(1) The amplitude, wavelength, velocity, and frequency of wave are 4 cm, 16 cm, 32 cm/s and 1 Hz, respectively, with wave propagating along $+x$ direction.

(2) The amplitude, wavelength, velocity, and frequency of wave are 4 cm, 32 cm, 16 cm/s, and 0.5 Hz, respectively, with wave propagating along $-x$ direction.

(3) Two positions occupied by the particle at time interval of 0.4 s have a phase difference of 0.4 π radian.

(4) Two positions occupied by the particle at separation of 12 cm have a phase difference of 135°.

15. A wire of 9.8×10^{-3} kg/m passes over a frictionless light pulley fixed on the top of a frictionless inclined plane which makes an angle of 30° with the horizontal. Masses m and M are tied at the two ends of wire such that m rests on the plane and M hangs freely vertically downwards. The entire system is in equilibrium and a transverse wave propagates along the wire with a velocity of 100 m/s.

(1) $m = 20$ kg (2) $M = 5$ kg

(3) $\frac{m}{M} = \frac{1}{2}$ (4) $\frac{m}{M} = 2$

16. A transverse sinusoidal wave of amplitude a, wavelength λ and frequency f is travelling on a stretched string. The maximum speed at any point on the string is $(v/10)$ where v is the speed of propagation of the wave. If $a = 10^{-3}$ m and $v = 10$ m/s, then λ and f are given by:

(1) $\lambda = 2\pi \times 10^{-2}$ m (2) $\lambda = 10^{-3}$ m

(3) $f = 10^3/(2\pi)$ Hz (4) $f = 10^4$ Hz

17. Let a disturbance y be propagated as a plane wave along the x-axis. The wave profiles at the instants $t = t_1$ and $t = t_2$ are represented respectively as: $y_1 = f(x_1 - vt_1)$ and $y_2 = f(x_2 - vt_2)$. The wave is propagating without change of shape.

(1) The velocity of the wave is v.

(2) The velocity of the wave is $v = (x_2 + x_1)/(t_2 + t_1)$.

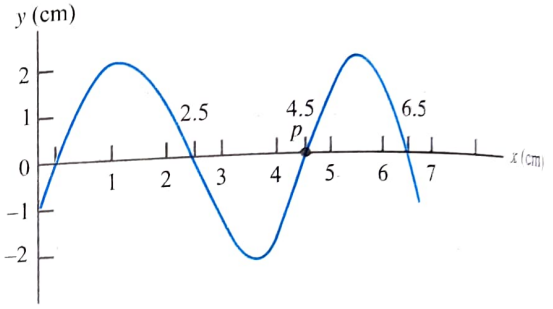
(3) The particle velocity is $v_P = -vf'(x - vt)$.

(4) The phase velocity of the wave is v.

Linked Comprehension Type

For Problems 1–4

Consider a sinusoidal travelling wave shown in the given figure. The wave velocity is +40 cm/s.



1. Find the frequency of the wave.

(1) 20 Hz (2) 30 Hz

(3) 25 Hz (4) 10 Hz

2. Find the phase difference between points 2.5 cm apart.

(1) $3\pi/4$ rad (2) $5\pi/4$ rad

(3) $7\pi/4$ rad (4) $9\pi/4$ rad

3. How long does it take for the phase at a given position change by 60° ?

(1) 1/30 s (2) 1/60 s

(3) 1/20 s (4) 1/40 s

4. Find the velocity of a particle at point P at the instant shown

(1) 1.26 m/s upward (2) 1.26 m/s downward

(3) 3.52 m/s upward (4) None of these

For Problems 5–7

A sinusoidal wave is propagating in negative x-direction on a string stretched along x-axis. A particle of string at $x = 2$ cm is found at its mean position and it is moving in positive y-direction at $t = 1$ s. The amplitude of the wave, the wavelength and the angular frequency of the wave are 0.1 m, $\pi/4$ m and 4π rad/s, respectively.

5. The equation of the wave is

(1) $y = 0.1\sin(4\pi(t-1) + 8(x-2))$

(2) $y = 0.1\sin((t-1) - (x-2))$

(3) $y = 0.1\sin(4\pi(t-1) - 8(x-2))$

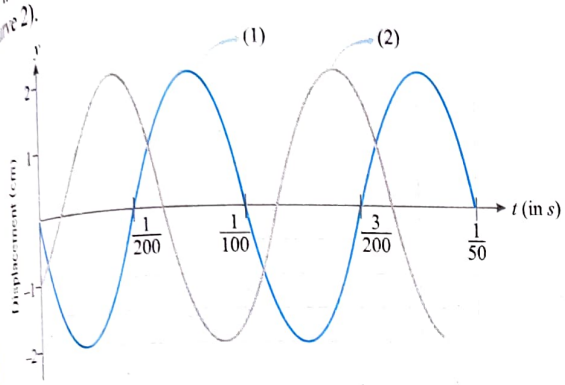
(4) none of these

The speed of particle at $x = 2$ m and $t = 1$ s is
(1) 0.2π m/s (2) 0.6π m/s
(3) 0.4π m/s (4) 0

The instantaneous power transfer through $x = 2$ m and $t = 1.125$ s is
(1) 10 J/s (2) $4\pi/3$ J/s
(3) $2\pi/3$ J/s (4) 0

For Problems 8–9

A simple harmonic plane wave propagates along x-axis in a medium. The displacement of the particles as a function of time is shown in figure, for $x = 0$ (curve 1) and $x = 7$ cm (curve 2).



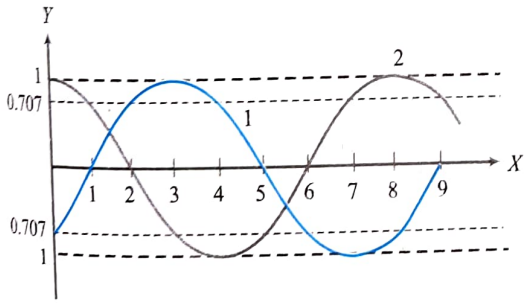
The two particles are within a span of one wavelength.

8. The wavelength of the wave is
(1) 6 cm (2) 24 cm
(3) 12 cm (4) 16 cm

9. The speed of the wave is
(1) 12 m/s (2) 24 m/s
(3) 8 m/s (4) 16 m/s

For Problems 10–11

Figure shows two 'snapshots' of medium particle supporting a plane progressive wave travelling along positive x-axis, corresponding to instants $t = 0.002$ s and $t = 0.008$ s, respectively shown by curves numbered 1 and 2.



Assume that the interval between the two snapshots is less than the time period.

10. Velocity of the wave is
(1) $\frac{1700}{3}$ m/s (2) $\frac{1700}{5}$ m/s
(3) $\frac{2500}{7}$ m/s (4) $\frac{2500}{3}$ m/s

11. The frequency of the wave
(1) 52 s^{-1} (2) 205 s^{-1}
(3) 104 s^{-1} (4) 84 s^{-1}

For Problems 12–14

A pulse is started at a time $t = 0$ along the $+x$ direction on a long, taut string. The shape of the pulse at $t = 0$ is given by function $f(x)$ with

$$f(x) = \begin{cases} \frac{x}{4} + 1 & \text{for } -4 < x \le 0 \\ -x + 1 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

here f and x are in centimeters. The linear mass density of the string is 50 g/m and it is under a tension of 5 N.

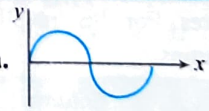
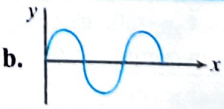
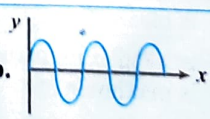
12. The shape of the string is drawn at $t = 0$ and the area of the pulse enclosed by the string and the x-axis is measured. It will be equal to
(1) 2 cm^2
(2) 2.5 cm^2
(3) 4 cm^2
(4) 5 cm^2

13. The vertical displacement of the particle of the string at $x = 7$ cm and $t = 0.01$ s will be
(1) 0.75 cm (2) 0.5 cm
(3) 0.25 cm (4) zero

14. The transverse velocity of the particle at $x = 13$ cm and $t = 0.015$ s will be
(1) –250 cm/s (2) –500 cm/s
(3) 500 cm/s (4) –1000 cm/s

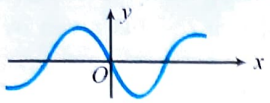
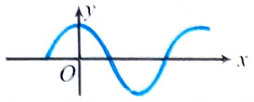
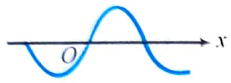
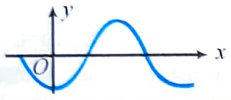
Matrix Match Type

1. Three travelling sinusoidal waves are on identical strings having same tension. The mathematical form of the waves are $y_1 = A \sin(3x - 6t)$, $y_2 = A \sin(4x - 8t)$ and $y_3 = A \sin(6x - 12t)$.

Column I	Column II
i. Speed of each wave is	a. 
ii. y_1 is best represented by	b. 
iii. y_2 is best represented by	b. 
iv. y_3 is best represented by	d. 2 m/s

2. For four sine waves, moving on a string along positive x-direction, displacement distance curves (y–x curves) as shown at time $t = 0$. In the right column, expressions

for y as function of distance x and time t for sinusoidal waves are given. All terms in the equations have their usual meanings. Correctly match y-x curves with corresponding equations.

Column I	Column II
i. 	a. $y = A\cos(\omega t - kx)$
ii. 	b. $y = -A\cos(kx - \omega t)$
iii. 	c. $y = A\sin(\omega t - kx)$
iv. 	d. $y = A\sin(kx - \omega t)$

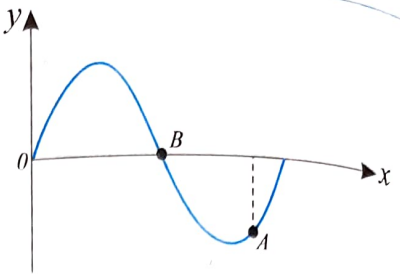
3. Suppose a wave pulse has been created at free end of a taut string by moving the hand up and down once. The string is attached at its other end to a distant wall.

Column I	Column II
i. Moving hand more quickly but still up and down once by the same amount indifferent time,	a. the amplitude changes
ii. Moving hand more quickly but still up and down once by more amount in same time,	b. the width of the pulse changes
iii. Moving hand at same speed, but still up and down once by same more amount,	c. the wave speed changes
iv. Moving hand more quickly, but still up and down once by less amount.	d. the particle speed changes

4. The equation of a travelling wave is given by: $y = (4\,\text{cm})\sin[\pi t + 2\pi x]$. Here, t is in second and x in metres. For this travelling wave, match the following two columns.

Column I	Column II
i. at $x = 0$, particle velocity is maximum at $t =$	a. 0.5 s
ii. at $x = 0$, particle acceleration is maximum at $t =$	b. 1.0 s
iii. at $x = 0.5$ m, particle velocity is maximum at $t =$	c. zero
iv. at $x = 0.5$ m, particle acceleration is maximum at $t =$	d. 1.5 s

5. For a travelling wave, displacement of medium particles vs. position graph (y-x graph) at a given instant is shown in figure. Match the following two columns.



Column I	Column II
i. Velocity of particle A	a. Positive
ii. Acceleration of particle A	b. Negative
iii. Velocity of particle B	c. zero
iv. Acceleration of particle B	d. Can't be calculated

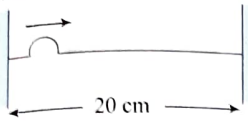
6. In column (I) the equations of travelling waves are given while in column (II) information about wave motion particle velocity and particle acceleration are given. Match the following two columns.

Column I	Column II
i. $y = A\sin(\omega t - kx)$	a. Travelling in positive x-direction
ii. $y = A\sin(\omega t - kx)$	b. Travelling in negative x-direction
iii. $y = A\sin(\omega t - kx)$	c. At $t = 0$, velocity of particle is positive at $x = 0$
iv. $y = A\sin(\omega t - kx)$	d. At $t = 0$ acceleration of particle is positive at $x = 0$

Numerical Value Type

1. Two wires of different densities but same area of cross section are soldered together at one end and are stretched at a tension T. The velocity of a transverse wave in the first wire is double of that in the second wire. Find the ratio of the density of the first wire to that of the second wire.
2. A string of length 40 cm and weighing 10 g is attached to a spring at one end and to a fixed wall at the other end. The spring has a spring constant of 160 N/m and is stretched by 1.0 cm. If a wave pulse is produced on the string near the wall. How much time (in sec) will it take to reach the spring?
3. A transverse wave of amplitude 0.50 mm and frequency 100 Hz is produced on a wire stretched to a tension of 100 N. If the wave speed is 100 m/s, what average power (in mW) is the source transmitting to the wire? (Take $\pi^2 = 10$)
4. A travelling wave is given by $y = \dfrac{0.8}{(3x^2 + 24xt + 48t^2 + 4)}$ where x and y are in metres and t is in seconds. Find the velocity in m/s.
5. A string of length 20 cm and linear mass density 0.40 g/cm is fixed at both ends and is kept under a tension of 16 N. A

wave pulse is produced at $t = 0$ near an end as shown in figure which travels towards the other end.



when will the string have the shape shown in the figure again? (in $\times 10^{-2}$ s)

The speed of a transverse wave, going on a wire having a length 50 cm and mass 5 g is 80 m/s. The area of cross section of the wire is 1.0 mm^2 and its Young's modulus is 8×10^{11} N/m^2. Find the extension (in $\times 10^{-2}$ mm) of the wire over its natural length.

A particle on a stretched string supporting a travelling wave, takes 5.0 ms to move from its mean position to the extreme position. The distance between two consecutive particles, which are at their mean positions, is 3.0 cm. Find the wave speed (in m/s).

A plane progressive wave is given by

$$y = (40 \text{ cm}) \cos (50\pi t - 0.02\pi y)$$

where y is in cm and t in s. The particle velocity at $y = 25$ m in time $t = \frac{1}{100}$ s will be $10\pi\sqrt{n}$ m/s .What is the value of n?

9. An ant with mass m is standing peacefully on top of a horizontal, stretched rope. The rope has mass per unit length μ and is under tension F. Without warning, a student starts a sinusoidal transverse wave of wavelength λ propagating along the rope. The motion of the rope is in a vertical plane. What minimum wave amplitude (in mm) will make the ant feel weightless momentarily? Assume that m is so small that the presence of the ant has no effect on the propagation of the wave.

[Given: $\lambda = 0.5$ m, $\mu = 0.1$ kg/m, $F = 3.125$ N, take $g = \pi^2$]

10. A 4.0 kg block is suspended from the ceiling of an elevator through a string having a linear mass density of 19.2×10^{-3} kg/m. Find the speed (in $\times$ 10 m/sec) (with respect to the string in m/s) with which a wave pulse can proceed on the string if the elevator accelerate up at the rate of 2.0 m/s^2.

Archives

JEE MAIN

Single Correct Answer Type

1. The equation of a wave on a string of linear mass density 0.04 kg/m is given by

$$y = 0.02(\text{m}) \sin\left[2\pi\left(\frac{t}{0.04(\text{s})} - \frac{x}{0.50(\text{m})}\right)\right]$$

The tension in the string is

(1) 4.0 N (2) 12.5 N

(3) 0.5 N (4) 6.25 N **(AIEEE 2010)**

2. The transverse displacement $y(x, t)$ of a wave on a string is given by $y(x,t) = e^{-(ax^2+bt^2+2\sqrt{ab}\,xt)}$

This represents a

(1) wave moving in $+x$ direction with speed $\sqrt{\frac{a}{b}}$

(2) wave moving in $-x$ direction with speed $\sqrt{\frac{b}{a}}$

(3) standing wave of frequency $\sqrt{b}$

(4) standing wave of frequency $\frac{1}{\sqrt{b}}$ **(AIEEE 2011)**

3. A uniform string of length 20 m is suspended from a rigid support. A short wave pulse is introduced at its lowest end. It starts moving up the string. The time taken to reach the support is: (take $g = 10$ ms^{-2})

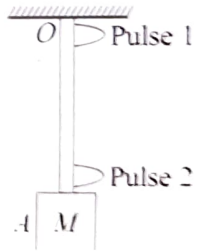
(1) $2\pi\sqrt{2}$ s (2) 2 s

(3) $2\sqrt{2}$ s (4) $\sqrt{2}$ s **(JEE Main 2016)**

JEE ADVANCED

Multiple Correct Answers Type

1. A block M hangs vertically at the bottom end of a uniform rope of constant mass per unit length. The top end of the rope is attached to a fixed rigid support at O. A transverse wave pulse (Pulse 1) of wavelength λ_0 is produced at point O on the rope. The pulse takes time T_{OA} to reach point A. If the wave pulse of wavelength λ_0 is produced at point A (Pulse 2) without disturbing the position of M it takes time T_{AO} to reach point O. Which of the following options is/are correct?



(1) The velocities of the two pulses (Pulse 1 and Pulse 2) are the same at the midpoint of rope

(2) The velocity of any pulse along the rope is independent of its frequency and wavelength

(3) The wavelength of Pulse 1 becomes longer when it reaches point A

(4) The time $T_{AO} = T_{OA}$ **(JEE Advance 2017)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (4)	2. (3)	3. (3)	4. (2)	5. (1)
6. (4)	7. (2)	8. (4)	9. (1)	10. (2)
11. (4)	12. (2)	13. (3)	14. (2)	15. (2)
16. (1)	17. (4)	18. (3)	19. (2)	20. (2)
21. (2)	22. (2)	23. (3)	24. (2)	25. (2)
26. (4)	27. (1)	28. (1)	29. (1)	30. (3)
31. (3)	32. (4)	33. (2)	34. (4)	35. (2)
36. (1)	37. (1)	38. (1)	39. (2)	40. (3)
41. (2)	42. (4)	43. (3)	44. (1)	45. (1)
46. (2)	47. (3)	48. (2)	49. (2)	50. (3)
51. (4)				

Multiple Correct Answers Type

1. (1),(2),(4)	2. (1),(2),(3),(4)	3. (1),(2),(3)
4. (1),(2),(3),(4)	5. (2),(3),(4)	6. (2),(3),(4)
7. (2),(4)	8. (1),(2),(3),(4)	9. (2),(4)
10. (2),(3)	11. (1),(2),(3)	12. (1),(2)
13. (1),(2),(4)	14. (2),(3),(4)	15. (1),(4)
16. (1),(3)	17. (1),(3),(4)	

Linked Comprehension Type

1. (4)	2. (2)	3. (2)	4. (2)	5. (1)
6. (3)	7. (4)	8. (3)	9. (1)	10. (4)
11. (3)	12. (2)	13. (3)	14. (1)	

Matrix Match Type

1. i. → d.; ii. → a.; iii. → b.; iv. → c.
2. i. → c.; ii. → a.; iii. → d.; iv. → b.
3. i. → b., d.; ii. → a., d.; iii. → a., b.; iv. → a., b., d.
4. i. → b., c.; ii. → a., d.; iii. → b., c.; iv. → a., d.
5. i. → d.; ii. → a.; iii. → d.; iv. → c.
6. i. → a., c.; ii. → a.; iii. → b., d.; iv. → a., d.

Numerical Value Type

1. (0.25)	2. (0.050)	3. (50)	4. (4)	5. (2)
6. (4)	7. (3)	8. (2)	9. (2)	10. (5)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (4)	2. (2)	3. (3)

JEE Advanced

Multiple Correct Answers Type

1. (2),(4)

7 Superposition and Standing Waves

INTRODUCTION

In previous chapter we have been dealing with a single wave travelling in an elastic medium. We encounter many situations in which we are concerned with two or more waves travelling in the same medium and we are interested in finding the resultant displacement at different points. In case of stringed musical instruments (like sitar, violin etc.) and organ pipes (like flute, bugle etc.), we have to describe the combined effect of an incident wave and its reflected wave. To analyze such wave combinations, we make use of the principle of superposition. This principle can be applied to many types of waves, including waves on strings, sound waves, surface water waves and electromagnetic waves. It embodies one of the most important concepts in physics, and the remainder of this chapter deals with examples related to it.

PRINCIPLE OF SUPERPOSITION FOR WAVES

In many occasions it happens that two or more waves pass simultaneously through the same region. When we listen to a concert, for example, sound waves from many instruments fall simultaneously on our eardrums. We can hear them simultaneously and detect them individually.

They move in same medium together without losing their identity just like you move in a crowd without losing your "self". Each wave will push or pull the particles of the medium as if they were acting along, if the displacement of the particle or deformation of the medium is very small, the net displacement (or deformation) y of a particle can be given as the vector sum of the displacements y_1, y_2 .... etc, due to the respective waves. this is called principle of superposition, which states that

When two or more waves pass through a medium simultaneously, the particles of the medium are subjected to simultaneous displacements due to each wave. As a result of this, a new wave is formed. The phenomenon of intermixing of two or more waves to produce a new wave is called superposition of waves.

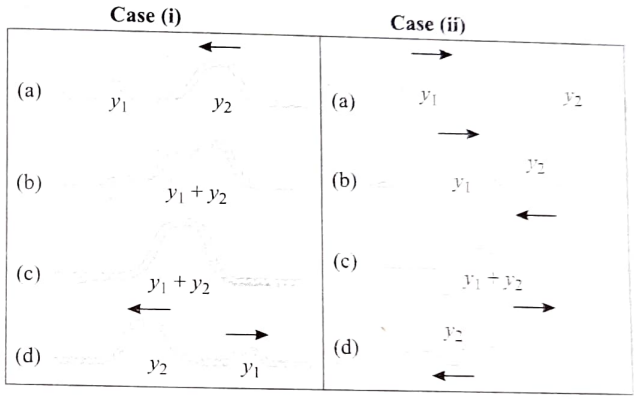
Let $y_1(x, t), y_2(x, t), y_3(x, t)$ ..... be the displacements separately at a point in the medium, the resultant displacement y is given by

$$y(x,t) = y_1(x,t) + y_2(x,t) + y_3(x,t) + ... \quad ...(i)$$

Since each individual displacement is a function of time, so is the resultant displacement. The resultant displacement may be greater than or smaller than the individual displacements, depending upon the phase relations amongst different displacements.

This particle is valid when the deformation are small or the waves are linear as the medium obeys Hooke's law. However, the medium does not obey Hooke's law for larger deformation (or amplitudes). Hence, the waves will be non-linear and the principle of superposition does not hold good.

Figure of case (i) and (ii) shows a sequence of snapshots of two pulses travelling in opposite directions on the same stretched string. When they overlap each other the displacement of particle on string is the algebraic sum of the two displacements as the displacements of the two pulses. Moreover, the overlapping waves do not in any way alter the travel of each other. Case(ii) also shows the similar situation when the wave pulses are in opposite sides.



Three important cases, based on the principle of superposition are:

(a) Interference. Here two waves of same frequency and hence wavelength, superpose and lead to what we call the interference of waves.

(b) Beats. Here two wave trains of nearly same frequency and moving in the same direction, superpose and give rise to the phenomenon of beats.

(c) Stationary waves. Here two waves of same frequency and amplitude moving along the same path, but in opposite directions, superpose and we get stationary waves.

ILLUSTRATION 7.1

Two pulses travelling on the same string are described by

$$y_1 = \frac{5}{(3x-4t)^2+2} \quad \text{and} \quad y_2 = \frac{-5}{(3x+4t-6)^2+2}$$

(a) In which direction does each pulse travel?

(b) At what instant do the two cancel everywhere?

(c) At what point do the two pulses always cancel?

Sol.

(a) At constant phase, $\phi = 3x - 4t$ will be constant. Then $x = (\phi + 4t)/3$ will change: the wave moves. As t increases in this equation, x increases, so the first wave moves to the right.

In the same way, in the second case $x = (\phi - 4t + 6)/3$. As t increases, x must decrease, so the second wave moves to the left.

(b) We require that $y_1 + y_2 = 0$.

$$\frac{5}{(3x-4t)^2+2} + \frac{-5}{(3x+4t-6)^2+2} = 0$$

This can be written as $(3x - 4t)^2 = (3x + 4t - 6)^2$

Solving for the positive root, $t = 0.750$ s

(c) The negative root yields $(3x - 4t) = -(3x + 4t - 6)$

The time terms cancel, leaving $x = 1.00$ m. At this point, the waves always cancel.

The total wave is not a standing wave, but we could call the point at $x = 1.00$ m a node of the superposition.

ILLUSTRATION 7.2

Two waves passing through a region are represented by

$$y = (1.0\text{ m})\sin[(\pi\text{ cm}^{-1})x - (50\,\pi\text{ s}^{-1})t]$$

and $\quad y = (1.5\text{ cm})\sin[(\pi/2\text{ cm}^{-1})x - (100\,\pi\text{ s}^{-1})t]$.

Find the displacement of the particle at $x = 4.5$ cm at time $t = 5.0$ ms.

Sol. According to the principle of superposition, each wave produces its disturbance independent of the other and the resultant disturbance is equal to the vector sum of the individual disturbance. The displacements of the particle at $x = 4.5$ cm at time $t = 5.0$ ms due to the two waves are,

$$y_1 = (1.0\text{ cm})\sin[(\pi\text{ cm}^{-1})(4.5\text{ cm}) - (50\,\pi\text{ s}^{-1})(5.0\times10^{-3}\text{ s})]$$

$$= (1.0\text{ cm})\sin\left[4.5\pi - \frac{\pi}{4}\right]$$

$$= (1.0\text{ cm})\sin\left[4\pi + \frac{\pi}{4}\right] = \frac{1.0\text{ cm}}{\sqrt{2}}$$

and $y_2 = (1.5\text{ cm})\sin[(\pi/2\text{ cm}^{-1})(4.5\text{ cm}) - (100\,\pi\text{ s}^{-1})(5.0\times10^{-3}\text{ s})]$

$$= (1.5\text{ cm})\sin\left[2.25\pi - \frac{\pi}{2}\right] = (1.5\text{ cm})\sin\left[2\pi - \frac{\pi}{4}\right]$$

$$= -(1.5\text{ cm})\sin\frac{\pi}{4} = -\frac{1.5\text{ cm}}{\sqrt{2}}$$

The net displacement is $y = y_1 + y_2 = -0.5/\sqrt{2}\text{ cm} = -0.35$ cm

ILLUSTRATION 7.3

Two travelling sinusoidal waves described by the wave functions

$$y_1 = (5.00\text{ m})\sin[\pi(4.00x - 1200t)]$$

and $\quad y_2 = (5.00\text{ m})\sin[\pi(4.00x - 1200t - 0.250)]$

where x, y_1 and y_2 are in metres and t is in seconds. **(a)** what is the amplitude of the resultant wave? **(b)** What is the frequency of resultant wave?

Sol. Graphs for these wave functions are like continuous sine curves. Think of water waves produced by continuously vibrating pencils on the water surface and seen through the side wall of an aquarium.

The pair of waves travelling in the same direction, perhaps separately created, combine to give a single travelling wave. We can represent the waves symbolically as

$$y_1 = A_0\sin(kx - \omega t) \quad \text{and} \quad y_2 = A_0\sin(kx - \omega t - \phi)$$

with $A_0 = 5.00$ m, $\omega = 1200\,\pi$ s^{-1} and $\phi = 0.250\,\pi$ rad

According to the principle of superposition, the resultant wave function has the form

$$y = y_1 + y_2 = 2A_0\cos\left(\frac{\phi}{2}\right)\sin\left(kx - \omega t - \frac{\phi}{2}\right)$$

(a) with amplitude $A = 2A_0\cos\left(\frac{\phi}{2}\right)$

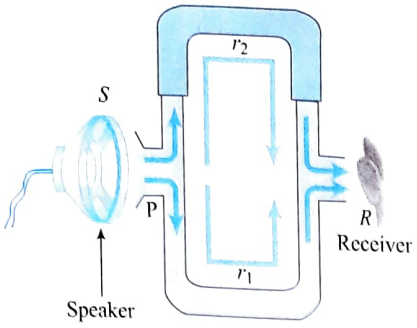
$$= 2(5.00)\cos\left(\frac{\pi}{8.00}\right) = 9.24\text{ m}$$

(b) and frequency $f = \frac{\omega}{2\pi} = \frac{1200\pi}{2\pi} = 600$ Hz

INTERFERENCE OF THE WAVES

When two waves of equal frequency and nearly equal amplitude travelling in same direction in a medium superimpose, the intensity is different at different points. At some points intensity is large even greater than the sum of the intensity of individual waves, whereas at other points it is nearly zero. It is to be noted that there is no creation or destruction of energy in the medium. In fact, energy has been redistributed due to its transference from regions of low intensity region to the regions of high intensity. This redistribution of energy due to superposition of two (or more) wave trains is called interference.

One simple device for demonstrating interference of sound waves is illustrated in figure. Sound from a loudspeaker S is sent into a tube at point P, where there is a T-shaped junction. Half the sound energy travels in one direction, and the other half travels in the opposite direction. Therefore, the sound waves that reach the receiver R can travel along either of the two paths. The distance along any path from speaker to receiver is called the path length r. The lower path length r_1 is fixed, but the upper path length r_2 can be varied by sliding the U-shaped tube, which is similar to that on a slide trombone. When the difference in the path lengths $\Delta r = |r_2 - r_1|$ is either zero or some integer multiple of the wavelength λ (that, $\Delta r = n\lambda$, where $n = 0, 1, 2, 3, \ldots$) the two waves reaching the receiver at any instant are in phase and interfere constructively as shown for this case, a maximum in the sound intensity is detected at the receiver. If the path length r_2 is adjusted such that the path difference $\Delta r = \lambda/2, 3\lambda/2, \ldots$ $n\lambda/2$ (for n odd), of the two waves is exactly π rad, or 180°, the two waves are out of phase at the receiver and hence cancel each other. This is the case of destructive interference, no sound is detected at the receiver. This simple experiment demonstrates that a phase difference may arise between two waves generated by the same source when they travel along paths of unequal lengths. This important phenomenon will be indispensable in our investigation of the interference of light waves.



MATHEMATICAL ANALYSIS OF INTERFERENCE

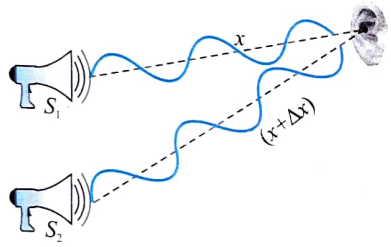
Consider two harmonic waves of same frequency (coherent waves). Suppose A_1 and A_2 be the amplitudes of the waves and ϕ is the phase difference between them. It is assumed that the waves are plane and move almost along a line. Thus wave equations are

$$y_1 = A_1 \sin(kx - \omega t) \quad ...(i)$$

or $$y_2 = A_2 \sin(kx - \omega t + \phi) \quad ...(ii)$$

here $$\phi = \left(\frac{2\pi}{\lambda}\right) \Delta x$$

When both the waves travel simultaneously, the resultant wave at P can be obtained by principle of superposition.



$$y = y_1 + y_2$$
$$= A_1 \sin(kx - \omega t) + [A_2 \sin(kx - \omega t)\cos\phi + A_2 \cos(kx - \omega t)\sin\phi]$$
$$= (A_1 + A_2 \cos\phi)\sin(kx - \omega t) + A_2 \sin\phi \cos(kx - \omega t)$$

Let $$A_1 + A_2 \cos\phi = R\cos\theta \quad ...(iii)$$

and $$A_2 \sin\phi = R\sin\theta \quad ...(iv)$$

Squaring and adding Eqs. (iii) and (iv), we get

$$R^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\phi \quad ...(v)$$

As $I \propto A^2$, $\therefore I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \quad ...(vi)$

Also $$\tan\theta = \frac{A_2\sin\phi}{A_1 + A_2\cos\phi} \quad ...(vii)$$

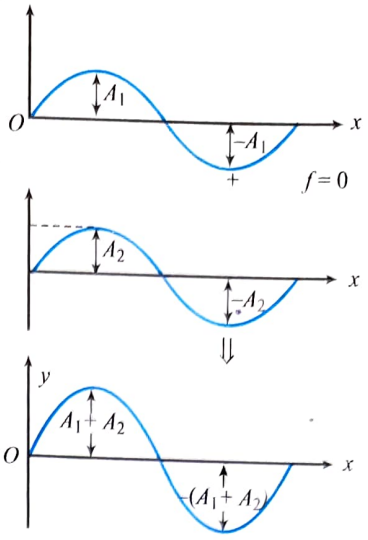
The resultant wave becomes $y = R\sin[(kx - \omega t) + \theta]$ (viii)

clearly the resultant wave has the same frequency and speed as the interfering waves

TWO TYPES OF INTERFERENCE

Constructive interference: If two waves are moving to the medium such that they are in phase as shown in figure. The resulting amplitude due to the superposition of the waves will be summation of the individual amplitudes of the waves.

For constructive interference the resultant intensity will be greater than the sum of the individual intensities of the waves. For maximum intensity using Eq. (vi), we get



Constructive interference

$$\cos\phi = 1$$

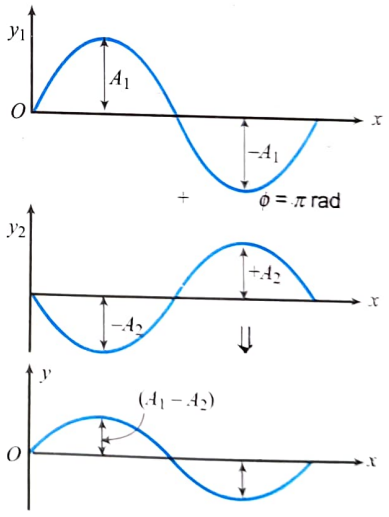
or $\phi = 2\pi n$, when $n = 0, 1, 2, 3, \ldots$

As 2π phase difference is equal to a path difference λ, so

$$\Delta x = n\lambda$$

Thus from Eq. (v) $R_{max} = A_1 + A_2$

Destructive interference: If two waves are moving to the medium such that they are out of phase (figure). The resultant amplitude of the wave may be less than the sum of individual amplitudes of the waves. For minimum amplitude using Eq. (v)



Destructive interference

$$\cos\phi = -1$$

or $\phi = (2n - 1)\pi$, where $n = 1, 2, 3, \ldots$

and $\Delta x = (2n - 1)\lambda/2$

Thus from Eq. (v), $R_{min} = A_1 - A_2$

The ratio of maximum to minimum intensities

$$\frac{I_{max}}{I_{min}} = \frac{(A_1 + A_2)^2}{(A_1 - A_2)^2} = \frac{\left(\sqrt{I_1} + \sqrt{I_2}\right)^2}{\left(\sqrt{I_1} - \sqrt{I_2}\right)^2} \quad ...(ix)$$

Note: The distance between maximum and next minima = $\lambda/2$.

Conditions of sustained interference:

1. Mathematically, interference phenomenon can take place between two waves of same frequency and different

amplitudes. But for observable interference, the amplitude of the waves should be equal. In this case,

$$A_1 = A_2 = A$$

$$\therefore \quad R^2 = A^2 + A^2 + 2AA\cos\phi$$

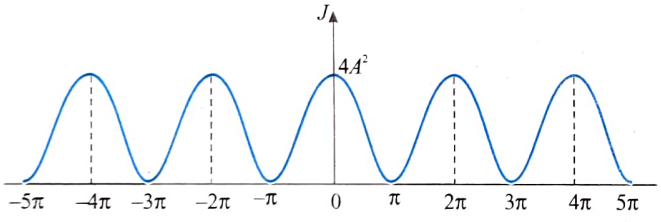
$$\text{or} \quad R^2 = 2A^2(1+\cos\phi) = 2A^2 \times 2\cos^2\frac{\phi}{2}$$

$$R^2 = 4A^2\cos^2\frac{\phi}{2} \qquad \ldots(x)$$

Write $R^2 = I$ and $4A^2 = I_0$

$$\therefore \quad I = I_0\cos^2\frac{\phi}{2} \qquad \ldots(xi)$$

From Eq. (xi), the maximum intensity is $4A^2$ and minimum intensity is zero. In the phenomenon of interference, the energy is not destroyed but is only redistributed from the positions of minimum intensity to those of maximum intensity. At the maximum intensity positions the intensity due to the two waves should be $2A^2$ but it actually $4A^2$. As shown in figure, the intensity varies from 0 to $4A^2$ and the average is still $2A^2$. It is equal to uniform intensity of $2A^2$ which will be present in the absence of interference phenomenon between two waves. Hence the formation of maxima and minima is due to interference of waves in accordance with the law of conservation of energy.



For sustained interference, the phase difference between the two waves must remain constant, it should not change with time. If the phase difference between the waves changes continuously, then the positions of the maximum and minimum intensities do not remain fix.

ILLUSTRATION 7.4

Two sound sources of same frequency produce sound intensities I_0 and $4I_0$ at a point P when used separately. Now, they are used together so that the sound waves from them reach P with a phase difference ϕ. Determine the resultant intensity at P for

(a) $\phi = 0$ **(b)** $\phi = 2\pi/3$ **(c)** $\phi = \pi$

Sol. We know that $I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$

Thus,

(a) for $\phi = 0$; $I = I_0 + 4I_0 + 2\sqrt{(I_0)(4I_0)} = 9I_0$

(b) for $\phi = 2\pi/3$; $I = I_0 + 4I_0 + 2\sqrt{(I_0)(4I_0)}\cos\frac{2\pi}{3}$

or $I = 3I_0$

(c) For $\phi = \pi$; $I = I_0 + 4I_0 + 2\sqrt{(I_0)(4I_0)}\cos\pi$

or $I = I_0$

ILLUSTRATION 7.5

Two identical sources of sound S_1 and S_2 produce intensity I_0 at a point P equidistant from each source.

(a) Determine the intensity of each source at the point P.

(b) If the power of S_1 is reduced to 64% and phase difference between the two sources is varied continuously, then determine the maximum and minimum intensities at the point P.

(c) If the power of S_1 is reduced by 64%, then determine the maximum and minimum intensities at the point P.

Sol.

(a) Both the sources produce maximum at the point P.

Thus, $I_{max} = I_0 = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$

Since the sources are identical, therefore, $I_1 = I_2 = I$.

$I_0 = 4I$ or $I = I_0/4$

(b) Now $I_1 = 0.64I = 0.16I_0$

And $I_2 = I = 0.25\,I_0$

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = 0.81\,I_0$$

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = 0.01\,I_0$$

(c) Now $I_1 = (1-0.64)I_0 = 0.36I = 0.09I_0$

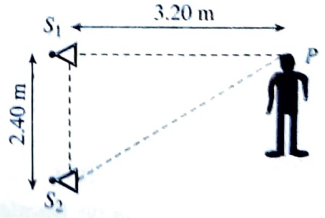
And $I_2 = I = 0.25\,I_0$

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = 0.64\,I_0$$

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = 0.04\,I_0$$

ILLUSTRATION 7.6

Two stereo speakers S_1 and S_2 are separated by a distance of 2.40 m. A person (P) is at a distance of 3.20 m directly in front of one of the speakers as shown in figure. Find the frequencies in the audible range (20–20,000 Hz) for which the listener will hear a minimum sound intensity. Speed of sound in air = 320 m/s.



Sol. Waves from S_1 to S_2 reach at point P.

Therefore, the path difference between waves reaching from S_1 to S_2 will be $\Delta = S_2P - S_1P$

Given $S_1P = 3.20$ m

$$S_2P = \sqrt{(2.40)^2 + (3.20)^2} = 4.0 \text{ m}$$

$$\therefore \quad \Delta = 4.0 - 3.20 = 0.80$$

For minima, path difference $\Delta = (2r-1)\frac{\lambda}{2}$; $r = 1, 2, 3, \ldots$

We have $\lambda = v/n$, where n is the frequency $\Delta = (2r-1)\frac{v}{2n}$

$$0.80 = (2r-1)\frac{v}{2n}$$

Frequency $n = \frac{(2r-1)v}{2\times 0.80} = (2r-1)\frac{320}{1.6}$ Hz $= (2r-1)200$ Hz

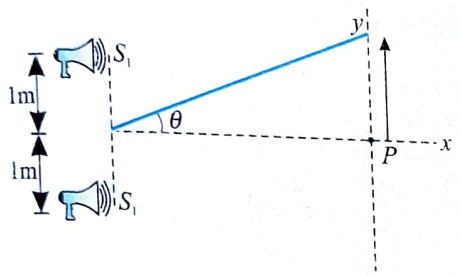
Therefore, the required frequencies are $n = (2r-1)\times 200$ with $r = 1, 2, \ldots 50$.

Putting the values of r in the above equation, we get

$n = 200$ Hz, 600 Hz, 1000 Hz, 1400 Hz, 1800 Hz, …, 19800 Hz

ILLUSTRATION 7.7

Two speakers S_1 and S_2 derived by the same amplifier and placed at $y = 1.0$ m and $y = -1.0$ m (figure) The speakers vibrate in phase at 600 Hz. A man stands at a point on the x-axis at a very large distance from the origin and starts moving parallel to the Y-axis. The speed of sound in air is 330 m/s.

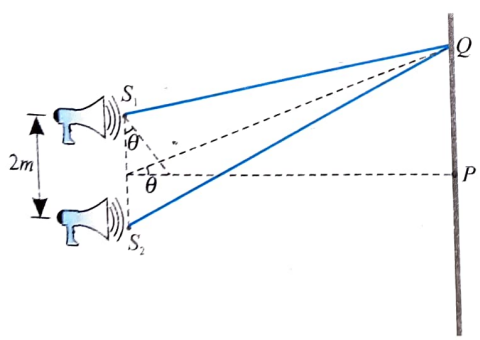


(a) At what angle will the intensity of sound drop to a minimum for the first time?

(b) At what angle will the sound intensity be maximum for the first time?

(c) If he continues to walk along the line, how many more maxima can he hear?

Sol. When the man is at a large distance from the speaker, the path differences between waves starting from S_1 and S_2 and reaching at Q will be $\Delta = S_2Q - S_1Q = S_1S_2 \sin\theta = 2\sin\theta \approx 2\tan\theta = 2\theta$



Also $\lambda = \frac{v}{n} = \frac{330}{600} = 0.55$ m

(a) For the first minima

Path difference $\Delta = \frac{\lambda}{2}$

$$2\theta = \frac{0.55}{2}$$

$$\theta = \frac{0.55}{4} = 0.1375 \text{ rad} = 7.9°$$

(b) For the first maxima, $\Delta = \lambda \Rightarrow 2\theta = 0.55$

$$\theta = \frac{0.55}{2} \text{ rad} = 15.8°$$

(c) Maximum path difference = 2 m

If order of maxima is r, then $2 = r\lambda$

$$r = \frac{2}{\lambda} = \frac{2}{0.55} = 3.6 \approx 3$$

Number of more maxima = 3 – 1 = 2

ILLUSTRATION 7.8

Figure shows a tube structure in which a sound signal is sent from one end and is received at the other end. The semicircular part has a radius of 20.0 cm. The frequency of the sound source can be varied electronically between 1000 and 4000 Hz. Find the frequencies at which maxima of intensity are detected. The speed of sound in air = 342 m/s.

Sol. The sound waves reach to detector by two paths simultaneously, one by straight path other by semicircular track. The wave through the straight part travels a distance, hence path length, $l_1 = 2 \times 20$ cm and the wave through the curved part travels a path length $l_2 = \pi(20 \text{ cm}) = 62.8$ cm before they meet again and travel to the receiver. The path difference between the two waves received is, therefore,

$$\Delta l = l_2 - l_1 = 62.8\text{cm} - 40 \text{ cm} = 22.8 \text{ cm} = 0.228 \text{ m}$$

The wavelength of either wave : $\lambda = \frac{v}{f} = \frac{342}{f}$. For constructive interference, path difference $\Delta\ell = N\lambda$, where N is an integer.

or $$0.228 = N\left(\frac{342}{f}\right)$$

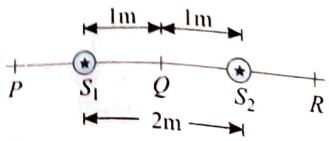
or $$f = N\left(\frac{342}{0.228}\right) = N(1500) \text{ Hz} = N(1500) \text{ Hz}$$

Thus, the frequencies within the specified range which cause maxima of intensity are 1500 Hz and 3000 Hz.

ILLUSTRATION 7.9

Two sound sources S_1 and S_2, separated by 2.0 m, vibrate according to equation $y_1 = 0.03 \sin \pi t$ and $y_2 = 0.02 \sin \pi t$ where y_1, y_2 and t are in SI. units. They send out waves of velocity 1.5 m/s. Calculate the amplitude of the resultant motion of the particle co-linear with S_1 and S_2 and located at a point **(a)** to the right of S_2, **(b)** to the left of S_1 and **(c)** in the middle of S_1 and S_2.

Sol. The situation can be represented as shown in figure.



The frequency of both sources is $f = \frac{\pi}{2\pi} = \frac{1}{2} = 0.5$ Hz

Now wavelength of each wave $\lambda = \frac{v}{n} = \frac{1.5}{0.5} = 3.0$ m

(a) The path difference for all points P_2 to the right of S_2 is

$$\Delta x = (S_1P_2 - S_2P_2) = S_1S_2 = 2\text{m}$$

Phase difference $\phi = \frac{2\pi}{\lambda} \times$ Path difference

$$= \frac{2\pi}{\lambda} \times 2.0 = \frac{4\pi}{3}$$

The oscillations y_1 and y_2 have amplitudes $A_1 = 0.03$ m and $A_2 = 0.02$ m respectively.

The resultant amplitude for this point is given by

$$A_R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi}$$

$$= \sqrt{(0.03)^2 + (0.02)^2 + 2\times 0.03\times 0.02\times\cos(4\pi/3)}$$

Solving we get $R = 0.0265$m

(b) The path difference for all points P, to the left of source S_1

$$\Delta x = (S_2P - S_1P) = S_1S_2 = 2.0\text{m}$$

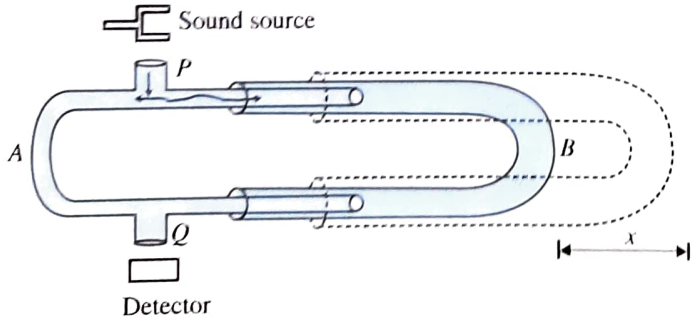
Hence the resultant amplitude for all points to the left of S_1 is also 0.0265 m

(c) For a point Q, midway between S_1 and S_2, the path difference is zero i.e., $\phi = 0$. Hence constructive interference takes place at Q, thus amplitude at this point is maximum and given as

$$R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2} = A_1 + A_2 = 0.03 + 0.02 = 0.05\text{m}$$

QUINCK'S TUBE

This is an apparatus used to demonstrate the phenomenon of interference and also used to measure velocity of sound in air. This is made up of two U-tubes A and B as shown in figure. Here the tube B can slide in and out from the tube A. There are two openings P and Q in the tube A. At opening P, a tuning fork or a sound source of known frequency n_0 is placed and at the other opening a detector is placed to detect the resultant sound of interference occurred due to superposition of two sound waves coming from the tubes A and B.



Initially tube B is adjusted so that detector detects a maximum. At this instant if length of paths covered by the two waves from P to Q from the side of A and side of B are l_1 and l_2, respectively then for constructive interference we must have

$$l_2 - l_1 = N\lambda \qquad \ldots(\text{i})$$

If now tube B is further pulled out by a distance x so that next maximum is obtained and the length of path from the side of B is l_2' then we have

$$l_2' = l_2 + 2x \qquad \ldots(\text{ii})$$

where x is the displacement of the tube. For next constructive interference of sound at point Q, we have

$$l_2' - l_1 = (N+1)\lambda \qquad \ldots(\text{iii})$$

From Eqs. (i), (ii) and (iii), we get $l_2' - l_2 = \lambda$

or $2x = \lambda$ or $x = \frac{\lambda}{2}$ $\qquad \ldots(\text{iv})$

Thus, by experiment we get the wavelength of sound as for two successive points of constructive interference, the path difference must be λ. As the tube B is pulled out by x, this introduces a path difference $2x$ in the path of sound wave through tube B. If the frequency of the source is n_0, the velocity of sound in the air filled in tube can be given as

$$V = n_0\lambda = 2n_0x \qquad \ldots(\text{v})$$

ILLUSTRATION 7.10

In a Quinck's experiment, the sound intensity being detected at an appropriate point, changes from minimum to maximum for the second time, when the slidable tube is drawn apart by 9.0 cm. If the speed of sound in air be 336 m/s, then what is the frequency of this sounding source?

Sol. As the sliding tube is pulled out by a distance of 9.0 cm, the path difference effected is 2×9.0 cm $= 18.0$ cm.

Initially, the position of the tube corresponds to the minimum intensity. When maximum intensity is heard for the first time, the path difference increased by $\lambda/2$ and when heard for the second time, there is an additional path difference of λ.

Therefore, the total path difference $= \frac{\lambda}{2} + \lambda = \frac{3\lambda}{2}$

$$\frac{3\lambda}{2} = 18.0 \text{ cm}$$

$$\lambda = 12.0 \text{ cm}$$

The frequency of sound emitted by the source will be

$$f = \frac{v}{\lambda} = \frac{336\text{ m/s}}{12.0\times10^{-2}\text{ m}} = 2.8 \text{ kHz}$$

ILLUSTRATION 7.11

In an experiment related to interference, Quinck's tube was employed to determine the speed of sound in air. A tuning fork of frequency 1328 Hz was used as the sounding source. Initially, the apparatus, yielded a maximum sound intensity. Later, when the slidable tube was drawn by a distance of 12.5 cm, the intensity was found to be maximum for the first time. Determine the speed of sound in air.

Sol. A distance of 12.5 cm drawn aside implies a double path difference of 25 cm.

Since the shortest path difference, which enables the intensity to become maximum once again is, obviously, λ.

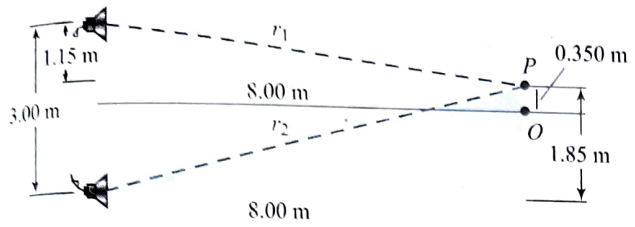
so $\lambda = 25$ cm

$$v = 1328\text{ s}^{-1} \times 25\text{ cm} = 332\text{ m/s}$$

Therefore, the speed of sound in air is 332 m/s.

ILLUSTRATION 7.12

Two identical loudspeakers placed 3.00 m apart are driven by the same oscillator as shown in figure. A listener is originally at point O, located 8.00 m from the centre of the line connecting the two speakers. The listener then moves to point P, which is a perpendicular distance 0.350 m from O, and she experiences the first minimum sound intensity. What is the frequency of the oscillator?



Sol. In this example, signal representing the sound is electrically split and sent to two different loudspeakers. After leaving the speakers, the sound waves recombine at the position of the listener. Despite the difference in how the splitting occurs, the path difference discussion related to the figure can be applied here.

Because the sound waves from two separate sources combine, we apply the waves in interference analysis model.

Figure shows the physical arrangement of the speakers, along with two shaded right triangles that can be drawn on the basis of the lengths described in the problem. The first minimum occurs when the two waves reaching the listener at point P are 180° out of phase, in other words, when their path difference Δr equals $\lambda/2$.

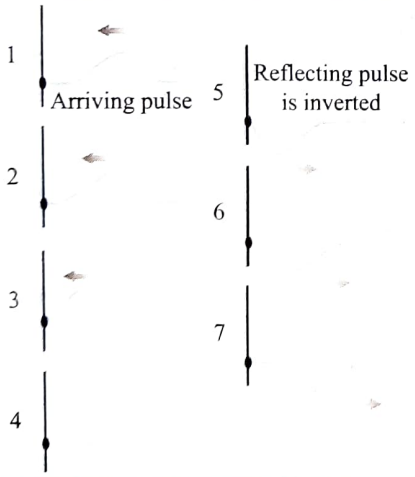
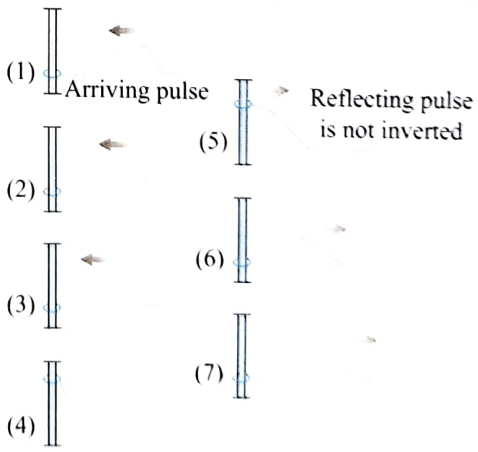
From the shaded triangles, find the path lengths from the speakers to the listener:

$$r_1 = \sqrt{(8.00\text{ m})^2 + (1.15\text{ m})^2} = 8.08\text{ m}$$

$$r_2 = \sqrt{(8.00\text{ m})^2 + (1.85\text{ m})^2} = 8.21\text{ m}$$

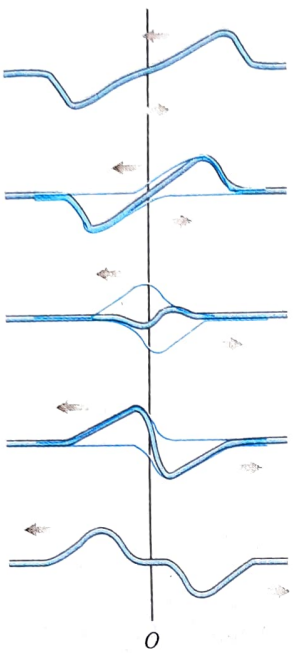
Hence, the path difference is $r_2 - r_1 = 0.132$ m. Because this path difference must equal $\lambda/2$ for the first minimum, $\lambda = 0.26$ m. To obtain the oscillator frequency, use equation $v = \lambda f$, where v is the speed of sound in air, 343 m/s.

REFLECTION OF WAVES AT FIXED END AND FREE END

Reflection of waves—Fixed end	Reflection of waves—Free end
• When the wave reaches the fixed end it exerts an upward pull on the end. According to Newton's third law, the fixed point exerts an equal and opposite force downward on the string. Thus, there is phase change of π for the reflected wave, the wave is inverted as shown in the figure. • Whenever a travelling wave reaches a boundary, some or all of the wave is reflected. • When it is reflected from a fixed end, the wave is inverted.	• If the end of the string is free to move vertically, the free end overshoots twice the amplitude as shown in figure. • When a travelling wave reaches a boundary, all or part of it is reflected. • When reflected from a free end, the pulse is not inverted.
	
The formation of the reflected pulse is similar to the overlap of two pulses travelling in opposite direction. Figure shows two pulses with the same shape, one inverted with respect to the other, travelling in opposite directions. As the pulses overlap and pass each other, the total displacement of the string is the algebraic sum of the displacements at that point in the individual pulses. Because these two pulses have the same shape, the total displacement at point O in the middle of the figure is zero at all times. Thus the motion of the left half of the string would be the same if we cut	Figure shows two pulses with the same shape, travelling in opposite directions but not inverted relative to each other. The displacement at point O in the middle of the figure is not zero, but the slope of the string at this point is always zero. According to equation, this corresponds to the absence of any transverse force at this point. In this case the motion of the left half of the string would be the same as if we cut the string at point O and attach the end to a frictionless sliding ring that maintains

Reflection of waves—Fixed end

the string at point O, threw away the right side, and held the end at O fixed. The two pulses on the left side and on the right side then correspond to the incident and reflected pulses, combining so that the total displacement at O is always zero. For this to occur, the reflected pulse must be inverted relative to the incident pulse.



Reflection of waves—Free end

tension without exerting any transverse force. In other words, this situation corresponds to reflection of a pulse at a free end of a string at point O. In this case the reflected pulse is not inverted.

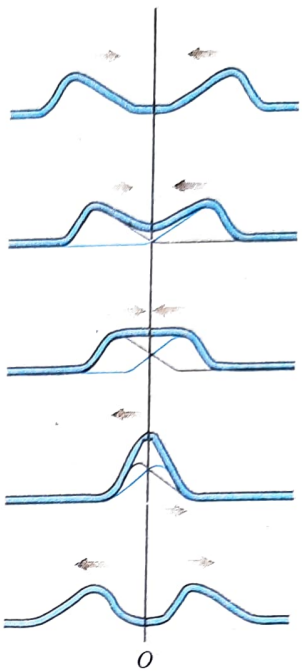
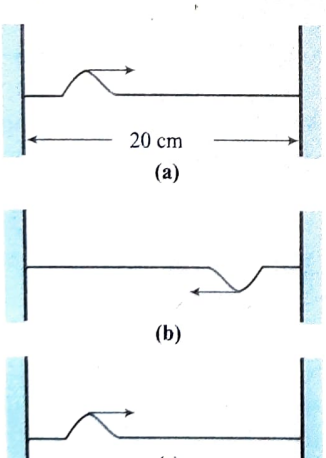


ILLUSTRATION 7.13

A string of length 20 cm and linear mass density 0.40 g/cm is fixed at both ends and is kept under a tension of 16 N. A wave pulse is produced at $t = 0$ near an end as shown in Fig. (b), which travels towards the other end. When will the string have the shape shown in the Fig. (c).



Sol. Given $\mu = 0.40$ g/cm $= \dfrac{0.40}{1000} \times 100 = 0.040$ kg/m

Wave speed in the stretched string, $v = \sqrt{\dfrac{F}{\mu}} = \sqrt{\dfrac{16}{0.040}} = 20$ m/s

The string will have the same shape after the pulse travels a distance = 20 + 20 = 40 cm

Thus time period $t = 0.40/20 = 0.02$ s

ILLUSTRATION 7.14

Consider a string fixed at one end. A travelling wave given by the wave equation $y = A \sin(\omega t - kx)$ is incident on it.

Show that at the fixed end of a string the wave suffers a phase change of π, i.e., as it travels back as if the wave is inverted.

Sol. Wave equation of incident wave is

$y_1 = A \sin(\omega t - kx)$ (positive x-direction)

When the wave strikes the fixed end it must be reflected. The wave will now be travelling in the negative x-direction, and so its equation is

$y_2 = A \sin(\omega t + kx + \phi)$ (negative x-direction)

The phase constant has been added to account for any phase change after reflection. Let us take $x = 0$ at the fixed end. The behaviour of a wave at particular positions is governed by appropriate boundary conditions. For fixed ends, the boundary condition is that the end point is a node.

The resultant motion due to incident and reflected wave is

$y = y_1 + y_2$

$y = A[\sin(\omega t - kx) + \sin(\omega t + kx + \phi)]$ ...(i)

The boundary condition that must be satisfied by the resultant wave on string is $y = 0$ at $x = 0$

On substituting these values in Eq. (i), we obtain

$\sin\omega t + \sin(\omega t + \phi) = 0$

$\sin\omega t = -\sin(\omega t + \phi)$ ...(ii)

Equation (ii) must be satisfied at all times. For the sake of convenience, we take $\omega t = \pi/2$.

$$\sin\left(\frac{\pi}{2} + \phi\right) = -1$$

$\sin\theta = -1$ implies θ can have any of the following values.

$3\pi/2, 7\pi/2, 11\pi/2, \ldots$

Therefore ϕ can have any of the values $\pi, 3\pi, 5\pi$, etc. All these values are physically possible and distinguishable. We choose the simplest one, so $\phi = \pi$.

ILLUSTRATION 7.15

Consider the following wave functions:

(a) $y = A\sin(\omega t - kx)$, (b) $y = A\sin(kx - \omega t)$

(c) $y = A\cos(\omega t - kx)$, (d) $y = A\cos(kx - \omega t)$

(e) $y = A\sin(\omega t + kx)$, (f) $y = A\cos(\omega t + kx)$

Write the equations of reflected wave after reflection from a free and a fixed boundary. Also find the resulting stationary waves formed by the superposition of its reflected wave.

Sol.

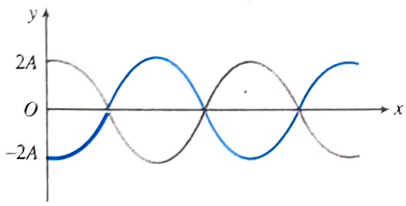
	Incident wave $\omega = \frac{2\pi}{T}, k = \frac{2\pi}{\lambda}$	Reflected wave from free boundary, $\phi = 0$	Reflected wave from fixed boundary, $\phi = \pi$
(a)	$y = A\sin(\omega t - kx)$	$y = A\sin(\omega t + kx)$	$y = A\sin(\omega t + kx + \pi)$ $= -A\sin(\omega t + kx)$
(b)	$y = A\sin(kx - \omega t)$	$y = A\sin(-kx - \omega t)$ $= -A\sin(kx + \omega t)$	$y = A\sin(-kx - \omega t + \pi)$ $= A\sin(kx + \omega t)$
(c)	$y = A\cos(\omega t - kx)$	$y = A\cos(\omega t + kx)$	$y = A\cos(\omega t + kx + \pi)$ $= -A\cos(\omega t + kx)$
(d)	$y = A\cos(kx - \omega t)$	$y = A\cos(-kx - \omega t)$ $= A\cos(kx + \omega t)$	$y = A\cos(-kx - \omega t + \pi)$ $= -A\cos(kx + \omega t)$
(e)	$y = A\sin(\omega t + kx)$	$y = A\sin(\omega t - kx)$	$y = A\sin(\omega t - kx + \pi)$ $= -A\sin(\omega t - kx)$
(f)	$y = A\cos(\omega t + kx)$	$y = A\cos(\omega t - kx)$	$y = A\cos(\omega t - kx + \pi)$ $= -A\cos(\omega t - kx)$

Stationary wave by superposition of reflected wave

$y_1 = A\sin(\omega t - kx)$ and $y_2 = A\sin(\omega t + kx)$

$y = y_1 + y_2 = A[\sin(\omega t - kx) + \sin(\omega t + kx)]$

$= 2A\cos kx \sin \omega t$



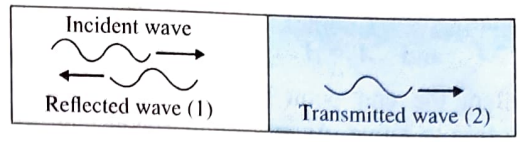
The similar treatment can be done for other combination of wave to get stationary wave.

REFLECTION AND REFRACTION OF WAVE

Consider a plane progressive wave travelling in a certain medium (say 1) having the equation

$y_1 = A_i\sin(\omega t - k_1 x)$ ...(i)

where A_i is the amplitude.



If A_r and A_t be the amplitudes of the reflected and transmitted waves, then their respective equations can be written as

$y_2 = A_r\sin(\omega t + k_1 x)$ ...(ii)

$y_3 = A_t\sin(\omega t - k_2 x)$ ...(iii)

For simplicity let us assume that the wave gets reflected at $x = 0$.

Since the wave is continuous, the resultant displacement at the two sides of the interface should be equal, i.e.,

$y_1 + y_2 = y_3$ at $x = 0$

From Eqs. (i), (ii) and (iii), we get on substituting $x = 0$

$A_i + A_r = A_t$ ...(iv)

From the condition of continuity of slope at the interface (i.e., $x = 0$), we have,

$$\frac{d(y_1)}{dx} + \frac{d(y_2)}{dx} = \frac{d(y_3)}{dx} \quad \text{...(v)}$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{d(y_1)}{dx} = -A_i k_1\cos(\omega t - k_1 x)$$

$$\frac{d(y_2)}{dx} = A_r k_1\cos(\omega t + k_1 x)$$

$$\frac{d(y_3)}{dx} = -A_t k_2\cos(\omega t - k_2 x)$$

Substituting these in Eq. (v) and remembering that $x = 0$, we get, $-A_i k_1 + A_r k_1 = -A_t k_2$

$$A_i\left(\frac{2\pi}{\lambda_1}\right) - A_r\left(\frac{2\pi}{\lambda_1}\right) = A_t\left(\frac{2\pi}{\lambda_2}\right) \quad \text{....(vi)}$$

Since $v = f\lambda$ and f is constant, so, $\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$ ...(vii)

From Eqs. (vi) and (vii), $A_i - A_r = \frac{v_1}{v_2}A_t$ ...(viii)

Solving Eqs. (iv) and (viii) for A_r and A_t, we have

$$A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)A_i \quad \text{and} \quad A_t = \left(\frac{2v_2}{v_2 + v_1}\right)A_i \quad \text{...(ix)}$$

Thus, we get the amplitudes of the reflected and transmitted waves, in terms of the amplitude of the incident wave.

When $v_1 > v_2$, i.e., medium 1 is rare and 2 is denser, A_r = negative and $|A_r|$ and $|A_t|$ both are individually less than $|A_i|$. Negative value of A_r indicates that the reflected wave suffers a phase change of π.

If opposite is the case, i.e., $v_1 < v_2$, both A_r and A_t are positive. Also $A_t > A_i$.

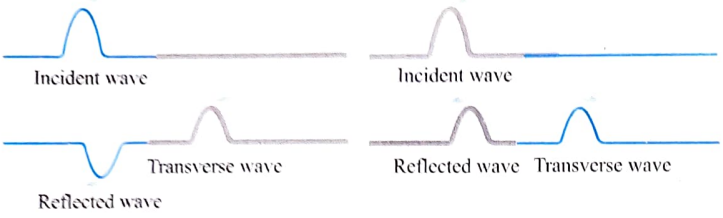
Fixed end of a string: Since the fixed end is equivalent to a string of infinite linear mass density, $v_2 = \sqrt{(T/\mu_2)} = 0$, and we obtain,

$A_t = 0$ and $A_r = -A_i$

Free end of a string: In this case $\mu_2 \to 0$, so $v_2 \to \infty$ and we can show that,

$$A_t = 2A_i \quad \text{and} \quad A_r = A_i$$

Quite often, the end point is neither completely fixed nor completely free to move. As an example, consider a light string attached to a heavier string as shown in Fig. (a). If a wave pulse is produced on the light string moving towards the junction, a part of the wave is reflected and a part is transmitted on the heavier string. The reflected wave is inverted with respect to the original one.



(a) (b)

On the other hand, if the wave is produced on the heavier string [Fig. (b)] which moves towards the junction, a part will be reflected and a part transmitted, no inversion of wave shape will take place.

The rule about the inversion at reflection may be stated in terms of the wave velocity. The wave velocity is smaller for the heavier string ($v = \sqrt{\frac{F}{\mu}}$) and larger for the lighter string. The above observation may be stated as follows.

If a wave enters a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.

ILLUSTRATION 7.16

A progressive wave gets reflected at a boundary such that the ratio of amplitudes of the reflected and incident wave is 1:2. Find the percentage of energy transmitted.

Sol. Given $\frac{A_r}{A_i} = \frac{1}{2}$ $\Rightarrow$ Ratio of intensities is $\frac{I_1}{I_2} = \left(\frac{1}{2}\right)^2$

Hence the fraction of energy transmitted $E_1 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

The percentage of energy transmitted (of course per unit time) is $3/4 \times 100 = 75\%$.

ILLUSTRATION 7.17

A progressive wave travels in a medium M_1 and enters into another medium M_2 in which its speed decreases to 75%. What is the ratio of the amplitude of the

(a) Reflected and the incident waves, and

(b) Transmitted and the incident waves?

Sol. Let A_i, A_r and A_t be the amplitudes of the incidents, reflected, and transmitted waves.

Given that, velocity in the medium refracted is 75% of that in the initial medium.

$$v_2 = \frac{3}{4} v_1$$

From Eqs. (viii) and (ix)

(a) $$\frac{A_r}{A_i} = \frac{v_2 - v_1}{v_2 + v_1} = \frac{\frac{v_2}{v_1} - 1}{\frac{v_2}{v_1} + 1} = \frac{\frac{3}{4} - 1}{\frac{3}{4} + 1} = -\frac{1}{7}$$

i.e., the required ratio is $\left|\frac{A_r}{A_i}\right| = 1:7$

(b) $$\frac{A_t}{A_i} = \frac{2v_2}{v_2 + v_1} = \frac{2v_2/v_1}{\frac{v_2}{v_1} + 1} = \frac{2(3/4)}{\left(\frac{3}{4}\right) + 1} = \frac{6}{7}$$

i.e., the required ratio is $\left|\frac{A_t}{A_i}\right| = 6:7$

ILLUSTRATION 7.18

A long wire PQR is made by joining two wires PQ and QR of equal radii. PQ has length 4.8 m and mass 0.06 kg. QR has length 2.56 m and mass 0.2 kg. The wire PQR is under a tension of 80 N. A sinusoidal wave pulse of amplitude 3.5 cm is sent along the wire PQ from the end? No power is dissipated during the propagation of the wave-pulse. Calculate

(a) The time taken by the wave pulse to reach the other end R of the wire, and

(b) The amplitude of the reflected and transmitted wave pulses after the incident wave pulse crosses the joint Q.

Sol. Given that $m_1 = 0.06$ kg, $m_2 = 0.2$ kg

Let m' be the mass per unit length then $m'_1 = 0.0125$ kg/m. $m'_2 = 0.078125$ kg/m.

Wire PQR is under a tension of 80 N = T_0. A sinusoidal wave pulse is sent from P.

(a) v_1 = velocity of wave on PQ $= \sqrt{\frac{T}{m'_1}} = \sqrt{\frac{80}{0.0125}} = 80$ m/s

v_2 = velocity of wave on QR $= \sqrt{\frac{T}{m'_2}} = \sqrt{\frac{80}{0.078125}} = 32$ m/s

Total time taken for wave pulse to reach from P to R

$$= \frac{PQ}{v_1} + \frac{QR}{v_2} = \left(\frac{4.8}{80} + \frac{2.56}{32}\right) \text{s} = 0.06 + 0.08 = 0.14 \text{s}$$

(b) Amplitude of reflected wave:

$$A_r = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_i = \left(\frac{32 - 80}{32 + 80}\right) 3.5 = -1.5 \text{ cm}$$

A_r is –ve, so reflected wave is inverted

Amplitude of transmitted wave:

$$A_t = \left(\frac{2v_2}{v_2 + v_1}\right) A_i = \left(\frac{2 \times 32}{32 + 80}\right) 3.5 = 2 \text{ cm}$$

ILLUSTRATION 7.19

A harmonic wave is travelling on string 1. At a junction with string 2 it is partly reflected and partly transmitted. The linear mass density of the second string is four times that of the first string, and that the boundary between the two strings is

at $x = 0$. If the expression for the incident wave is $y_i = A_i \cos(k_1 x - \omega_1 t)$.

(a) What are the expressions for the transmitted and the reflected waves in terms of A_i, k_1 and ω_1?

(b) Show that the average power carried by the incident wave is equal to the sum of the average power carried by the transmitted and reflected waves.

Sol.

(a) Since $v = \sqrt{T/\mu}$, $T_2 = T_1$ and $\mu_2 = 4\mu_1$

We have, $v_2 = \frac{v_1}{2}$ ...(i)

Since the frequency does not change, that is,

$\omega_1 = \omega_2$ ... (ii)

Also, because $k = (\omega/v)$, the numbers of the harmonic waves in the two strings are related by

$$k_2 = \frac{\omega_2}{v_2} = \frac{\omega_1}{v_1/2} = 2\frac{\omega_1}{v_1} = 2k_1 \quad \text{...(iii)}$$

The amplitudes are $A_t = \left(\frac{2v_2}{v_1 + v_2}\right) A_i$

$$= \left[\frac{2(v_1/2)}{v_1 + (v_1/2)}\right] A_i = \frac{2}{3} A_i \quad \text{...(iv)}$$

and $A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) A_i = \left[\frac{(v_1/2) - v_1}{v_1 + (v_1/2)}\right] A_i = \frac{-A_i}{3}$ ...(v)

Now with Eqs. (ii), (iii) and (iv), the transmitted wave can be written as

$$y_t = (2/3) A_i \cos(2k_1 x - \omega_1 t)$$

Similarly the reflected wave can be expressed as

$$y_x = -\frac{A_i}{3}\cos(k_1 x + \omega_1 t) = \frac{A_i}{3}\cos(k_1 x + \omega_1 t + \pi)$$

(b) The average power of a harmonic wave on a string is given by

$$P = \frac{1}{2}\rho A^2 \omega^2 s v = \frac{1}{2} A^2 \omega^2 \mu v \quad (\text{as } \rho s = \mu)$$

Now $P_i = \frac{1}{2}\omega_1^2 A_i^2 \mu_1 v_1$ ...(vi)

$$P_t = \frac{1}{2}\omega_1^2 \left(\frac{2}{3} A_i\right)^2 (4\mu_1)\left(\frac{v_1}{2}\right) = \frac{4}{9}\omega_1^2 A_i^2 \mu_1 v_1 \quad \text{... (vii)}$$

$$P_r = \frac{1}{2}\omega_2^2 \left(-\frac{A_i}{3}\right)^2 (\mu_1)(v_1) = \frac{1}{18}\omega_1^2 A_i^2 \mu_1 v_1 \quad \text{... (viii)}$$

From Eqs. (vi), (vii) and (viii), we can show that $P_i = P_t + P_r$.

ILLUSTRATION 7.20

In a stationary wave pattern that forms as a result of reflection of waves from an obstacle, the ratio of the amplitude at an antinode and a node is 1.5. What percentage of the energy passes across the obstacle?

Sol. As we have studied that when incident wave and reflected wave superimpose to produce stationary wave, the ratio of amplitudes at antinode and at node is given by

$$\frac{A_{max}}{A_{min}} = \frac{A_i + A_r}{A_i - A_r}$$

This ratio is given as 1.5 or 3/2.

$$\frac{A_i + A_r}{A_i - A_r} = \frac{3}{2} \quad \text{or} \quad \frac{1 + \frac{A_r}{A_i}}{1 - \frac{A_r}{A_i}} = \frac{3}{2}$$

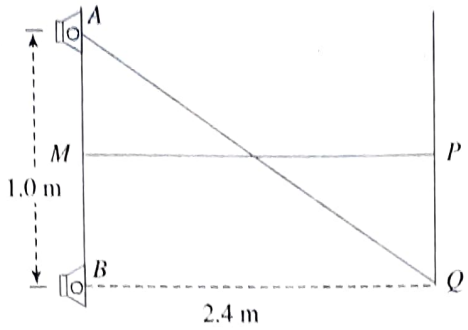
Solving this equation, we get

$$\frac{A_r}{A_i} = \frac{1}{5} \Rightarrow \frac{I_r}{I_i} = \left(\frac{A_r}{A_i}\right)^2 = \frac{1}{25} \quad \text{or,} \quad I_r = 0.04\, I_i$$

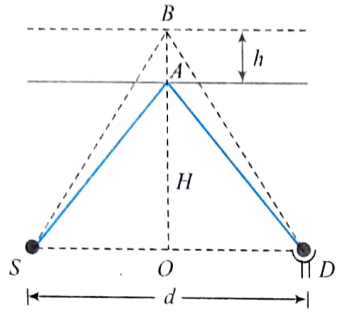
this means 4% of the incident energy is reflected or 96% energy passes across the obstacle.

CONCEPT APPLICATION EXERCISE 7.1

1. Two sound waves with amplitude 4 cm and 3 cm interfere with a phase difference of
   (a) 0 **(b)** $\pi/3$ **(c)** $\pi/2$ **(d)** π
   Find the resultant amplitude in each case.
2. Find the resultant amplitude and the phase difference between the resultant wave and the first wave, in the event the following waves interfere at a point: $y_1 = (3\text{ cm})\sin\omega t$, $y_2 = (4\text{ cm})\sin\left(\omega t + \frac{\pi}{2}\right)$; $y_3 = (5\text{cm})\sin(\omega t + \pi)$
3. A and B are two point sources of sound (of same frequency) and are kept at a separation. At a point P, the intensity of sound is observed to be I_0 when only source A is put on. With only B on the intensity is observed to be $2I_0$. The distance AP is higher than distance BP by half the wavelength of the sound. Find the intensity recorded at P with both sources on. Give your answer for following cases:
   (a) The sources are coherent and in phase.
   (b) The sources are coherent and 180° out of phase.
   (c) The sources are incoherent.
4. Two identical sinusoidal waves with wavelengths of 3.00 m travel in the same direction at a speed of 2.00 m/s. The second wave originates from the same point as the first, but at a later time. The amplitude of the resultant wave is the same as that of each of the two initial waves. Determine the minimum possible time interval between the starting moments of the two waves.
5. Two small loudspeakers A, B (1 m apart) are connected to the same oscillator so that both emit sound waves of frequency 1700 Hz in phase. A sensitive detector, moving parallel to the line AB along PQ 2.40 m away, detects a maximum wave at P on the perpendicular bisector MP of AB and another maximum wave when it first reaches a point Q directly opposite to B. Calculate the speed of the sound waves in air.



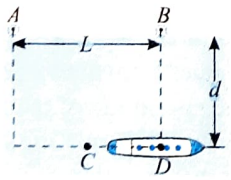
6. A source and a detector D of high frequency waves are a distance d apart on the ground. Maximum signal is received at D when the reflecting layer is at a height H. When the layer rises a distance h, no signal is detected at D. Neglecting absorption in the atmosphere, find the relation between d, h, H, and the wavelength λ of the waves.



7. Two waves have the same frequency. The first has intensity I_0. The second has intensity $4I_0$ and lags behind the first in phase by $\pi/2$. When they meet, find the resultant intensity, and the phase relationship of the resultant wave with the first wave.

8. A travelling wave has speeds 50 m/s and 200 m/s in two different media A and B. Such a wave travelling through A, gets incident normally on a plane boundary, separating A and B. Find the ratio of amplitudes of the reflected and transmitted waves.

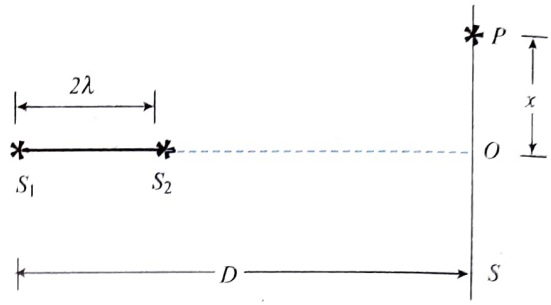
9. The ship in figure travels along a straight line parallel to the shore and a distance $d = 600$ m from it. The ship's radio receives simultaneous signals of the same frequency from antennas A and B, separated by a distance $L = 800$ m. The signals interfere constructively at point C, which is equidistant from A and B. The signal goes through the first minimum at point D, which is directly outward from the shore from point B. Determine the wavelength of the radio waves.



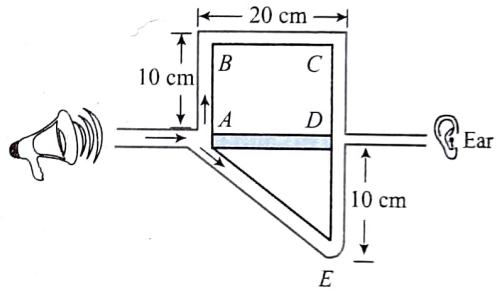
10. Two wires of different linear mass densities are soldered together end to end and then stretched under a tension F. The wave speed in the first wire is thrice that in the second. If a harmonic wave travelling in the first wire is incident on the junction of the wires and if the amplitude of the incident wave is $A = \sqrt{13}$ cm, find the amplitude of reflected wave.

11. The pulse shown in figure has a speed of 5 cm/s. If the linear mass density of the right string is 0.5 that of the left string, find the ratio of height of the transmitted pulse to that of incident pulse.

12. Two coherent narrow slits emitting wavelength λ in the same phase are placed parallel to each other at a small separation of 2λ, the sound is detected by moving a detector on the screen S at a distance D ($\gg\lambda$) from the slit S_1 as shown in the figure. Find the distance x such that the intensity at P is equal to the intensity at O.



13. Figure shows a tube structure in which signal is sent from one end and is received at the other end. The frequency of the sound source can be varied electronically between 2000 and 5000 Hz. Find the frequencies at which maxima of intensity are detected. The speed of sound in air 340 m/s.



14. In Quinck's acoustic interferometer, it is found that the sound intensity has a minimum value of 100 units at one position of the sliding tube, and continuously climbs to a maximum of 900 units at a second position 1.65 cm from the first. Find **(a)** the frequency of the sound emitted by the source and **(b)** the relative amplitudes of the two waves arriving at the detector. Velocity of sound in air = 340 m/s.

ANSWERS

1. (a) 7 cm (b) $\sqrt{37}$ cm (c) 5 cm (d) 1 cm

2. $2\sqrt{5}$ cm, $\frac{\pi}{2}+\tan^{-1}\left(\frac{1}{2}\right)$

3. (a) $0.2I_0$ (b) $3(3+2\sqrt{2})I_0$ (c) $3I_0$ **4.** 0.500 s

5. 340 m/s **6.** $\lambda = 2\sqrt{4(H+h)^2+d^2} - 2\sqrt{4H^2+d^2}$

7. $5I_0$, The resultant wave lags behind the Ist wave by phase angle $\phi = \tan^{-1}(2)$.

8. $\frac{3}{8}$ **9.** 800 m **10.** $\frac{\sqrt{13}}{2}$ cm **11.** $\frac{2\sqrt{2}}{1+\sqrt{2}}$

12. $\sqrt{3}\,D$ **13.** 4450 Hz **14.** (a) 5152 Hz (b) $\frac{2}{1}$

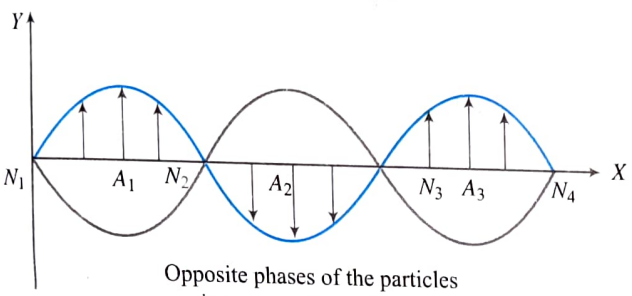
CHARACTERISTICS OF STATIONARY WAVES

(i) In a stationary wave, the disturbance does not move in any direction. The conditions of crests and troughs merely appear and disappear in fixed positions to be followed by opposite conditions after every $T/2$.

(ii) All the particles of the medium, except those at nodes, execute simple harmonic motion with the particle of the wave about their mean position.

(iii) During the formation of a stationary wave, the medium is broken into loops between equally spaced points called nodes which remain at rest and in between them are points of maximum displacement called antinodes.

(iv) The amplitude of the particles are different at different points. The amplitude varies gradually from zero at the nodes to the maximum at the antinodes.



Opposite phases of the particles in consecutive segments

(v) The maximum velocity is different at different points. Its value is zero at the nodes and gradually increases towards the antinodes. All the particles attain their maximum velocities simultaneously when they pass through their mean positions.

(vi) All the particles in a particular segment between two nodes vibrate in the same phase but the particles in the neighbouring segments vibrate in opposite phases, as shown in figure.

(vii) The energy becomes alternately wholly potential and wholly kinetic twice in each cycle. It is wholly potential when particles are at their positions of maximum displacement and wholly kinetic when the particle pass through the mean positions.

(viii) A stationary wave has the same wavelength and time period as the component waves.

(ix) The distance between two consecutive nodes and antinodes is $\lambda/2$. The distance between nodes and next antinode is $\lambda/4$.

ILLUSTRATION 7.21

Can two waves of the same frequency and amplitude travelling in the same direction give rise to a stationary wave after superposition?

Sol. Let the two superposing waves be

$y_1 = A \sin(\omega t - kx)$

$y_2 = A \sin(\omega t - kx + \phi)$

where ϕ is the phase difference between the two waves.

From principle of superposition the resultant wave function is

$$y = y_1 + y_2 = A \sin(\omega t - kx) + A \sin(\omega t - kx + \phi)$$

$$= 2A \cos\frac{\phi}{2} \sin\left(\omega t - kx + \frac{\phi}{2}\right)$$

This is the expression for a travelling wave with amplitude $2A \cos \phi/2$. Hence superposition of two waves travelling in the same direction give rise to another travelling wave with different amplitude, which depends on the phase difference between the two waves.

ILLUSTRATION 7.22

Two travelling waves of equal amplitudes and equal frequencies move in opposite direction along a string. They interfere to produce a standing wave having the equation.

$y = A \cos kx \sin \omega t$

in which $A = 1.0$ mm, $k = 1.57$ cm^{-1} and $\omega = 78.5$ s^{-1}. **(a)** Find the velocity and amplitude of the component travelling waves. **(b)** Find the node closest to the origin in the region $x > 0$. **(c)** Find the antinode closest to the origin in the region $x > 0$. **(d)** Find the amplitude of the particle at $x = 2.33$ cm.

Sol.

(a) The standing wave is formed by the superposition of the waves

$$y_1 = \frac{A}{2} \sin(\omega t - kx) \text{ and } y_2 = \frac{A}{2} \sin(\omega t + kx)$$

The wave velocity (magnitude) of either of the waves is

$$v = \frac{\omega}{k} = \frac{78.5 \text{ s}^{-1}}{1.57 \text{ cm}^{-1}} = 50 \text{ cm/s};$$

Amplitude = 0.5 mm

(b) For a node, $\cos kx = 0$.

The smallest positive x satisfying this relation is given by

$kx = \pi/2$

or, $$x = \frac{\pi}{2k} = \frac{3.14}{2 \times 1.57 \text{ cm}^{-1}} = 1 \text{ cm}$$

(c) For an antinode, $|\cos kx| = 1$.

The smallest positive x satisfying this relation is given by

$kx = \pi$

or, $$x = \frac{\pi}{k} = 2 \text{ cm}$$

(d) The amplitude of vibration of the particle at x is given by $|A \cos kx|$. For the given point,

$$kx = (1.57 \text{ cm}^{-1})(2.33 \text{ cm}) = 1.57\left[2 + \frac{1}{3}\right]$$

$$= \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

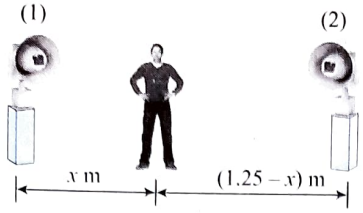
Thus, the amplitude will be

$$(1.0 \text{ mm}) |\cos(\pi + \pi/6)| = \frac{\sqrt{3}}{2} \text{ mm} = 0.86 \text{ mm}$$

ILLUSTRATION 7.23

Two identical loudspeakers are driven in phase by a common oscillator at 800 Hz and face each other at a distance of 1.25 m. Locate the points along the line joining the two speakers where relative minima of sound pressure amplitude would be expected. (Use $v = 343$ m/s.)

Sol. Let two loudspeakers (1) and (2) be separated by a distance 1.25 m, and observer moves between the two speakers along the line joining two as shown in figure.



Two identical waves moving in opposite directions constitute a standing wave. We must find the nodes.

The wavelength is $\lambda = \frac{v}{f} = \frac{343 \text{ m/s}}{800 \text{ Hz}} = 0.429 \text{ m}$

The two waves moving in opposite directions along the line between the two speakers will add to produce a standing wave with this distance between nodes.

Distance node to node = $\lambda/2 = 0.214$ m

Because the speakers vibrate in phase, air compressions from each will simultaneously reach the point halfway between the speakers, to produce antinode of pressure here. A node of pressure will be located at this distance on either side of the midpoint.

Distance node to antinode = $\lambda/4 = 0.107$ m

Therefore nodes of sound pressure will appear at these distances from either speaker:

$[1/2(1.25 \text{ m}) + 0.107 \text{ m} = 0.732 \text{ m}$

and $1/2(1.25 \text{ m}) - 0.107 \text{ m} = 0.518 \text{ m}]$

The standing wave contains a chain of equally spaced nodes at distances from either of the speakers

$0.732 \text{ m} + 0.214 \text{ m} = 0.946 \text{ m}$

$0.947 \text{ m} + 0.214 \text{ m} = 1.16 \text{ m}$

also at $0.518 \text{ m} - 0.214 \text{ m} = 0.303 \text{ m}$

$0.303 \text{ m} - 0.214 \text{ m} = 0.089 \text{ m}$

The standing wave exists only along the line segment between the speakers. No nodes or antinodes appear at distances greater than 1.25 m or less than 0, because waves add to give standing wave only if they are travelling in opposite directions and not in the same direction. Thus, the distances from either of the speakers to the nodes of pressure between the speakers are 0.089 m, 0.303 m, 0.518 m, 0.732 m, 0.946 m and 1.16 m.

ILLUSTRATION 7.24

Two sinusoidal waves combining in a medium are described by the wave functions

$y_1 = (3.0 \text{ cm}) \sin\pi(x + 0.60t)$

$y_2 = (3.0 \text{ cm}) \sin\pi(x - 0.60t)$

where x is in centimetres and t is in seconds. Determine the maximum transverse position of an element of the medium at **(a)** $x = 0.250$ cm, **(b)** $x = 0.500$ cm and **(c)** $x = 1.50$ cm. **(d)** Find the three smallest values of x corresponding to antinodes.

Sol. The answers for maximum transverse position must be between 6 cm and 0. The antinodes are separated by half a wavelength, which we expect to be a couple of centimetres.

According to the waves in interference model, we write the function $y_1 + y_2$ and start evaluating things.

We get $y = y_1 + y_2 = (6.0 \text{ cm}) \sin(\pi x) \cos(0.60\pi t)$

Since cos (0) = 1, we can find the maximum value of y b setting $t = 0$.

$y_{max}(x) = y_1 + y_2 = (6.0 \text{ cm}) \sin(\pi x)$

(a) At $x = 0.250$ cm, $y_{max} = (6.0 \text{ cm}) \sin(0.250\pi) = 4.24$ cm

(b) At $x = 0.50$cm, $y_{max} = (6.0 \text{ cm}) \sin(0.500\pi) = 6.00$ cm

(c) At $x = 1.50$ cm, $y_{max} = |(6.0 \text{ cm}) \sin(1.50\pi)| = +6.00$ cm

(d) The antinodes occur when $x = n\lambda/4$ for $n = 1, 3, 5, \ldots$.

But $k = 2\pi/\lambda = \pi$, so $\lambda = 2.00$ cm

and $x_1 = \lambda/4 = (2.00 \text{ cm})/4 = 0.500$ cm

$x_2 = 3\lambda/4 = 3(2.00 \text{ cm})/4 = 1.50$ cm

$x_3 = 5\lambda/4 = 5(2.00 \text{ cm})/4 = 2.50$ cm

Two of our answers in (d) can be read from the way th amplitude had its largest possible value in parts (b) and (c). Not again that an amplitude is defined to be always positive, as th maximum absolute value of wave function.

STATIONARY WAVES

In case of interference, we discussed two sinusoidal waves of th same wavelength and amplitude traveling in the same directio along a stretched string. In this section we will learn, wha happens if they travel in opposite directions? We can again fin the resultant wave by applying the superposition principle.

Let us learn the situation graphically. The given figure show two combining waves, one travelling to the left in Fig. (a), the other to the right in Figs. (b) and (c) show the summation of waves. obtained by applying principle of superposition graphically.

(i) At $t = 0$, the positions of waves (W_1) and (W_2) are as shown in Fig. (a) and (b).The crest of one falls on the crest of the other and trough falls on trough. The resultant displacement at every point is, therefore, the sum of the component displacements. The resultant wave (W) is represented by a thick line curve in Fig. (c).

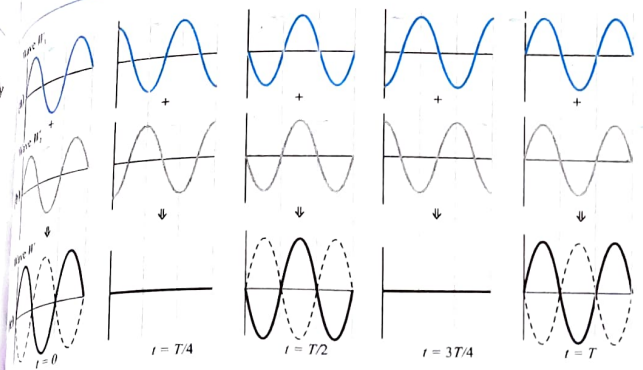
(ii) At $t = T/4$ (where T is the period of each wave), the wave (W_1) moves forward, i.e., towards right through a distance = $\lambda/4$, and the wave (W_2) moves towards left through the same distance (= $\lambda/4$). The crest of one wave falls on the trough of the other and vice-versa. The component displacements at each point being equal and opposite, cancel each other. The resultant is, therefore, represented by the thick horizontal straight line in Fig. (c). Clearly, at this instant, all the particles are passing through their respective mean positions.

(iii) At $t = T/2$, the wave (W_1) moves further through a distance = $\lambda/4$ towards right, and the reflected wave (W_2) through an equal distance towards the left. The thick line curve represents the resultant wave (W) in Fig. (c). The state of affairs is precisely the same as that at $t = 0$, except that the direction of displacement is reversed.

(iv) At $t = 3T/4$, the positions of the incident, reflected and the resultant waves are represented in Figs. (a) and (b). All the particles are passing through their mean positions in a direction opposite to the one at $t = T/4$.

(v) At $t = T$, i.e., after one complete time period, the positions of the waves is the same at $t = 0$.

The whole cycle is repeated time and again and the stationary waves are formed.



The outstanding feature of the resultant wave is that there are places along the string, called **nodes,** where the string never moves. Four such nodes are marked by dots in Fig. (c). Halfway between adjacent nodes are **antinodes,** where the amplitude of the resultant wave is a maximum. Wave patterns such as that of Fig. (c) are called **standing waves** because the wave patterns do not move left or right; the locations of the maxima and minima do not change.

When two waves of the same type (longitudinal or transverse), having same amplitude and same time period, travel with the same speed, along the same straight line, in opposite directions, they superimpose upon each other to give rise to a new type of waves called stationary waves or standing waves.

The waves are said to be stationary or standing because there is no transference of energy in the medium along the waves.

The stationary waves are of two types:

(i) **Transverse stationary waves.** These are formed when the two component waves are transverse in nature. For example, the stationary waves produced in stringed instruments (like a sitar and violin etc.) are transverse.

(ii) **Longitudinal stationary waves.** These are formed when the two component waves are longitudinal in nature. For example, the stationary waves produced in organ pipes (like a flute or a bugle etc.)

For the production of stationary waves, we need two similar waves trains travelling in opposite directions. Therefore, stationary waves are usually formed when a progressive wave is reflected at a point and the reflected wave interacts with the progressive wave. Because it is easier to visualize transverse than the longitudinal waves, we shall discuss the formation of stationary waves in terms of transverse wave trains.

ANALYTICAL TREATMENT OF STATIONARY WAVES

Let us consider two plane progressive harmonic waves of the same amplitude, time period and speed travelling in the opposite directions, say along $+X$-axis and $-X$-axis. We may represent these waves as

$$y_1 = A\sin(kx - \omega t) \quad \ldots(i)$$

and $$y_2 = A\sin(kx + \omega t) \quad \ldots(ii)$$

The equation of the resultant stationary wave is

$$y = y_1 + y_2 = A\left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$$

or $$y = (2A\sin kx)\cos\omega t \quad \ldots(iii)$$

Equation (iii) can be rewritten as

$$y = R\cos\omega t \quad \ldots(iv)$$

Where $R = 2A\sin kx$ ... (v)

Here equation (iv) is an equation of SHM. It implies that after superposition of the two waves the medium particles execute SHM with same frequency ω_0 and amplitude R which is given by equation (v). Here we can see that the oscillation amplitude of medium particles depends on x i.e. the position of medium particles. Thus, on superposition of two waves travelling in opposite direction the resulting interference pattern, we call **stationary waves**, the oscillation amplitude of the medium particle at different positions is different.

Important observation

The resulting wave in Eq. (iii) does not have the term $(\omega t \pm kx)$, this cannot be a travelling wave.. Let us verify that the wave function for a standing wave given in Eq. (iii) is a wave function, $y = (2A\sin kx)\cos\omega t$ is a solution of the general linear wave equation, $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

From equation (iii), we find , $\dfrac{\partial y}{\partial x} = 2Ak\cos kx\cos\omega t$

$$\frac{\partial^2 y}{\partial x^2} = -2Ak^2\sin kx\cos\omega t \quad \ldots(i)$$

and $$\frac{\partial y}{\partial t} = -2A\omega\sin kx\sin\omega t$$

$$\frac{\partial^2 y}{\partial t^2} = -2A\omega^2\sin kx\cos\omega t \quad \ldots(ii)$$

Substitution the values from Eqs. (i) and (ii) into the general wave equation, we get

$$-2Ak^2\sin kx\cos\omega t = \left(\frac{1}{v^2}\right)\left(-2A\omega^2\sin kx\cos\omega t\right)$$

This is satisfied, provide that $v = \dfrac{\omega}{k}$. But this is true, because

$$v = \lambda f = \frac{\lambda}{2\pi}2\pi f = \frac{\omega}{k}$$

NODES AND ANTINODES

Nodes: The positions where the oscillation amplitude of medium particles is always zero.

From Eq. (v) we can observe, when x is such that $kx = n\pi$

or $$x = \frac{n\pi}{k} = \frac{n\pi}{2\pi/\lambda} = n\left(\frac{\lambda}{2}\right),$$

Thus, at points for which $x = 0, \dfrac{\lambda}{2}, \dfrac{3\lambda}{2}, \dfrac{5\lambda}{2}$..... displacement is zero. Such points where displacement (i.e., amplitude) is zero are called displacement nodes (or simply nodes) and are spaced one-half wavelength apart.

Antinodes: The positions where the oscillation amplitude of medium particles is maximum.

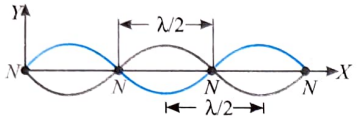
From Eq. (v) we can observe, when x is such that

$$kx = (2n+1)\frac{\pi}{2}$$

or $x=\dfrac{(2n+1)\pi}{2k}=\dfrac{(2n+1)\pi}{2\times(2\pi/\lambda)}=(2n+1)\dfrac{\lambda}{4}$,

From Eq. (v), $y=\pm 2A$ $\left[as\ \sin(2n+1)\dfrac{\pi}{2}=\pm 1\right]$

Thus, at points for which $x=\dfrac{\lambda}{4},\dfrac{3\lambda}{4},\dfrac{5\lambda}{4}$.....; displacement (i.e., amplitude) is maximum ($\pm$ 2A) Such points where displacement (i.e., amplitude) is maximum are called displacement antinodes (or simply antinodes) and are spaced one-half wavelength apart.



Important points:

- The separation between a node and an adjacent antinode is one-quarter wavelength.
- In oscillations of a stationary wave in a region, some points are always at rest (nodes) and some oscillates with maximum amplitudes (antinodes). All other medium particles oscillate with amplitudes less then those of antinodes
- In the region of a stationary wave during one complete oscillation all the medium particles come in the form of a straight line twice.
- Energy cannot flow past the nodal points which are permanently at rest and hence the energy remains "standing" although it alternates between kinetic energy and potential energy. Hence, the waves have been named stationary or standing waves.

COMPARISON BETWEEN PROGRESSIVE AND STATIONARY WAVES

We have already studied the characteristics of progressive as well as the stationary waves and the difference between the two types of waves. The table of characteristics given below reveals the difference between the two types of waves.

	Progressive Waves		Stationary Waves
1.	The disturbance travels onward, it being handed on from one particle to the adjoining particle.	1.	The disturbance is confined to a particular region and there is no onward motion.
2.	Energy is transferred in the medium along the waves.	2.	No energy transference occurs in the medium.
3.	The amplitude of vibration of each particle is the same.	3.	The amplitude of vibration of the particles changes from zero (at nodes) to maximum (at antinodes).
4.	There is a continuous change of phase amongst the various particles.	4.	All the particles in a segment (i.e., between two successive nodes) are in the same phase. The particles in two consecutive segments differ in phase by 180°.
5.	No particle of the medium is permanently at rest.	5.	The particles of the medium at nodes are permanently at rest.
6.	All the particles are never at the mean position simultaneously.	6.	Twice during each vibration, all particles pass through their mean positions simultaneously.
7.	Particles cross the mean position with maximum velocity which is the same for all the particles.	7.	The velocity of the particles is maximum while crossing their mean positions. However, the maximum velocity is different for different particles; it being zero at nodes and highest at the antinodes.

ILLUSTRATION 7.25

A standing wave set up in a medium, the equation of wave is given by $y=4\cos\dfrac{\pi x}{3}\sin 40\pi t$ where x and y are in cm and t in sec. Find the **(i)** amplitude and the velocity of the two component waves **(ii)** distance between adjacent nodes. **(iii)** the velocity of a medium particle at x = 3 cm at time 1/8 s ?

Sol.

(i) The given equation is: $y=4\cos\dfrac{\pi x}{3}\sin 40\pi t$

or $y=2\times 2\cos\dfrac{2\pi x}{6}\sin\dfrac{2\pi(120)t}{6}$ ...(i)

The standard equation of a stationary wave given by

We know that $y=2A\cos\dfrac{2\pi x}{\lambda}\sin\dfrac{2\pi vt}{\lambda}$ ...(ii)

Comparing the Eqs. (i) and (ii), we get

$A=2$ cm, $\lambda=6$ cm and $v=120$ cm/s

The component waves are $y_1=A\sin\dfrac{2\pi}{\lambda}(vt-x)$

and $y_2=A\sin\dfrac{2\pi}{\lambda}(vt+x)$

(ii) Distance between two adjacent nodes $=\dfrac{\lambda}{2}=\dfrac{6}{2}=3$ cm

(iii) The velocity of medium particle

$$\frac{\partial y}{\partial t}=4\cos\frac{\pi x}{3}\cos(40\pi t)\times 40\pi$$

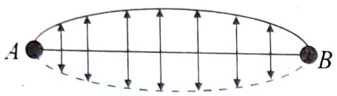
$$\Rightarrow\quad v_P=160\pi\cos\left(\frac{\pi x}{3}\right)\cos 40\pi t$$

At $x = 3$ m, $t = 1/8$ s, the particle velocity is ,

$$v_P\big|_{x=3\text{m and } t=\frac{1}{8}\text{s}} = 160\pi\cos\pi\cos\left(40\pi\times\frac{1}{8}\right) = 160\pi \text{ cm/s}$$

FORMATION OF STATIONARY WAVES ON STRINGS

Consider a string which is fixed between the points A and B. If we pluck clamped string at its mid point and release it will start oscillating as shown in figure. Actually when we pluck the string at its mid point and release, two transverse waves are generated and start moving toward the clamps A and B from middle portion of string. These are reflected from the clamps and superpose on the length of the string and stationary wave is formed between A and B. As being rigid clamps the string particles are not allowed to oscillate at these points. Here at A and B nodes must be formed and there is one antinode at the mid point of the string.



Comments: Whereas the vibrations of the strings are transverse (and that too stationary) the sound so produced reaches the listener through longitudinal (progressive) waves. This is due to the reason that as the string vibrates, it strikes the air molecules all round it, thereby producing longitudinal waves.

NORMAL MODES OF VIBRATION OF A STRETCHED STRING

A clamped string under tension, can be made to oscillate in more than one segment giving rise to various modes of vibration. But due to boundary condition, the string can oscillate only in special pattern which are called normal modes. The frequencies of vibration of a string for all these modes of vibration are different as it is clear from the following discussion. In fact, a string is said to vibrate in a normal mode if it vibrates according to the equation

$$y = (2A\sin kx)\cos\omega t \qquad \ldots(i)$$

From Eq. (i), for a point to be a node,

$$x = \frac{n\lambda}{2} \text{ where } n = 0.1.2,3, \ldots \qquad \ldots(ii)$$

As two ends to which the string (of length L) are fixed hence they will be nodes. We call this type of end condition as similar ends condition. If we choose one of the ends as $x = 0$, then the other is $x = L$ For this end ($x = L$) will be a node, from Eq. (ii)

$$L = \frac{n\lambda}{2} \text{ or } \lambda = \frac{2L}{n} \text{ where } n = 1, 2, 3, \qquad \ldots(iii)$$

Clearly, $\lambda_1 = 2L, \lambda_2 = L, \lambda_3 = \frac{2L}{3}\ldots.$

The frequencies corresponding to these wavelengths are given by

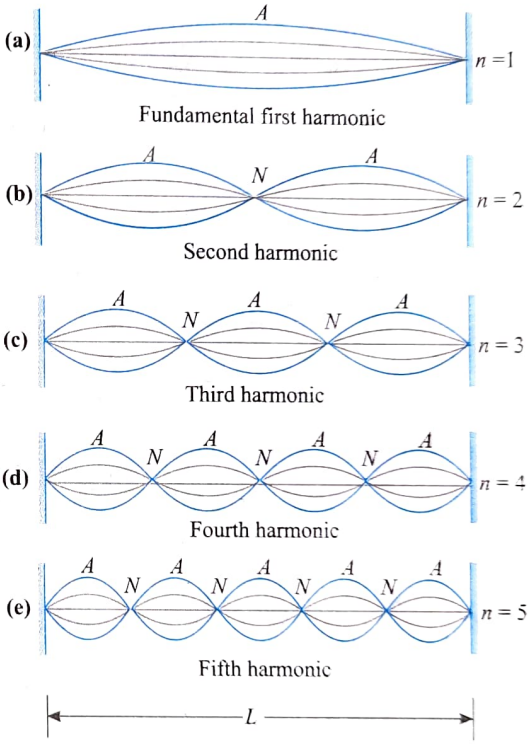
$$f = \frac{v}{\lambda} = \frac{v}{2L/n} = n\left(\frac{v}{2L}\right) \text{ where } n = 1, 2,\ldots \qquad \ldots(iv)$$

where v is the speed of travelling waves on the string.

(i) If $n = 1$, $f = \frac{v}{2L}$ and it is called the fundamental frequency or first harmonic. The corresponding mode is called fundamental mode of vibration. In this case, the string vibrates in one segment as shown in Fig. (a).

(ii) If $n = 2, f_1 = 2\left(\frac{v}{2L}\right) = \frac{v}{L}$ and it is called second harmonic or first overtone. It is obvious that $f_1 = 2f$. In this case, the string vibrates in two segments as shown in Fig. (b).

(iii) If $n = 3, f_2 = 3\left(\frac{v}{2L}\right) = \frac{3v}{2L}$ and it is called third harmonic or second overtone. Clearly, $f_3 = 3f$. In this case, the string vibrates in three segments as shown in Fig. (c). Similarly we can find other modes of oscillation.



The velocity of wave in string is given by $v = \sqrt{T/\mu}$, where T is the tension in the string and μ is its linear mass density, we can also express Eq. (v) the natural frequencies of a taut string as

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}; \quad n = 1,2,3,\ldots \qquad \ldots(v)$$

Comments:

- The collection of all frequencies is called the harmonic series.
- A plucked string vibrates in a complicated way with many of its harmonics contributing to its motion. In fact, it is this combination of harmonics that gives a vibrating string its distinctive sound. What is more, the particular combination of harmonics present in vibration depends on where it is plucked (as in the case of a sitar or a guitar) or bowed (as in the case of a violin).

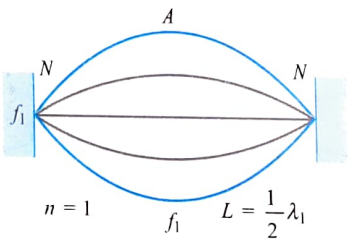
ILLUSTRATION 7.26

A string on a cello vibrates in its first normal mode with a frequency of 220 Hz. The vibrating segment is 70.0 cm long and has a mass of 1.20 g. **(a)** Find the tension in the string. **(b)** Determine the frequency of vibration when the string vibrates in three segments.

Sol. The tension should be less than 500 N (~100 lb) since excessive force on the four cello strings would break the neck of the instrument. If a string vibrates in three segments, there will be three antinodes (instead of one for the fundamental mode), so the frequency should be three times greater than the fundamental.

From the string's length, we can find the wavelength. We then can use the wavelength with the fundamental frequency to find the wave speed. Finally, we can find the tension from the wave speed and the linear mass density of the string.

When the string vibrates in the lowest frequency mode, the length of sting forms a standing wave where $L = \lambda/2$ so the fundamental harmonic wavelength is $\lambda = 2L = 2(0.700\text{ m}) = 1.40\text{ m}$ and the speed is



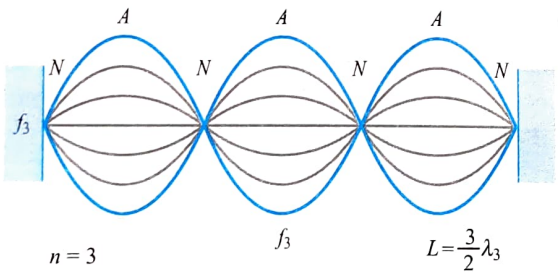
$$v = f\lambda = (220\text{ s}^{-1})(1.40\text{ m}) = 308\text{ m/s}$$

(a) From the tension equation $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{m/L}}$

We get $T = v^2 m/L$

or $$T = \frac{(308\text{ m/s})^2(1.20\times10^{-3}\text{ kg})}{0.700\text{ m}} = 163\text{ N}$$

(b) For the third harmonic, the tension, linear density and speed are the same, but the string vibrates in three segments. Thus, the wavelength is one-third of the fundamental wavelength.



$$\lambda_3 = \lambda_1/3$$

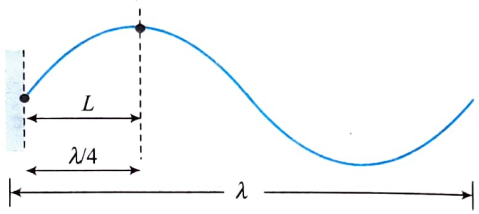
From the equation $v = f\lambda$, we find that the frequency is three times as high.

$$f_3 = \frac{v}{\lambda_3} = 3\frac{v}{\lambda_1} = 3f_1 = 660\text{ Hz}$$

The tension seems reasonable, and the third harmonic is three times the fundamental frequency as expected. Related to part (b), some stringed instrument players use a technique to double the frequency of a note by playing a natural harmonic, in effect cutting a vibrating string in half. When the string is lightly touched at its midpoint to form a node there, the second harmonic is formed, and the resulting note is one octave higher (twice the original fundamental frequency).

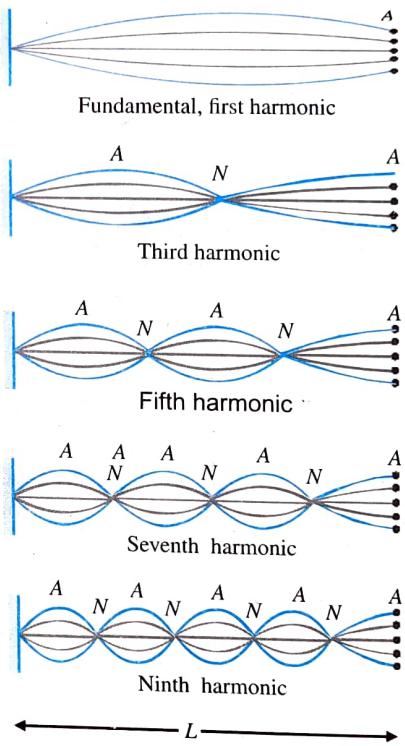
STRING FIXED AT ONE END AND FREE FROM OTHER END

Figure shows a string that has one end fixed and the other end free. In the fundamental mode the free end is an antinode; the length of string $L = \lambda/4$.



In the next mode $L = \frac{3\lambda}{4}$

In general, $L = n\frac{\lambda}{4}$, $n = 1, 3, 5, \ldots$



The resonance frequencies are given by

$$f_n = n\frac{v}{4L} = nf_1;\quad n = 1, 3, 5, \ldots$$

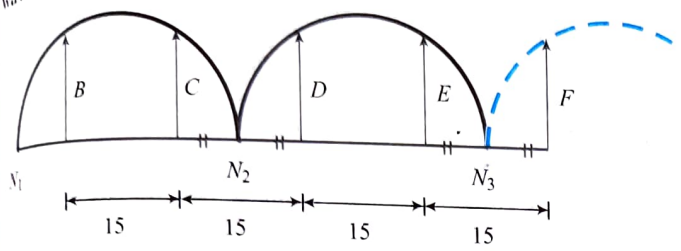
where $f_1 = v/4L$ (fundamental frequency)

The natural frequencies occur in the ratios: 1:3:5:7: ...

ILLUSTRATION 7.27

A string 120 cm in length sustains a standing wave with the points of the string at which the displacement amplitude is equal to 3.5 mm being separated by 15.0 cm. Find the maximum displacement amplitude. To which overtone do these oscillations correspond?

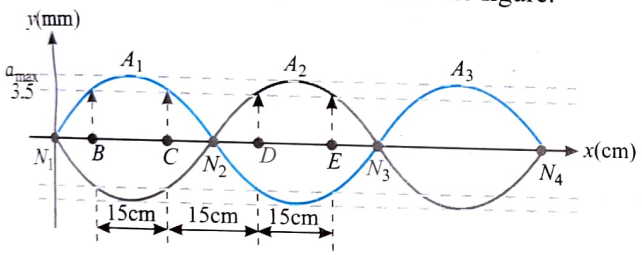
Sol. When a string fixed at both the ends sustains a standing wave, there are n (a whole number) segments of the length of the string, where particles in each segment of the string oscillate in phase with each other and are out of phase with the particles of the adjacent segment, the string contains n loops between its ends. Under this condition the amplitude of the oscillation varies from point to point on the string as a harmonic function (a sine function or a cosine function). A typical equation for the standing wave can be given as $y = y_{max} \sin kx \cos \omega t$.



Let the string be lying along the x-axis between $x = 0$ and $x = 120$ cm. Let the amplitude variation along the length of the string be given by $a = a_{max} \sin kx$, where k is the propagation constant.

First Method: $\left(k = \frac{\omega}{v} = \frac{2\pi}{\lambda}\right)$

Let the string be oscillating in its nth harmonic. The string will contain n loops with end points as node. The points in a loop, where the displacement amplitudes are 3.5 mm will be located symmetrically about antinode, as shown in the figure.



Now $BC = DE = \ldots = 15$ cm and $CD = EF = \ldots = 15$ cm

(A sine function given by $y = \sin x (0 \le x \le \pi)$ is symmetrical about $x = \pi/2$, etc.)

$$CD = CN_2 + N_2D = \left(\frac{120}{2n} - 7.5\right) + \left(\frac{120}{2n} - 7.5\right) = 2\left(\frac{120}{2n} - 7.5\right)$$

Thus, $2\left(\frac{120}{2n} - 7.5\right) = 15$ or $n = 4$

The oscillations correspond to the 4th harmonic (the 3rd overtone). The string will oscillate in 4 loops. There are a total of 5 nodes (including the nodes on the end of the string) on the string. Recalling that from a node of the next node the distance is $\lambda/2$, the length of the string $l = 4\lambda/2$.

or $\lambda/2 = l/4 = 30$ cm or $\lambda = 60$ cm

$$k = \frac{2\pi}{\lambda} = \frac{\pi}{30}$$

Amplitude as function of x is $a = a_{max} \sin kx = a_{max} \sin\left(\frac{\pi x}{30}\right)$

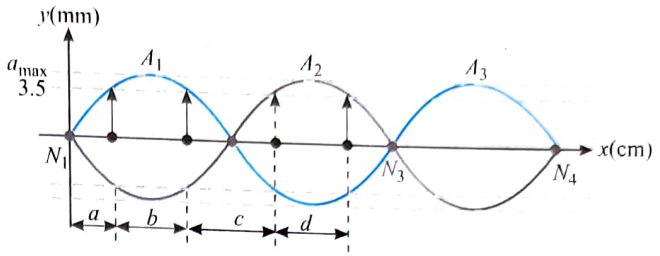
Now at $x = 7.5, 12.5$ cm, $a = 3.5$ mm

or $3.5 = a_{max} \sin\left(7.5\frac{\pi}{30}\right)$

or $3.5 = a_{max} \sin\frac{\pi}{4} = \frac{a_{max}}{\sqrt{2}}$

$$a_{max} = 3.5\sqrt{2} = 4.935 = 5 \text{ mm}$$

Second Method: Suppose that the points shown in figure represent the points which have equal amplitudes and are equally spaced. Now, distance between two consecutive nodes is $\lambda/2$. Therefore,



$a + b + c = \lambda/2$ ...(i)

From symmetry considerations $a = c = d$ ...(ii)

Now, for such points to be equidistant throughout the string,

$b = c + d$ ...(iii)

From Eqs. (ii) and (iii), $b = 2a$

From Eq. (i), we get $a + 2a + a = \lambda/2$

or $a = \lambda/8$ ...(iv)

$b = 2a = \lambda/4$. The points where displacement amplitudes are 3.5 mm are $\lambda/4$ apart.

$\therefore$ $\lambda/4 = 15$ or $\lambda = 60$ cm

Hence, $a = \lambda/8 = 7.5$ cm

[Note that only one value for b, i.e., distance between two such consecutive points exist, hence only one such set of points exist on the string and such points are $\lambda/4$ apart]

Now length of string = 120 cm

$$\text{No. of loops} = \frac{2L}{\lambda} = \frac{2\times120}{60} = 4$$

The string is in the 4th harmonic or 3rd overtone.

ILLUSTRATION 7.28

Three successive frequencies for a string are 75, 125, 175 Hz.

(a) State whether the string is fixed out at one end or at both ends.

(b) What is the fundamental frequency?

(c) To which harmonics do these frequencies correspond?

(d) Taking the speed of the transverse wave on the string as 400 m/s, determine the length of the string.

Sol.

(a) The ratio of three successive frequencies are 3: 5:7, the odd harmonics are present, therefore, the string is fixed at one end.

(b) As the ratio of three successive frequencies are $f_n : f_{n+1} : f_{n+2} = 75:125:175 = 3\times25 : 5\times25 : 7\times25$

Hence the fundamental frequency is $f = 25$ Hz

(c) The given frequencies are the third, fifth and seventh harmonics.

(d) The string is fixed t at one end, it is having dissimilar end condition. The fundamental frequency in this case is given by, $f = \frac{v}{4l} \Rightarrow l = \frac{v}{4f} = \frac{400}{4\times25} = 4$ m

ILLUSTRATION 7.29

A steel wire of length 1 m and density 8000 kg m^{-3} is stretched tightly between two rigid supports. When vibrating in its fundamental mode, its frequency is 200 Hz.

(a) What is the velocity of transverse waves along this wire?

(b) What is the longitudinal stress in the wire?

(c) If the maximum acceleration of the wire is 800 ms^{-2}, what is the amplitude of vibration at the mid-point?

Sol.

(a) Fundamental frequency of string when vibrating in its fundamental mode,

$f_0 = \frac{v}{2l}$ or $v = 2f_0 l$, here v is the velocity of transverse waves along this wire

$\therefore \quad v = 2 \times 200 \times 1 = 400 \text{ ms}^{-1}$

(b) the velocity of transverse waves in string, $v = \sqrt{\frac{T}{\mu}}$ ...(i)

Tension in string = Stress in wire × Area of cross section of wire

$\Rightarrow \quad T = \sigma \times \pi r^2$

Linear mass density of the wire = Density of material of wire × Area of cross of wire

$\Rightarrow \quad \mu = \rho \times \pi r^2$

Hence, $v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{\sigma \times \pi r^2}{\rho \times \pi r^2}} = \sqrt{\frac{\sigma}{\rho}}$

$\therefore$ Stress $\sigma = v^2 \rho = 400^2 \times 8000 = 1.28 \times 10^9 \text{ Nm}^{-2}$

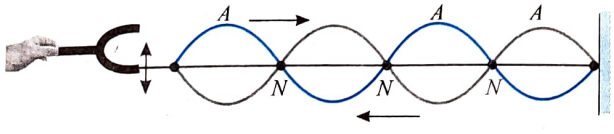
(c) The acceleration of a particle on the wire $= -\omega^2 \times$ displacement amplitude

$\Rightarrow \quad a_{max} = \omega^2 A$ where A = amplitude

$\therefore \quad 800 = (2\pi f)^2 \times A^2$ or $A = \frac{\sqrt{800}}{2\pi \times 200} = 0.02 \text{ m}$

RESONANCE IN STRING OSCILLATION

If we shake one end of a string of length l periodically through some mechanical oscillator. The transverse travelling wave pulses go and get reflected at the fixed end by changing their phase by 180° and come back to the source end and again get reflected changing the phase by 180° at the source end. If we neglect the time of reflection, a pulse can overlap with another pulse in phase given by the source at time $t = \frac{2l}{v}$, where v = speed of the pulse in the string. This condition can be met if the source vibrates once, a twice or thrice or in general n times completely during the time t. If T = time period of oscillation of the source.



$t = nT$, where $t = \frac{2l}{v}$

or $\frac{2l}{v} = nT$ ...(i)

The wavelength of the wave can be written as, $\lambda = vT$ ...(ii)

From Eqs. (i) and (ii), we have $l = \frac{n\lambda}{2}$

From above expression it is clear that, the whole number (integral) multiples of half wavelength of the waves released by the source must fit into the segment of the string between the source and rigid wall.

Each time because of constructive superposition of the added waves, the amplitudes of the resulting travelling waves increases gradually to a large value generating a standing wave of large amplitude compared to the amplitude of the oscillating source. The condition in which every time the travelling waves are added in phase with the reflected waves is called **"resonance"**. And the stationary waves formed in the string is called **"resonant stationary waves"**.

Resonant frequencies: The frequency of source is equal to the frequency of oscillating string , which is given as

$f = \frac{v}{\lambda}$, where $\lambda = \frac{2l}{n}$

or $f = \frac{nv}{2l}$, where $v = \sqrt{\frac{T}{\mu}}$

$f = \frac{n}{2l}\sqrt{\frac{T}{\mu}}$

where n = number of loops = 0,1,2.....

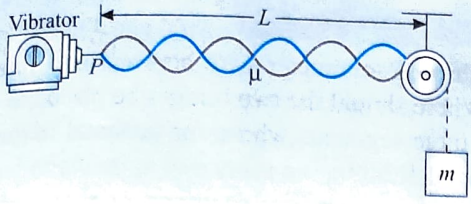
It means, the stretched string can be set into oscillations with resonant stationary waves with some discrete frequencies called resonant frequencies, *i.e.*, f_0, $2f_0$, $3f_0$ ..... nf_0 where f_0 = fundamental frequency. For a particular set (T, μ are same) we can not have any intermediate frequency to form resonant stationary waves in the string. The many mode of oscillations of the string is possible due to the presence of distributed mass in the form of countless molecules attached with each other by molecular spring inside the string.

At resonance, the frequency of the driving agent must be equal to any one resonant frequency of the oscillating system (string) given as, $f = nf_0$; where $f_0 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$ and n = 1, 2, 3, ... for a stretched string fixed at both ends.

ILLUSTRATION 7.30

An object of mass m is hung from a thin taut string (with linear mass density μ = 2 g/m) that passes over a light pulley as shown in figure. The string is connected to a vibrator which can be vibrated with constant frequency f, and the length of the string between vibrator and the pulley is L = 2.0 m. The standing waves are formed when the mass m of the object is either 16.0 kg or 25.0 kg. It is observed that no standing waves are formed with any mass between these values, however. What is the **(a)** frequency of the vibrator? **(b)** largest object mass for which standing waves could be observed?

(use g=9.8m/s^2)



Sol.

(a) Let p be the number of loops in the standing wave resulting from the 25.0-kg mass. Then $p + 1$ is the number of loops for the standing wave resulting from the 16.0-kg mass. The wavelength for standing waves in similar end condition (N.N), $\lambda = \frac{2L}{n}$ and the frequency is $f = \frac{v}{\lambda}$.

The frequency does not change as the masses are changed.

Thus, $f = \frac{p}{2L}\sqrt{\frac{T_p}{\mu}}$ and also $f = \frac{p+1}{2L}\sqrt{\frac{T_{n+1}}{\mu}}$

Equating the expressions for f, we have

$$\frac{p+1}{p} = \sqrt{\frac{T_p}{T_{p+1}}} = \sqrt{\frac{25.0\times g}{16.0\times g}} = \frac{5}{4}$$

Therefore, $4p + 4 = 5p$, or $p = 4$. Using either expression for f, we find

$$f = \frac{4}{2\times 2.0}\sqrt{\frac{25.0\times 9.8}{0.00200}} = 350 \text{ Hz}$$

(b) The tension in the string for n loops formed, $T_n = mg$, Hence frequency,

$$f = \frac{n}{2L}\sqrt{\frac{T_n}{\mu}} = \frac{n}{2L}\sqrt{\frac{mg}{\mu}} \Rightarrow m = \frac{4L^2 f^2 \mu}{n^2 g} \quad \ldots(i)$$

For largest mass m, the number of loops should be least, $n = 1$:

Hence, $m = \frac{4(2.0)^2(350)^2\times 0.00200}{(1)^2\times 9.80} = 400$ kg

ILLUSTRATION 7.31

A wire having a linear density of 5 gm/m is stretched between two rigid supports with a tension of 450 N. It is observed that the wire resonates at a frequency of 420 Hz. The next highest frequency at which the same wire resonates is 490 Hz. Find the length of the wire.

Sol. Let the frequency 420 Hz corresponds to p^{th} harmonic. The formula for p^{th} harmonic is given by

$$n_p = \frac{p}{2l}\sqrt{\left(\frac{T}{\mu}\right)}$$

Hence, $420 = \frac{p}{2l}\sqrt{\left(\frac{T}{\mu}\right)}$ ...(i)

For the next higher frequency p is $(p + 1)$, hence

$$490 = \frac{p+1}{2l}\sqrt{\left(\frac{T}{\mu}\right)} \quad \ldots(ii)$$

From Eqs. (i) and (ii), we have $\frac{490}{420} = \frac{p+1}{p}$

Solving we get, $p = 6$

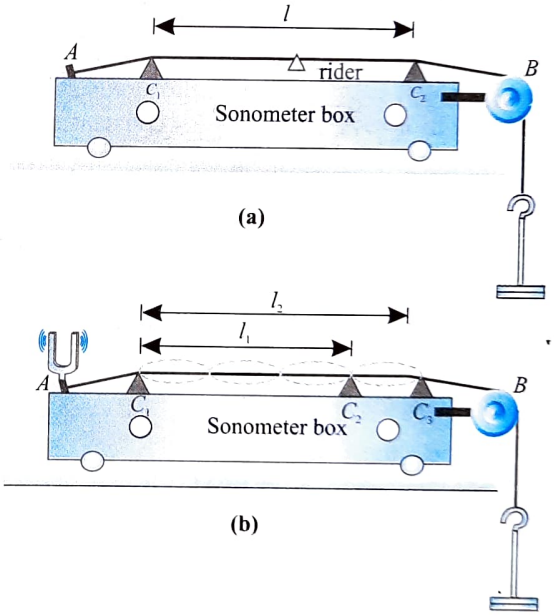
Substituting the value p in Eq. (i), we get

$$420 = \frac{6}{2l}\left(\frac{450}{5\times 10^{-3}}\right)^{1/2} \Rightarrow l = 2.14 \text{ m}$$

SONOMETER

It is a device used to measure velocity of transverse mechanical wave in a stretched metal wire. The principle of sonometer is based on resonance of string vibrations. Working oscillations are induced in a clamped string by an external source like a tuning fork or an oscillator and the corresponding oscillations in string will become stronger when resonance takes place, i.e., the frequency of oscillation of source matches with any of the harmonic of the string vibration.

Figure (a) shows basic structure and setup of a sonometer. It consists of a wooden box M on which a wire AB is stretched, by a hanging weight as shown in Fig. (a). On sonometer box there are two clamps C_1 and C_2 placed which can slide under the wire to change the length of wire between the clamps.



(a)

(b)

When an oscillating turning fork is placed in contact with the sonometer wire as shown in Fig. (b), same oscillations are transferred to the wire. If tension in wire is T and n_0 is the frequency of turning fork, the wavelength of wave in wire is

$$\lambda = \frac{v}{n_0} = \frac{1}{n_0}\sqrt{\frac{T}{\mu}} \quad \ldots(i)$$

When the length between clamps is an integral multiple of $\lambda/2$ then stationary waves are established in the portion of wire between C_1 and C_2. To adjust this, clamp C_1 is fixed and C_2 is displaced so that a small rider (a piece of paper) on wire start jumping violently on wire and falls indicating that the oscillation amplitude of wire is increasing and stationary waves are established. Let in this situation the length between clamps be l_1. Now again C_2 is displaced away from C_1 so that again resonance is obtained. This will happen again when the clamp reaches the

position C_3 and when next node of stationary waves is present as shown in Fig. (b). Let this length be l_2.

So we can say that if l_1 and l_2 are the two successive resonance lengths then we have

$$l_2 - l_1 = \frac{\lambda}{2}$$

So wavelength of wave is $\lambda = 2(l_2 - l_1)$.

As frequency n_0 of oscillating source is known, we can find the velocity of wave in wire as

$$v = n_0\lambda = 2n_0(l_2 - l_1) \quad ...(ii)$$

Equation (ii) gives the partially measured value of velocity of transverse waves in a stretched wire. This can be compared with the theoretical value of v given by $\sqrt{T/\mu}$.

ILLUSTRATION 7.32

The fundamental frequency of a sonometer wire increases by 6 Hz if its tension is increased by 44%, keeping the length constant. Find the change in the fundamental frequency of the sonometer wire when the length of the wire is increased by 20%, keeping the original tension in the wire constant.

Sol. Fundamental frequency of vibrations of string $n = \frac{1}{2l}\sqrt{\frac{T}{m}}$

At constant length, $n \propto \sqrt{T}$ or $\frac{n}{\sqrt{T}} = \text{constant}$ ...(i)

(i) New tension, $T' = T + \frac{44}{100}T = 1.44T$

If n' is new frequency, then $\frac{n'}{\sqrt{T'}} = \frac{n}{\sqrt{T}}$

$$n' = \left(\sqrt{\left(\frac{T'}{T}\right)}\right)n \quad ...(ii)$$

$\therefore \quad n' = n + 6$

From Eq. (iii), $n + 6 = \left(\sqrt{\left(\frac{1.44T}{T}\right)}\right)n$

or $\quad n + 6 = 1.2n$

$0.2n = 6 \quad$ or $\quad n = \frac{6}{0.2} = 30$ Hz

(ii) As tension is constant

$$n \propto \frac{1}{l} \quad \text{or} \quad nl = \text{constant} \quad ...(iii)$$

When length increase by 20%

New length $l' = l + \frac{20}{100}l = 1.2l$

As $\quad nl = \text{constant}$

Therefore, $nl = n'l'$

$$n' = \frac{l}{l'}n = \frac{l}{1.2l} \times 30 = 25 \text{ Hz}$$

Change in fundamental frequency

$$\Delta n = n' - n = 25 - 30 = -5 \text{ Hz}$$

Therefore, $\Delta n = 5$ Hz (decrease)

ILLUSTRATION 7.33

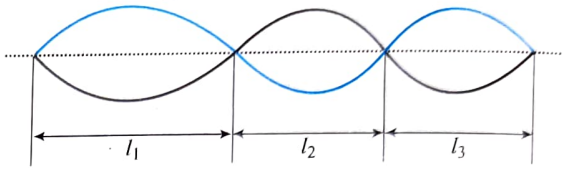
The length of a sonometer wire between two fixed ends is 1.10 m. Where should the two bridges be placed to divide the wire into three segments whose fundamental frequencies are in the ratio of 1:2:3?

Sol. Let l_1, l_2, l_3 be the lengths of three segments.

Given $\quad l_1 + l_2 + l_3 = 1.10 \quad ...(i)$

From relation $n = \frac{1}{2l}\sqrt{\left(\frac{T}{\mu}\right)}$

If tension T and mass per unit length μ are fixed, then $n \propto 1/l$



so $\quad nl = \text{constant}$

$$n_1l_1 = n_2l_2 = n_3l_3$$

$$\therefore \quad l_1 : l_2 : l_3 \equiv \frac{1}{n_1} : \frac{1}{n_2} : \frac{1}{n_3} \equiv \frac{1}{1} : \frac{1}{2} : \frac{1}{3} \equiv \frac{6:3:2}{6} \equiv 6:3:2$$

$\therefore \quad l_1 = 6k, l_2 = 3k, l_3 = 2k, k$ being a constant.

From Eq. (i) $6k + 3k + 2k = 1.10 \quad$ or $\quad 11k = 1.10$

$\therefore \quad k = 1.10/11 = 0.1$

$\therefore \quad l_1 = 6 \times 0.1 = 0.6$ m

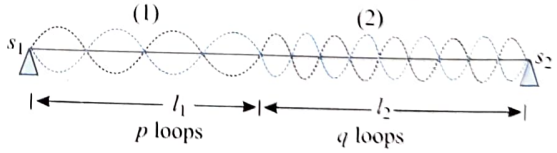
$l_2 = 3 \times 0.1 = 0.3$ m

$l_3 = 2 \times 0.1 = 0.2$ m

Therefore the bridges must be placed at distance 0.6 m and $(0.6 + 0.3) = 0.9$ m

VIBRATIONS OF COMPOSITE STRINGS

In previous section we learned the oscillation of a string clamped string, now we learn the formation of the stationary waves in a string having two different strings as shown in figure. Let two strings of different material and lengths are joined end to end and tied between two clamps S_1 and S_2 as shown. If we induce oscillations in this composite string, stationary waves are established only at those frequencies which matches with any one harmonic of both the independent string.



Let at certain frequency stationary waves established in this string so that string (1) oscillates in p loops and string (2) oscillates in q loops as frequency of both the string is equal, we have

$$f_{ext} = f_1 = f_2$$

Or $\quad f_{ext} = \frac{p}{2l_1}\sqrt{\frac{T}{\mu_1}} = \frac{p}{2l_2}\sqrt{\frac{T}{\mu_2}} \quad ...(i)$

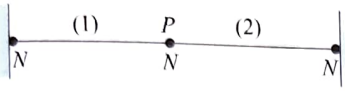
[As T = same in both the string]

Or $\quad \frac{p}{q} = \frac{l_1}{l_2}\sqrt{\frac{\mu_1}{\mu_2}} \quad ...(ii)$

Both p and q must be integers in above equation, thus the minimum values of p and q which satisfies Eq. (ii) will decide the fundamental frequency or the least frequency at which stationary waves can exist in this composite string and the oscillation frequency can be obtained from Eq. (i).

In this type of situation two cases are possible. Let us analyze these cases by one more approach.

Case-I: Let joint P acting as a node (N):



Here end condition for both the strings are (N, N) i.e., similar end condition. In case similar end condition, the number of loops should be integer. Let the number of loops and wavelength of the wave in string 1 be n_1 and λ_1 and for string 2 these conditions are n_2 and λ_2 respectively.

As the length of one loop should be $\frac{\lambda}{2}$, hence

for string '1': $n_1\frac{\lambda_1}{2}=l_1$

for string '2': $n_2\frac{\lambda_2}{2}=l_2$

from above Equation we get $\frac{n_1}{n_2}=\frac{l_1}{l_2}\times\frac{\lambda_2}{\lambda_1}$ ...(i)

Both the strings are connected together they will oscillate with same frequency.

We know $v=f\lambda$

Hence $\frac{v_1}{v_2}=\frac{\lambda_1}{\lambda_2}$ ...(ii)

where v_1 and v_2 are the velocities of wave in string '1' and '2' respectively. Here both the strings are connected together hence tension in both the string should be same.

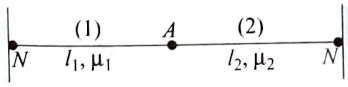
The velocity of wave in taut string is given by, $v=\sqrt{\frac{T}{\mu}}$

Hence $\frac{v_1}{v_2}=\sqrt{\frac{\mu_2}{\mu_1}}$ ...(iii)

From Eqs. (i), (ii) and (iii) $\frac{n_1}{n_2}=\frac{l_1}{l_2}\sqrt{\frac{\mu_1}{\mu_2}}$

- The values of λ_1 and λ_2, μ_1 and μ_2 will be given.
- The ratio $\frac{n_1}{n_2}$ can be calculated.
- If we know the value of n_1 or n_2 from Eqs. (i) and (ii), we can get the value of λ_1 and λ_2.
- After knowing the value of λ_1 or λ_2, we can calculate the fundamental frequency of oscillation using $v=f\lambda$.
- Then next higher frequency = $2f$ and so on.

Case-II: If joint P acts a antinode, both the strings have dissimilar and condition (N, A). In such type of condition the number of loop formed should be $\frac{(2n-1)}{2}$ loops where n is an integer.



For string '1': $\frac{(2n_1-1)}{2}\cdot\frac{\lambda_1}{2}=l_1$

For string '2': $\frac{(2n_2-1)}{2}\cdot\frac{\lambda_2}{2}=l_2$

from above equations we get $\frac{(2n_1-1)}{(2n_2-1)}=\frac{l_1}{l_2}\cdot\frac{\lambda_2}{\lambda_1}$ ...(i)

As both the wires vibrate with common frequency hence

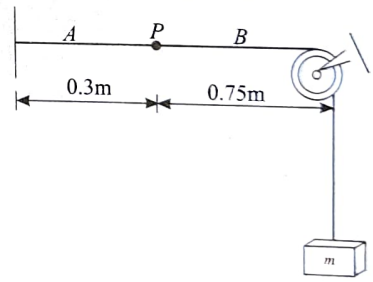
$\frac{\lambda_1}{\lambda_2}=\frac{v_1}{v_2}=\sqrt{\frac{\mu_2}{\mu_1}}$ ...(ii)

Hence from Eqs. (i) and (ii) we get $\frac{(2n_1-1)}{(2n_2-1)}=\frac{l_1}{l_2}\sqrt{\frac{\mu_1}{\mu_2}}$

- From Eq. (ii) we can calculate the value of n_1 or n_2.
- Once the value of n_1 or n_2 is calculated we can calculate the value of λ_1 or λ_2.
- After getting the value of λ_1 or λ_2 we can calculate the value of 'f' using the expression $v=f\lambda$.

ILLUSTRATION 7.34

Two metallic strings A and B of different materials are connected in series forming a joint. The strings have similar cross-sectional area a = 1 mm^2. The length of A is l_A = 0.3 m and that B is l_B = 0.75 m. One end of the combined string is tied with a support rigidly and the other end is loaded with a block of mass m = 25.2 kg passing over a frictionless pulley. Transverse waves are set up in the combined string using an external source of variable frequency, calculate



(a) the lowest frequency for which standing waves are observed such that the joint is a node

(b) the total number of antinodes at this frequency. The densities of A and B are 6.3×10^3 kg/m^3 and 2.8×10^3 kg/m^3, respectively.

Sol. **Approach-1:** Let p = number of loops formed in string A and q = number of loops formed in string B

(a) Both p and q will be integer numbers, as the one end is fixed rigidly, the other end is loaded and the joint P is a node.

For string A, $n_A=\frac{p}{2l_A}\sqrt{\frac{T}{\mu_A}}=\frac{p}{2l_A}\sqrt{\frac{T}{a\rho_A}}$

For string B, $n_B=\frac{q}{2l_B}\sqrt{\frac{T}{\mu_B}}=\frac{q}{2l_B}\sqrt{\frac{T}{a\rho_B}}$

As both the strings are to be exerted by the same source of variable frequency, hence $n_A=n_B$.

$\frac{p}{2l_A}\sqrt{\frac{T}{a\rho_A}}=\frac{q}{2l_B}\sqrt{\frac{T}{a\rho_B}}$

$$\frac{p}{q}=\frac{l_A}{l_B}\sqrt{\frac{d_A}{d_B}}=\frac{0.3}{0.75}\sqrt{\frac{6.3\times10^3}{2.8\times10^3}}$$

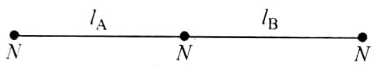
$$=\frac{2}{5}\sqrt{\frac{9}{4}}=\frac{2}{5}\times\frac{3}{2}=\frac{3}{5}$$

As we have to determine the lowest frequency hence the mode of variation will be

$$n_{\min}=\frac{3}{2l_A}\sqrt{\frac{T}{a\rho_A}}=\frac{3}{2\times0.3}\sqrt{\frac{25.2}{10^{-6}\times6.3\times10^3}}=2000\,\text{Hz}$$

Approach-2: As both the strings have similar end condition (N, N), hence integer number of loops will be formed in both the strings. Let n_1 and n_2 are the number of loops formed in string A and B, then

$$\frac{n_1}{n_2}=\frac{l_A}{l_B}\times\frac{\lambda_A}{\lambda_B}=\frac{l_A}{l_B}\times\sqrt{\frac{\mu_A}{\mu_B}} \quad \ldots(i)$$



μ_1 and μ_2 are the linear mass densities of the wires. As the area of cross section of the wires are same hence,

$$\frac{\mu_A}{\mu_B}=\frac{\rho_A}{\rho_B} \quad \ldots(ii)$$

From (i) and (ii), we have $\frac{n_1}{n_2}=\frac{l_A}{l_B}\times\sqrt{\frac{\rho_A}{\rho_B}}=\frac{0.3}{0.75}\times\sqrt{\frac{6.3\times10^3}{2.8\times10^3}}$

$$\Rightarrow \quad \frac{n_1}{n_2}=\frac{2}{5}\times\sqrt{\frac{9}{4}}=\frac{3}{5}$$

Hence in case of fundamental node of oscillation the string A and B will oscillate with 3 and 5 loops respectively.

It means for the lowest frequency of oscillation of the wavelength in string A should be

$$n_1\cdot\frac{\lambda_A}{2}=l_A\Rightarrow\lambda_A=\frac{2l_A}{n_1}$$

or $\lambda_A=\frac{2\times0.3}{3}=0.20\,\text{m}$

Velocity of wave in string 'A', $v_A=\sqrt{\frac{T}{\mu_A}}=\sqrt{\frac{mg}{a\rho_A}}$

$$\Rightarrow \quad v_A=\sqrt{\frac{25.2\times10}{10^{-6}\times6.3\times10^3}}=200\ \text{m/s}$$

Hence lowest frequency $f_A=f_B=\frac{v_A}{\lambda_A}=\frac{200}{0.2}=2000$ Hz

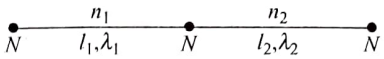
(b) There are total 8 loops, hence number of antinodes = number of loops = 8

Total number of nodes = number of loop + 1 = 8 + 1 = 9

ILLUSTRATION 7.35

Two strings of same material are joined to form a large string and is stretched between rigid supports. The diameter of the second string is twice that of the first. It was observed in an experiment that the whole string was oscillating in 4 loops with a node at the joint. Find the possible lengths of the second string if the length of first string is 90 cm.

Sol. The ends of strings are connected to fixed support and the joint of the strings is a node. It means both the strings have similar end condition (N, N)



For this condition we can write

$$\frac{n_1}{n_2}=\frac{l_1}{l_2}\cdot\sqrt{\frac{\mu_1}{\mu_2}}=\frac{l_1}{l_2}\cdot\sqrt{\frac{\rho a_1}{\rho a_2}} \quad \ldots(i)$$

ρ = density of material of wires a_1 and a_2 = area of cross sections of wires.

n_1 and n_2 = number of loops in wire 1 and 2

we can write Eq. (i) as

$$\frac{n_1}{n_2}=\frac{l_1}{l_2}\cdot\sqrt{\frac{\pi d_1^2/4}{\pi d_2^2/4}}=\frac{l_1}{l_2}\times\frac{d_1}{d_2} \quad \ldots(ii)$$

d_1 and d_2 = diameters of the wires.

$$\frac{n_1}{n_2}=\frac{90}{l_2}\times\frac{1}{2}=\frac{45}{l_2} \quad \ldots(iii)$$

According to problem the string is oscillating with 4 loops. There are 3 possibilities:

(1) When first string has 1 loop and second string has 3 loops.

then from Eq. (iii), $\frac{1}{3}=\frac{45}{l_2}\Rightarrow l_2=135$ cm

(2) When both the strings has 2 loops then from Eq. (iii)

$$\frac{2}{2}=\frac{45}{l_2}\Rightarrow l_2=45\,\text{cm}$$

(3) When the first string has 3 loop and second has 1 loop then from Eq. (iii)

$$\frac{3}{1}=\frac{45}{l_2}\Rightarrow l_2=15\,\text{cm}$$

MELDE'S EXPERIMENT

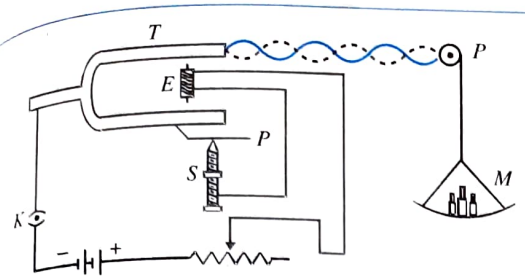
This is an experiment for demonstration of transverse stationary wave in stretched string.

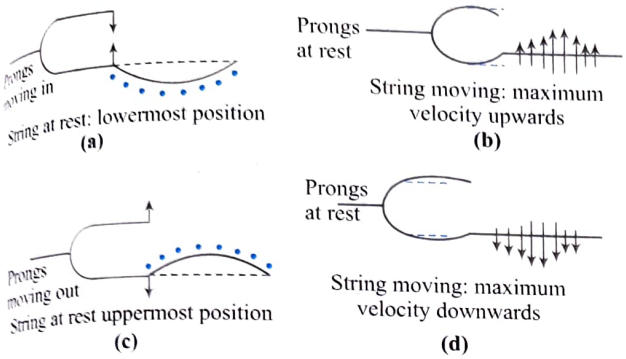
In Melde's experiment, the one end of the string is connected to the prong of an electrically oscillated tuning fork. The other end of the string is connected to the scale pan. The string passes over a smooth frictionless pulley. The distance between tuning fork and pulley can be adjusted. There are two different ways in which oscillations can be established in the string.

CASE I: TRANSVERSE MODE OF VIBRATION

As shown in the figure, tuning fork vibrates right angle to the length of the string. In this case the frequency of vibration of string is equal to the frequency of the tuning fork. First, we adjust the length of string so that stationary waves are formed in string. In this case the vibrations of string are enough so that the loop can be seen in the string. If string vibrates in p loops as shown in Fig. (a), the frequency of the oscillating string can be given as

$$n_s=\frac{p}{2l}\sqrt{\frac{T}{\mu}} \quad \ldots(i)$$



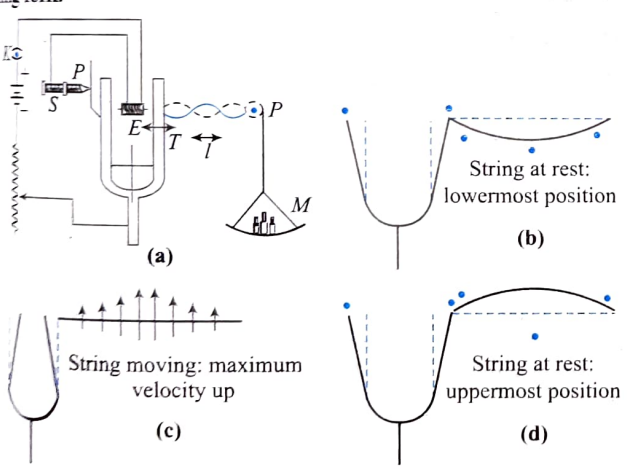


If p be the frequency of oscillation of tuning fork then we have

$$n_f = n_s = \frac{p}{2l}\sqrt{\frac{T}{\mu}} \qquad \ldots(ii)$$

CASE II: LONGITUDINAL MODE OF VIBRATIONS

In this case the vibrations of tuning fork are along the length of the string. The orientation of tuning fork is shown in the figure. In this case for one complete vibration of the tuning fork, the string completes only half of its vibrations so the frequency of vibration of string is half of that of the oscillation frequency of tuning fork.



We can see if the frequency of tuning fork remains constant from Eqs. (i) and (ii) we can write

$$p\sqrt{T} = \text{constant} \qquad [\text{As } \mu = \text{constant}] \quad \ldots(iii)$$

If the tension in the string is changed then the number of loops in the stationary wave formed varies according to Eq. (iii). This is called as Melde's law.

Thus, if in Melde's experiment, stationary waves are formed at two different values of tension T_1 and T_2 at which p_1 and p_2 loops are formed in the string then we can write

$$\frac{P_1}{P_2} = \sqrt{\frac{T_2}{T_1}}$$

ILLUSTRATION 7.36

In Melde's experiment, a string vibrates in 3 loops when 8 g were placed in the pan. What mass must be placed in the pan to make the string vibrate in 5 loops?

Sol. **Case I:** Stretching force, $T_1 = 8\text{g wt} = 8\times1000$ dyne

Number of loops, $n_1 = 3$

Case II: Stretching force, $T_2 = ?$

Number of loops, $n_2 = 5$

As $\quad f = \dfrac{n}{2L}\sqrt{\dfrac{T}{\mu}}$

Case I: $\quad f = \dfrac{3}{2L}\sqrt{\dfrac{8\times1000}{\mu}} \qquad \ldots(i)$

Case II: $\quad v = \dfrac{5}{2L}\sqrt{\dfrac{T_2}{\mu}} \qquad \ldots(ii)$

From Eqs. (i) and (ii), we get $\dfrac{3}{2L}\sqrt{\dfrac{8\times1000}{\mu}} = \dfrac{5}{2L}\sqrt{\dfrac{T_2}{\mu}}$

or $\quad 3\sqrt{8\times1000} = 5\sqrt{T_2} \qquad \ldots(iii)$

Squaring Eq. (iii) both sides, we get:

$$9(8\times1000) = 25T_2$$

or $\quad T_2 = \dfrac{9(8\times1000)}{25\times1000}\text{g wt} = 2.9\text{g wt}$

ILLUSTRATION 7.37

In Melde's experiment, when a string is stretched by a piece of glass it vibrates with 7 loops. When the glass piece is completely immersed in water the string vibrates in 9 loops. What is the specific gravity of glass?

Sol. If F_1 and F_2 are the tensions in the string in the two cases, then

$$F_1P_1^2 = F_2P_2^2$$

$$\frac{F_1}{F_2} = \frac{P_2^2}{P_1^2} = \frac{9^2}{7^2} = \frac{81}{49}$$

If F_b is the buoyant force on the glass piece, then $F_b = F_1 - F_2$

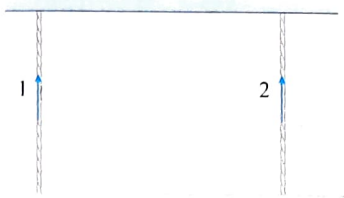
$$\text{Specific gravity of glass} = \frac{\text{Weight in air}}{\text{Loss in weight in water}}$$

$$= \frac{F_1}{F_b} = \frac{F_1}{F_1 - F_2} = \frac{1}{\left(1 - \dfrac{F_2}{F_1}\right)}$$

$$= \frac{1}{\left(1 - \dfrac{49}{81}\right)} = 2.53$$

CONCEPT APPLICATION EXERCISE 7.2

1. The equation of wave in a string fixed at both end is $y = 2 \sin \pi t \cos \pi x$. Find the phase difference between oscillations of two points located at $x = 0.4$ m and $x = 0.6$ m.
2. The third harmonic of a certain piano wire ($\mu = 0.01$ kg/m and $T = 500$N) has a frequency of 600 Hz. What is the wavelength of the fundamental?
3. Two wires of ratio of lengths 1 : 2 and tensions in the ratio 3 : 1 have the ratio of their fundamental frequencies having ratio 2:1. Find the ratio of their masses.
4. A string vibrates with a frequency 200Hz. Its length is doubled and its tension is altered until it begins to vibrate with frequency 300 Hz. What is the ratio of the new tension to the original tension?
5. A 2.0-m-long wire having a mass of 0.10 kg is fixed at both ends. The tension in the wire is maintained at 20.0 N. **(a)** What are the frequencies of the first three allowed modes of vibration? **(b)** If a node is observed at a point 0.40 m from one end, in what mode and with what frequency is it vibrating?
6. A wire of density 9×10^3 kg/m^3 is stretched between two clamps 1 m apart and is subjected to an extension of 4.9×10^{-4} m. What will be the lowest frequency of transverse vibration in the wire ? (Young's modulus of material $= 9 \times 10^{10}$ N/m^2).
7. A stationary wave is given by $y = 5 \sin \frac{\pi x}{3} \cos 40\pi t$ where x and y are in cm and t is in second.
   (a) What are the amplitude and velocity of the component wave whose superposition can give rise to this vibration?
   (b) What is the distance between the nodes?
   (c) What is the velocity of a particle of the string at the position $x = 1.5$ cm when $t = 9/8$ s?
8. A string of length 25 cm is stretched by a load of 10 kg. What is the highest overtone that a man of normal hearing capacity can detect? The mass of the string is 0.5 g.
9. A piano string 1.5 m long is made of steel of density 7.7×10^3 kg/m^3 and $Y = 2 \times 10^{11}$ N/m^2. It is maintained at a tension which produces an elastic strain of 1% in the string. What is the fundamental frequency of transverse vibration of the string?
10. A wire of density 9000 kg/m^3 is stretched between two clamps 100 cm apart while subjected to an extension of 0.05 cm. What is the lowest frequency of transverse vibrations in the wire, assuming Young's modulus of the material to be 9×10^{10} N/m^2?
11. A wire of diameter 0.04 cm and made of steel of density 8000 kg/m^3 is under a tension of 80 N. A fixed length of 50 cm is set into transverse vibrations. How would you cause vibrations of frequency 840 Hz to predominate in intensity?
12. A wave is given by the equation
    $y = 10 \sin 2\pi(100t - 0.02x) + 10 \sin 2\pi(100t + 0.02x)$
    Find the loop length, frequency, velocity and maximum amplitude of the stationary wave produced.
13. A uniform horizontal rod of length 0.40 m and mass 1.2 kg is supported by two identical wires as shown in figure. Where should a mass of 4.8 kg be placed on the rod, so that the same tuning fork may excite the wire on left into its fundamental vibrations and that on right into its first overtone? $g = 10$ m/s^2



ANSWERS

1. π **2.** $\frac{\sqrt{5}}{2}$ m **3.** 3:2 **4.** 9

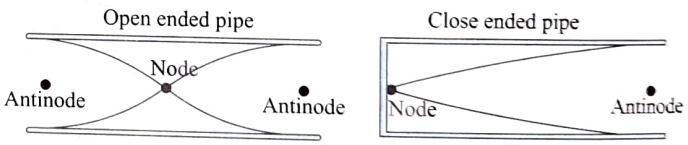
5. (a) 5.0 Hz, 10.0 Hz, 15.0 Hz
(b) fifth state at 25.0 Hz, tenth state at 50.0 Hz, the fifteenth state at 75.0 Hz and so on

6. 35 Hz **7.** (a) 2.5 cm, 1.2 m/s (b) 0.03 m (c) 0
8. 10 **9.** 170 Hz **10.** 35.4 Hz
11. wire should vibrate in third harmonic
12. 25 units, 100 units, 5000 units, 20 units
13. 5 cm from left wire

ORGAN PIPES

An organ pipe is a wind instrument which produces musical sound by setting the air column within it into vibration.

Trumpet, bugle, clarinet, flute, saxophone, shehnai etc. are all organ pipes. It consists of a hollow cylindrical tube of a suitable length. The other end of the tube may be open or closed. If both ends of the tube are open the pipe is called an open organ pipe and if one end is closed other is open, it is called a closed organ pipe.



REFLECTION OF LONGITUDINAL WAVES

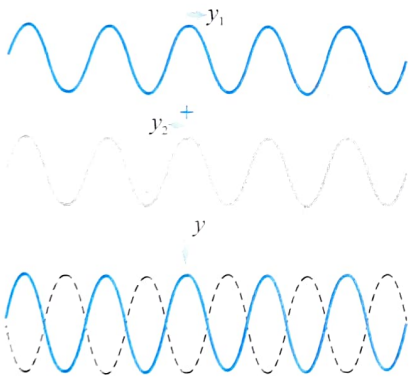
At rigid boundary (at closed end): If a congressional pulse collides with the rigid boundary, during collision it comes back in the compressed phase as the rigid boundary gives a reaction force. From the above argument it is evident that a compressional pulse gets reflected as a congressional pulse. Likewise a rarefied pulse gets reflected as a rarefied pulse at the rigid boundary. The incident and reflected pulses are both compression (or both rarefied) with zero phase difference Hence the incident and reflected compressional pulses superimpose constructively at the rigid boundary giving rise to a maximum compressional zone. In other words at the rigid boundary, the elements of air have zero displacement (maximum density and maximum pressure). Thus a displacement node, pressure antinode and density antinode are formed at the rigid boundary (closed end) of an organ pipe. At the rigid boundary a wave gets reflected without changing its phased

At the non-rigid boundary (open end): If a compressional pulse reach the open end of an organ pipe, the pressure of air in this pulse is greater than the atmospheric pressure, molecules from this zone rushes into the atmosphere. As a result the pressure of the pulse decreases gradually. When its pressure falls below the atmospheric pressure [we can call it an rarefied (expanded) pulse], the atmosphere starts pushing it back. This means that a compression pulse gets reflected as a rarefied pulse. Similarly a rarefied pulse gets reflected as a compression pulse as the air molecules from atmosphere rushes into this pulse. Then, we can say that a pulse gets reflected at the open boundary with a change in phase by 180°.

When a periodic wave reaches the open end, the incident and reflected wave get superimposed destructively to cause a zero-pressure difference at the end ($\Delta p = 0$) In other words, a pressure node is established at the open end. This is because the mighty atmosphere nullifies the pressure variation of air at the open end due to the incident and reflected wave by soaking (or injecting) the air molecules.

LONGITUDINAL STATIONARY WAVES

Let two identical plane progressive longitudinal (pressure) waves (neglecting the initial phases) $\Delta p_1 = \Delta p_0 \sin(\omega t - kx)$ and $\Delta p_2 = \Delta p_0 \sin(\omega t + kx)$ moving along $+x$ and $-x$ directions respectively interfere.



Then, the resultant wave is given as $\Delta p = \Delta p_1 + \Delta p_2$

$$\Delta p = \Delta p_0 \sin(\omega t - kx) + \Delta p_0 \sin(\omega t + kx)$$

$$= \Delta p_0 \{\sin(\omega t - kx) + \sin(\omega t + kx)\}$$

$$= 2\Delta p_0 \sin \omega t \cos kx$$

or $$\Delta p = (2\Delta p_0 \cos kx) \sin \omega t$$

where $2\Delta p_0 \cos kx$ = Amplitude of the resulting wave which depends on the position x of an element.

Note: Similarly, the equation of longitudinal standing wave can be given as

$$\Delta p = 2\Delta p_0 \sin kx \sin \omega t \quad \text{and} \quad \Delta p = 2\Delta p_0 \cos kx \cos \omega t$$

and $$\Delta p = 2\Delta p_0 \sin kx \sin \omega t$$

The points where maximum pressure is felt can be given as

$$\cos kx = 1 \text{ or, } x = n\lambda$$

The points where minimum pressure is felt can be given as

$$\cos kx = -1 \text{ or } x = (2n+1)\frac{\lambda}{2}$$

The points where atmospheric pressure is felt ($\Delta p = 0$) can be given as $\cos kx = 0$ or $x = (2n+1)\frac{\lambda}{4}$

Then the distance between consecutive pressure antinode and pressure node is

$$\Delta x = (2n+1)\frac{\lambda}{2} - (2n+1)\frac{\lambda}{4} = \frac{\lambda}{4}$$

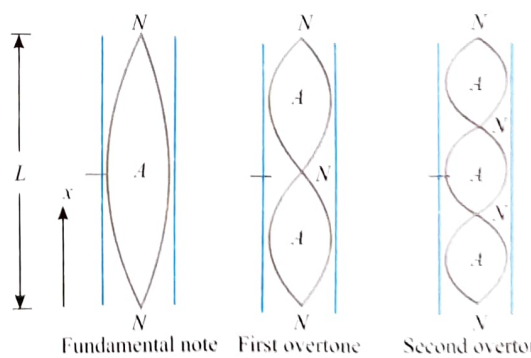
The consecutive distance between two successive pressure nodes or pressure antinode is $\Delta x = \frac{\lambda}{2}$.

At pressure antinodes, maximum pressure $P_0 + \Delta p_{max}$ and minimum pressure $P_0 - \Delta p_{max}$ is felt followed by compressions and rarefactions alternately followed by maximum pressure variation. At the pressure nodes, always the normal (atmospheric) pressure P_0 is felt followed by zero pressure variation $\Delta p = 0$. No energy escapes beyond the region confined between two pressure nodes. Hence, we call it stationary wave.

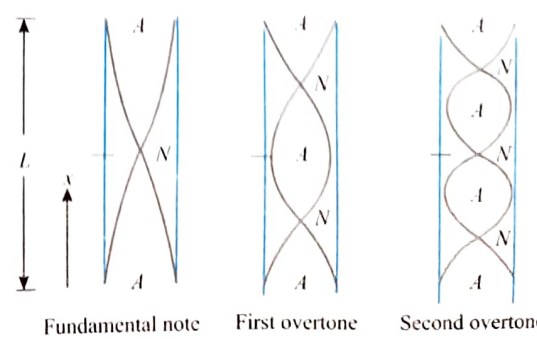
NORMAL MODES OF VIBRATION IN AN OPEN ORGAN PIPE

An open organ pipe can be made to vibrate in different modes by blowing harder and harder. Since such a pipe is open at both the ends, the air molecules at these ends can easily move out into the open space and are thus free to vibrate with maximum amplitude. Thus, when standing waves are set up in the air column in an open organ pipe, an antinode is formed at each end.

A sound wave can be thought of either as a displacement wave or as a pressure wave. The displacement and pressure variations in a sound wave are 90° out of phase. Thus, in a standing sound wave, displacement nodes (or simply nodes) are pressure antinodes and vice-versa. Consequently, an antinode near each end of the open organ pipe is a pressure node. Mathematically and physically, this boundary condition (presence of pressure nodes at the ends of an open organ pipe) is similar to the boundary condition in case of a string fixed at both ends. We call this type of end condition as similar ends condition.



Various modes of vibration showing pressure nodes and antinodes



Various modes of vibration showing dispacement nodes and antinodes

Thus, the standing wave condition is given by: $L=\frac{n\lambda}{2}$ where L is the length of the open organ pipe

Clearly, $\lambda=\frac{2L}{n}$ where n = 1, 2,3,.. ....(i)

If v is the speed of the travelling wave in the pipe, the corresponding frequencies are given by

$$f=\frac{v}{\lambda}=\frac{v}{2L/n}=n\left(\frac{v}{2L}\right) \text{ where } n=1,2,3,... \quad ...(ii)$$

When $n=1$ (first mode) $f_0=\frac{v}{2L}$ (Fundamental frequency or First harmonic)

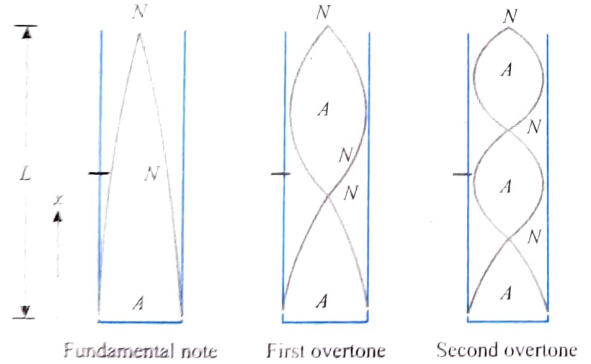
When $n=2$ (second mode) $f_1=\frac{v}{L}=2f_0$ (First overtone or Second harmonic)

When $n=3$ (third mode) $f_2=\frac{3v}{2L}=3f_0$ (Second overtone or Third harmonic)

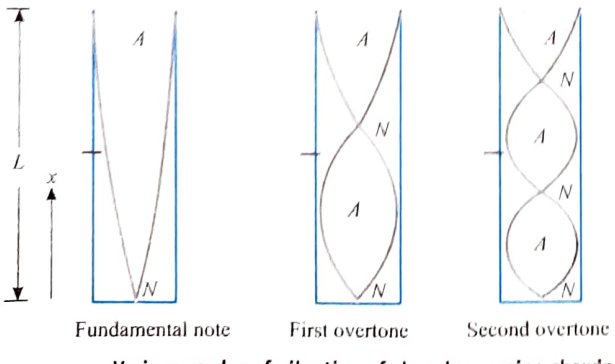
All these modes of vibration with their corresponding wavelengths are shown in figure. Thus, in an open organ pipe, the frequency of the fundamental mode is f_0, whereas overtones or harmonics have frequencies $2f_0$, $3f_0$,... etc. which are integral multiples of the fundamental frequency.

NORMAL MODES OF VIBRATION IN A CLOSED ORGAN PIPE

In a closed organ pipe, since one of the ends is open and the other is closed, the amplitude of vibrations of air particles is maximum at the open end and goes on decreasing till there is no vibration at the closed end. Thus, when standing waves are set up in the air column in a closed organ pipe, an antinode (displacement) is formed at the open end and a node (displacement) is formed at the closed end. We call this type of end condition as dissimilar ends condition.



Various modes of vibration of closed orgn pipe showing pressure nodes and antinodes



Various modes of vibration of closed orgn pipe showing dispacement nodes and antinodes

Mathematically and physically, this boundary condition (presence of pressure node at the end of an open organ pipe and antinode at closed end) is similar to the boundary condition in case of a string fixed at one end and free at other end.

Here, $L=(2n+1)\frac{\lambda}{4}$ where, $n=0,1,2,...$ or $\lambda=\frac{4L}{(2n+1)}$

As $v=f\lambda \Rightarrow f=\frac{v}{\lambda}$ or $f=\frac{v}{4L/(2n+1)}$ or $f=\frac{(2n+1)v}{4L}$

When $n=0$ (first mode) $f_0=\frac{v}{4L}$ (Fundamental frequency or First harmonic)

When $n=1$ (second mode) $f_1=\frac{3v}{4L}=3f_0$ (First overtone or Second harmonic)

When $n=2$ (third mode) $f_2=\frac{5v}{4L}=5f_0$ (Second overtone or Third harmonic)

Hence, in a closed organ pipe, the frequency of a fundamental note is $f_0=\frac{v}{4L}$ and the overtones has frequency $3f_0, 5f_0 ..... , 2f_0+$

i.e., odd integral multiples of the fundamental frequency.

Important points:

- Speed of longitudinal wave in fluid, $v=\sqrt{\frac{B}{\rho}}$
- Speed of longitudinal wave in solids, $v=\sqrt{\frac{Y}{\rho}}$

ILLUSTRATION 7.38

An organ pipe has a fundamental frequency of 330 Hz. The first overtone of a closed organ pipe has the same frequency as the first overtone of this open pipe. How long is each pipe? (Speed of sound in air = 330 m/s)

Sol. Fundamental frequency of an open organ pipe, $f_1=\frac{v}{2\ell_o}$

$$\Rightarrow \quad l_o=\frac{v}{2f_1}=\frac{330}{2\times 330}=0.50=50 \text{ cm}$$

Given, first overtone of closed organ pipe = first overtone of open organ pipe

Hence, $3\left(\frac{v}{4l_c}\right)=2\left(\frac{v}{2l_o}\right)$

$$l_c=\frac{3}{4}l_o=\left(\frac{3}{4}\right)(0.50)=0.375 \text{ m}=37.5 \text{ cm}$$

ILLUSTRATION 7.39

Two adjacent natural frequencies of an organ pipe are formed to be 550 Hz and 650 Hz. Calculate the fundamental frequency and length of this pipe. (Give velocity of sound in air 350 m/s).

Sol. In case of open organ pipe we have all types of harmonic, even as well as odd. While in case of close organ pipe we have only odd harmonic.

The ratio of two adjacent frequencies $\frac{f_n}{f_{n+1}}=\frac{550}{650}=\frac{50\times 11}{50\times 13}=\frac{11}{13}$

Hence the adjacent harmonic are 11th and 13th. Both harmonic are odd hence it is the case of close organ pipe. Clearly, the fundamental frequency should be 50 Hz.

The fundamental frequency in case of close organ pipe

$$f_0 = \frac{v}{4l} \Rightarrow 50 = \frac{350}{4 \times l}$$

Hence length of the pipe $l = \dfrac{350}{50 \times 4} = 1.75$ m

ILLUSTRATION 7.40

A pipe of 30.0 cm length is open at both ends. Which harmonic mode of the pipe resonates a 1.1 kHz source ? Will resonance with the same source be observed if one end of the pipe is closed ? Take the speed of sound in air as 330 m/s.

Sol. In case of an open organ pipe, fundamental frequency,

$$f_0 = \frac{v}{2L} = \frac{(330 \text{ m/s})}{(2 \times 0.3 \text{ m})} = 550 \text{ Hz}$$

Frequency of the second harmonic

$$= 2f_0 = 2 \times 550 \text{ Hz} = 1100 \text{ Hz} = 1.1 \text{ kHz}$$

Clearly, the given source of frequency 1.1 kHz resonates with second harmonic of the open organ pipe.

On closing one end of the given pipe, it becomes a closed organ pipe. In case of a closed organ pipe, fundamental frequency,

$$f_0' = \frac{v}{4L} = 275 \text{ Hz}$$

Since a closed organ pipe after first harmonic of frequency f_0' emits third harmonic (of frequency $3f_0' = 3 \times 275 = 825$ Hz), fifth harmonic (of frequency $5f_0' = 5 \times 275 = 1375$ Hz), there is no harmonic of frequency matching the given source, *i.e.*, 1100 Hz. Thus, no resonance is observed if one end of the pipe which is open at both ends is closed.

ILLUSTRATION 7.41

A student uses an audio oscillator of adjustable frequency to measure the depth of a water well. The student reports hearing two successive resonances at 52.0 Hz and 60.0 Hz. **(a)** How deep is the well? **(b)** How many antinodes are in the standing wave at 60.0 Hz? Take the speed of sound in air as 336 m/s.

Sol.

(a) The well acts like an open organ pipe, open at one end and closed at the other.

The normal modes of vibrations of such a pipe are odd harmonics of a fundamental. Let L be the depth of the well and v the speed of sound.

Then for some integer n,

$$L = (2n-1)\frac{\lambda_1}{4} = (2n-1)\frac{v}{4f_1} = \frac{(2n-1) \times 336}{4 \times 52}$$

and for the next resonance

$$L = [2(n+1)-1]\frac{\lambda_2}{4} = (2n+1)\frac{v}{4f_2} = \frac{(2n+1) \times 336}{4 \times 60}$$

Thus, $\dfrac{(2n-1) \times 336}{4 \times 52} = \dfrac{(2n+1) \times 336}{4 \times 60}$

and we require an *integer* solution to $\dfrac{2n+1}{60} = \dfrac{2n-1}{52}$

The equation gives $n = 7$, which gives us

$$L = \frac{[2(7)-1] \times 336}{4 \times 52} = 21.0 \text{ m}$$

(b) The first harmonic (fundamental frequency) of the well is

$$f_0 = \frac{v}{4L} = \frac{336}{4 \times 21} = 4.0 \text{ Hz}$$

Its pattern is the case of dissimilar end condition. We can see that such a pattern with n antinodes is the $(2n-1)^{th}$ harmonic. The frequency 60 Hz = 15(4 Hz) is the 15th harmonic: $15 = 2(8) - 1$, so the standing wave has 8 antinodes.

ILLUSTRATION 7.42

A section of drainage culvert 1.23 m in length makes a howling noise when the wind blows across its open ends.

(a) Determine the frequencies of the first harmonics of the culvert if it is cylindrical in shape and open at both ends. Take $v = 343$ m/s as the speed of sound in air.

(b) What are the three lowest natural frequencies of the culvert if it is blocked at one end?

Sol. The sound of the wind blowing across the end of the pipe contains many frequencies, and the culvert responds to the sound by vibrating at the natural frequencies of the air column.

This example is a relatively simple substitution problem.

(a) Let us first find the frequency of the first harmonic of the culvert, modelling it as an air column open at both ends.

$$f_1 = \frac{v}{2L} = \frac{343 \text{ m/s}}{2(1.23 \text{ m})} = 139 \text{ Hz}$$

Find the next harmonics by multiplying by integers.

$$f_2 = 2f_1 = 278 \text{ Hz}$$

$$f_3 = 3f_1 = 417 \text{ Hz}$$

(b) Find the frequency of the first harmonic of the culvert, modelling it as an air column closed at one end:

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(1.23 \text{ m})} = 69.7 \text{ Hz}$$

Find the next two harmonics by multiplying by odd integers.

$$f_3 = 3f_1 = 209 \text{ Hz}$$

$$f_5 = 5f_1 = 349 \text{ Hz}$$

ILLUSTRATION 7.43

Find the number of possible natural oscillations of air column in a pipe frequencies of which lie below $\nu_0 = 1250$ Hz. The length of the pipe is $l = 85$ cm. The velocity of sound is $v = 340$ m/s. Consider two cases

(i) the pipe is closed from one end,

(ii) the pipe is open from both ends.

Sol. First Method

(i) Pipe is closed from one end: An air column in a pipe closed from one end oscillates only odd harmonics [1st harmonic (fundamental mode), 3rd harmonic (1st overtone), 5th harmonic (2nd overtone), 7th harmonic (3rd overtone) etc.]

$$\text{Fundamental frequency} = \frac{v}{4l} = \frac{340}{4 \times \frac{85}{100}} = 100 \text{ Hz}$$

Other modes of oscillation are:

3rd harmonic frequency = 3 × 100 = 300 Hz
5th harmonic frequency = 5 × 100 = 500 Hz
7th harmonic frequency = 7 × 100 = 700 Hz
9th harmonic frequency = 9 × 100 = 900 Hz
11th harmonic frequency = 11 × 100 = 1100 Hz
13th harmonic frequency = 13 × 100 = 1300 Hz

Only those natural oscillations are to be counted frequencies of which lie below v_0 = 1250 Hz, the harmonics till 11th harmonic are to be counted.

Since number of possible natural oscillations = 1 (1st harmonic) + 1 (3rd harmonic) + 1 (5th harmonic) + 1(7th harmonic) + 1(9th harmonic) + 1 (11th harmonic) = 6

Second Method: All the frequencies possible are integral multiples of fundamental frequency which is 100 Hz. Using the fact that integer which is multiplied by fundamental frequency is the number of harmonic itself you get, highest harmonic predicted = [12.50/100] where [x] represents greatest integer less than or equal to x = [12.5] = 12.

Now for closed pipe, only odd harmonics are possible and the highest harmonic possible = 11th. The possible harmonics are 1, 3, 5, 7, 9, 11 which are six in number.

(ii) Pipe opened from both ends: Fundamental frequency

$$= \frac{V}{2l} = \frac{340}{2 \times 85} \times 100 = 200 \text{ Hz}$$

Frequency of the other natural modes of oscillational are:

2nd harmonic frequency = 2 × 200 = 400 Hz
3rd harmonic frequency = 3 × 200 = 600 Hz
4th harmonic frequency = 4 × 200 = 800 Hz
5th harmonic frequency = 5 × 200 = 1000 Hz
6th harmonic frequency = 6 × 200 = 1200 Hz
7th harmonic frequency = 7 × 200 = 1400 Hz

You have to count only those harmonics whose frequencies are below 1250 Hz. All the harmonics till 6th harmonic are possible, and obviously they are six in number.

Third Method

Fundamental frequency = 200 Hz

Frequencies possible = n × fundamental frequency = $n \times 200$ [n is 1, 2, ...]

Maximum value of n = [12.50/200] = 6 ([x] represents greater than or equal to x)

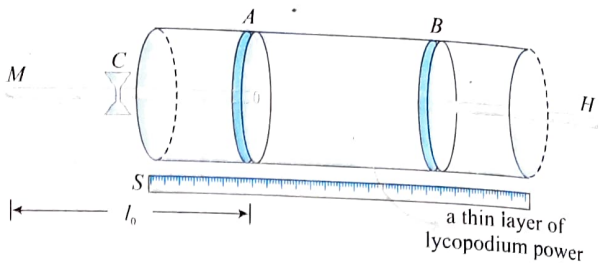
Now n is also equal to the number of harmonic for which frequency is being calculated, highest harmonic possible = 6th.

As all harmonics are possible in case of open tube, harmonics possible are 1st, 2nd, 3rd, 4th, 5th and 6th.

Number of harmonics possible in this case = 6.

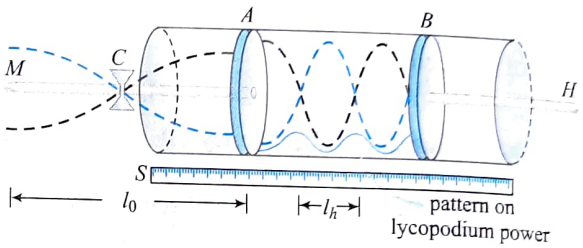
KUNDT'S TUBE

This is an apparatus used to find velocity of sound in a gaseous medium or in different materials. It consists of a glass tube as shown in figure one end of which is fitted with piston B which is attached to a wooden handle H which can be moved inside and outside the tube, and a rod M of the required material is fixed at clamp C in which the velocity of sound is required, at one end of rod a disc A is fixed as shown below.



In the tube, air is filled at room temperature and a thin layer of lycopodium powder is put along the length of the tube. It is a very fine powder particles which can be displaced by the air particles also.

When rod M is gently rubbed with a resin cloth or hit gently, it starts oscillating in fundamental mode as shown in figure, frequency of which can be given as



$$n_{rod} = \frac{v}{\lambda} = \frac{1}{2l_0}\sqrt{\frac{Y}{\rho}} \qquad \left[\text{As } l_0 = \frac{\lambda}{2}\right] \qquad \text{...(i)}$$

In fundamental mode, the free ends of the rod behave as antinodes and the clamped point C acts as a node, which we have discussed earlier. When these vibrations are transferred to the air column in the tube through the disc A, the air column starts oscillating at the same frequency. Now the piston B is adjusted so that the air column resonates with the vibrations of rod. In resonance condition, the lycopodium powder sets itself in the form of heaps at the position of nodes because at antinodes the air particles vibrate with maximum amplitude and displace the powder to the adjacent nodes. We can measure the length between successive heaps of powder with the help of a scale S attached to the tube, let it be l_h then we can write.

$$l_h = \frac{\lambda_a}{2} \quad [\lambda_a = \text{wavelength of sound in air}]$$

and we know that frequency of sound in air and rod is equal, thus

$$n_{rod} = n_{air} \quad \text{or} \quad \frac{v_{rod}}{\lambda_{rod}} = \frac{v_{air}}{\lambda_{air}}$$

$$\text{or} \quad v_{rod} = \left(\frac{\lambda_{rod}}{\lambda_{air}}\right) v_{air} = \frac{2l_0}{2l_h} \times v_{air} = \left(\frac{l_0}{l_h}\right) \times v_{air} \qquad \text{...(ii)}$$

If the Young's modulus and density of material of rod is known then using Eqs. (i) and (ii) we can find velocity of sound in air or in a gas which is filled in the tube, as

$$n_{rod} = n_{gas}$$

$$\frac{1}{2l_0}\sqrt{\frac{Y}{\rho}} = v_{gas}/2l_h \quad \text{or} \quad v_{gas} = \left(\frac{l_h}{l_0}\right)\sqrt{\frac{Y}{\rho}} \qquad \text{...(iii)}$$

ILLUSTRATION 7.44

In a Kundt's tube experiment, with a wooden rod 170 cm long, clamped at the middle, the lycopodium powder gets heated up at regular intervals of 13.4 cm, the experiment being performed with air. If the frequency of vibrations be 1270 Hz, find the velocity of sound in air and in the wooden rod.

Sol. The wavelengths of sound waves in air and in rod will be respectively,

$\lambda_a = 2 \times 13.4 \text{ cm} = 26.8 \text{ cm}$, and

$\lambda_r = 2 \times 170 \text{ cm} = 340 \text{ cm}$

Since velocity $v = \lambda f$

Velocity of sound in air $v_a = l_a f$

$= 26.8 \text{ cm} \times 1270 \text{ s}^{-1} = 340 \text{ m/s}$

Velocity of sound in wooden rod $v_r = \lambda_r f$

$= 340 \text{ cm} \times 1270 \text{ s}^{-1} = 4318 \text{ m/s}$

ILLUSTRATION 7.45

A Kundt's tube experiment is conducted with a 1 m long glass rod, twice, one with air and the other with hydrogen gas filled in the tube. In the first case, there were 11 heaps of lycopodium powder within a length of 64.4 cm between the first and the last. The corresponding parameters in the second case are 5 nodal heaps within 99.7 cm length. Find the velocity of sound in glass and in hydrogen, if that in air be 335 m/s.

Sol. In case of air, distance between two consecutive nodes (heaps of powder)

$$= \frac{64.4 \text{ cm}}{10} = 6.44 \text{ cm}$$

If λ_a and λ_r be the respective wavelengths of sound in air and in rod, then

$\lambda_a = 2 \times 6.44 \text{ cm}$ and $\lambda_r = 2 \times 100 \text{ cm}$

Since the frequency remains the same, so $f = \frac{v_a}{\lambda_a} = \frac{v_r}{\lambda_r}$,

$$v_r = \frac{\lambda_r}{\lambda_a} v_a = \frac{(2 \times 100 \text{ cm})}{(2 \times 6.44 \text{ cm})} \times 335 \text{ m/s} = 5.2 \text{ km/s}$$

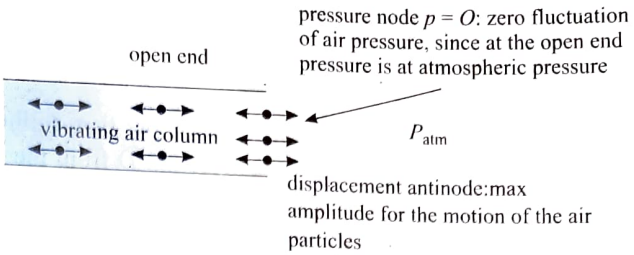
If v_h and λ_h correspond to the velocity and wavelength of sound waves in hydrogen respectively, then

$$\frac{v_a}{\lambda_a} = \frac{v_h}{\lambda_h} \quad \text{or} \quad v_h = \frac{\lambda_h}{\lambda_a} \times v_a$$

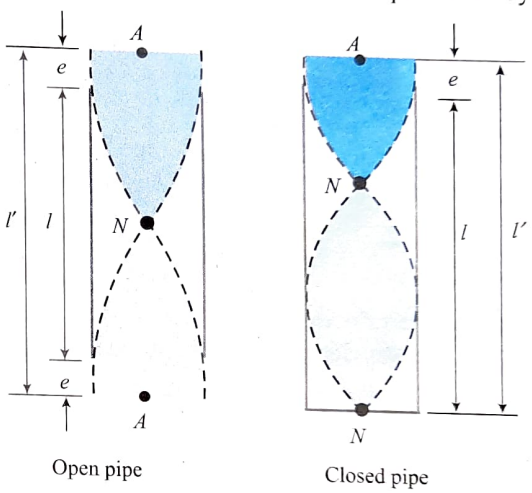
$$v_h = \frac{(99.7 \text{ cm}/4)}{6.44 \text{ cm}} \times 335 \text{ m/s} = 1.3 \text{ km/s}$$

END CORRECTION

Consider an organ pipe shown in figure. Here we consider a wave is propagating towards its open end. Due to longitudinal wave medium (air), particles oscillate along the length of pipe as shown in figure. But the oscillations are along the length of the pipe within the boundaries of the pipe. When wave reaches the open end, due to collisions the medium particles outside the pipe scatters in the direction away from pipe and due to this, medium (air) density reduces outside the pipe and from the region of this rarer medium the wave is reflected.



Here we can see that when a wave reaches the open end of pipe it penetrates atmosphere up to a small depth where the density is decreased and then it is reflected back into the pipe. Thus, the wave is not exactly reflected from the open end of the pipe. Hence in the formation of stationary waves in organ pipe we say that an antinode is formed always a little above the open end as shown in figure. The distance above the open end where an antinode formed is called the end correction and is represented by e.



It is observed that the end correction depends on the radius of organ pipe and is experimentally determined and expressed as

$$e = 0.6r \qquad \text{...(i)}$$

Thus, for a broad pipe the end correction is more than a narrow pipe. When we find the different harmonic frequencies of oscillations of air column in organ pipe, we must account the end corrections. Now taking into account the end correction the fundamental frequency of a closed pipe of length l can be given as

$$n_0 = \frac{v}{4(l+e)} \quad \text{[One end open]} \qquad \text{...(ii)}$$

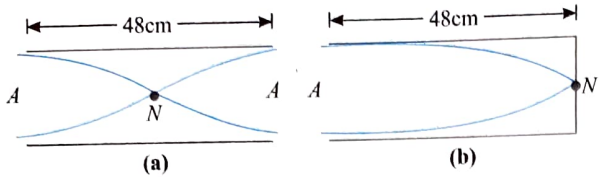
and fundamental frequency of an open pipe of length l is taken as

$$n_0 = \frac{v}{2(l+2e)} \quad \text{[Both ends open]} \qquad \text{...(iii)}$$

ILLUSTRATION 7.46

A tube of certain diameter and of length 48 cm is open at both ends. Its fundamental frequency of resonance is found to be 320 Hz. The velocity of sound in air / s320 m/s. Estimate the diameter of the tube. One end of the tube is now closed. Calculate the lowest frequency of resonance for the tube.

Sol. The displacement curves of longitudinal waves in a tube open at both ends is shown in Figs. (a) and (b).



Let D be the diameter of the tube. We know that antinodes occur slightly outside the tube at a distance 0.3 D from the tube end.

The distance between two antinodes is given by $\frac{\lambda}{2} = 48 + 2 \times 0.3D$

We have $\lambda = \frac{v}{n} = \frac{32000}{320} = 100\,\text{cm}$

or $50 = 48 + 0.6D$

Thus diameter of the tube is $D = \frac{10}{3}\text{cm} = 3.33\text{ cm}$

$$= 48 + 0.3 \times \frac{10}{3} = 49$$

or $\lambda = 4 \times 49 = 196\text{ cm}$

Hence the lowest frequency is, $f = \frac{v}{\lambda} = \frac{32000}{196} = 163.3\text{ Hz}$

ILLUSTRATION 7.47

An open pipe 40 cm long and a closed pipe 31 cm long, both having same diameter, are producing their first overtone, and these are in unison. Determine the end correction of these pipes.

Sol. Let v be the velocity of sound and f be the frequency of the note. Thus, the wavelength of sound produced is

$$\lambda = \frac{v}{f}$$

The wavelength of first overtone of open pipe is

$$\lambda_1 = l + 2e = 40 + 2e$$

$$\frac{v}{f} = 40 + 2e \qquad (\because \lambda_1 = \lambda)$$

The wavelength of first overtone of closed pipe is

$$\frac{3}{4}\lambda_1 = l + e = 31 + e$$

$$\frac{3}{4}\left(\frac{v}{f}\right) = 31 + e \qquad (\because \lambda_1 = \lambda)$$

By eliminating v/f from the above two equations, we get

$$\frac{3}{4}(40 + 2e) = 31 + e$$

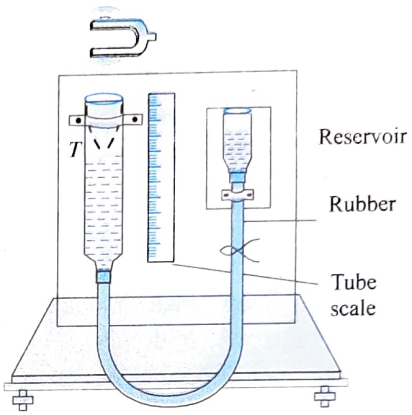
$\Rightarrow \quad e = 2\text{ cm}$

RESONANCE TUBE

This is apparatus used to determine velocity of sound in air experimentally and also to compare frequencies of two tuning forks.

It is a closed organ pipe having an air column of variable length. When a tuning fork is brought over it's mouth. It's air column vibrates with the frequency of the fork. If the length of the air column is varied until it's natural frequency equals the frequency of the fork, then the column resonants and emits a loud note.

A tuning fork of known frequency n_0 is struck gently on a rubber pad and brought near the open end of tube T due to which oscillations are transferred to the air column in the tube above water level. Now we gradually decrease the water level in the tube. This air column behaves like a closed organ pipe and the water level as closed end of pipe. As soon as water level reaches a position where there is a node of corresponding stationary wave, in air column, resonance takes place and maximum sound intensity is detected. Let at this position length of air column be l_1. If water level is further decreased, again maximum sound intensity is observed when water level is at another node i.e., at a length l_2 as shown in figure.



If l_1 and l_2 are lengths of first and second resonances, then we have

$$l_1 + e = \frac{\lambda}{4} \text{ and } l_2 + e = \frac{3\lambda}{4}$$

$$l_2 - l_1 = \frac{\lambda}{2} \Rightarrow \lambda = 2(l_2 - l_1)$$

Speed of sound in air at room (temperature) $v = n\lambda = 2n(l_2 - l_1)$

Also $\frac{l_2 + e}{l_1 + e} = 3 \Rightarrow l_2 = 3l_1 + 2e$ i.e. second resonance is obtained at length more than thrice the length of first resonance.

ILLUSTRATION 7.48

A tuning fork of frequency 340 Hz is vibrated just above a cylindrical tube. The length of the tube is L = 120 cm. Water is slowly poured into the tube. Determine the minimum height of water required for resonance. (Take velocity of sound in air v = 340 m/s)

Sol. Here, the tuning fork vibrates in resonance with the air column in the pipe closed at one end. The resonant frequency of the air column is given by

$$f = \frac{nv}{4l} \quad \text{where} \quad n = 1, 2, 3, \ldots$$

The length of the air column is given by

$$l = \frac{nv}{4f} = \frac{n(340)}{4(340)} = 0.25n\text{ m} = 25n\text{ cm}$$

l = 25 cm, 75 cm, 125 cm

Since the length of the tube is 120 cm, therefore, the possible lengths of air column are

l = 25 cm

or l = 75 cm

If h is the height of water filled in the tube, then

$l + h = 120$ cm

$h = 120 - l$

For minimum value of h, l is maximum, thus

$h_{min} = 120 - l_{max} = 120 - 75 = 45$ cm

ILLUSTRATION 7.49

The first two lengths of an air column, in a resonance column method, were found to be 32.1 cm and 99.2 cm, respectively. Determine the end correction for the tube. If it is known that velocity of sound in the laboratory is 332 m/s, then find the frequency of the vibrating tuning fork.

Sol. Here, $l_1 = 32.1$ cm and $l_2 = 99.2$ cm

from $e = \dfrac{l_2 - 3l_1}{2}$, the end correction is given by

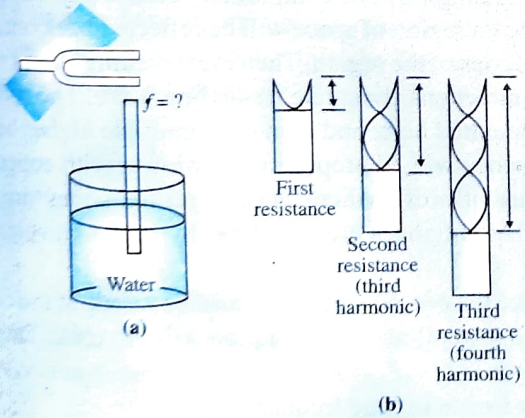
$$e = \frac{99.2 \text{ cm} - 3 \times 32.1 \text{ cm}}{2} = 1.45 \text{ cm}$$

Also, from $v = 2f(l_2 - l_1)$, the frequency of the vibrating tuning fork is given by

$$f = \frac{v}{2(l_2 - l_1)} = \frac{332 \text{ m/s}}{2(99.2 - 32.1) \times 10^{-2} \text{ m}} = 247.4 \text{ Hz}$$

ILLUSTRATION 7.50

A simple apparatus for demonstrating resonance in an air column is depicted in figure. A vertical pipe open at both ends is partially submerge in water, and a tuning fork vibration at an unknown frequency is placed near the top of the pipe. The length L of the air column can be adjusted by moving the pipe vertically. The sound waves generated by the fork are reinforced of the pipe. For a certain pipe, the smallest value of L for which a peak occurs in the sound intensity is 9.00 cm.



(a) What is the frequency of the tuning fork?

(b) What are the values of L for the next two resonance conditions?

Sol. Consider how this problem differs from the preceding problem. In the culvert, the length was fixed and the air column was presented with a mixture of many frequencies. The pipe in this example is presented with one single frequency from the tuning fork, and the length of the pipe is varied until resonance is achieved.

This example is a simple substitution problem. Although the pipe is open at its lower end to allow the water to enter, the water's surface acts like a barrier. Therefore, this setup can be modelled as an air column closed at one end.

(a) The fundamental frequency for $L = 0.090$ m:

$$f_1 = \frac{v}{4L} = \frac{343 \text{ m/s}}{4(0.090 \text{ m})} = 953 \text{ Hz}$$

Because the tuning fork causes the air column to resonate at this frequency, this frequency must also be that of the tuning fork.

(b) Use equation $v = \lambda f$ to find the wavelength of the sound wave from the tuning fork.

$$\lambda = \frac{v}{f} = \frac{343 \text{ m/s}}{953 \text{ Hz}} = 0.360 \text{ m}$$

Notice from Fig. (b) that the length of the air column for the second resonance is $3\lambda/4$.

$L = 3\lambda/4 = 0.270$ m

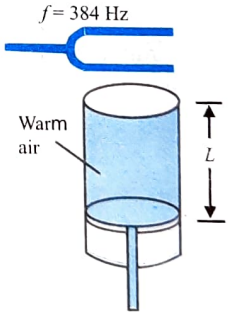
Notice from Fig. (b) that the length of the air column for the third resonance is $5\lambda/4$.

$L = 5\lambda/4 = 0.450$ m

ILLUSTRATION 7.51

An air column in a glass tube is open at one end and closed at the other by a movable piston. The air in the tube is warmed above room temperature, and a 384 Hz tuning fork is held at the open end. Resonance is heard when the piston is 22.8 cm from the open end and again when it is 68.3 cm from the open end. **(a)** What speed of sound is implied by these data? **(b)** How far from the open end will the piston be when the next resonance is heard?

Sol. The resonance is heard as amplification, when all the air in the tube vibrates in a standing-wave pattern along with the tuning fork.



The air vibration has a node at the piston and an antinode at the top end. For an air column closed at one end, resonances will occur when the length of the column is equal to $\lambda/4$, $3\lambda/4$, $5\lambda/4$ and so on. Thus, the change in the length of the pipe from one resonance to the next is $d_{NN} = \lambda/2$. In this case,

$\lambda/2 = (0.683 - 0.228)$ m $= 0.455$ m

and $\lambda = 0.910$ m

(a) $v = f\lambda = (384 \text{ Hz})(0.910 \text{ m}) = 349$ m/s

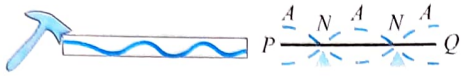
(b) $L = 0.683 \text{ m} + 0.455 \text{ m} = 1.14$ m

VIBRATION IN RODS

We have discussed the formation of stationary waves in string and organ pipes. A solid rod can be made to vibrate, both transversely and longitudinally. The mode of vibrations set up within the rod depends upon the manner of its striking and the way the bar is clamped.

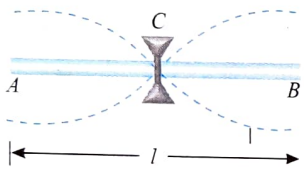
(a) Transverse vibrations: If a metal bar being struck in a direction perpendicular to its length. The deformation caused by the hammer propagates along the bar and a transverse wave is produced

Another notable case of transverse vibrations in rod occurs when a rod PQ of rectangular cross-section supported at two points within its free ends, is given a hammer blow at its midpoint, as shown in figure. The two points at which it is supported become nodes with antinodes at midpoint and at the two free ends, P and Q.



(b) Longitudinal vibrations: Longitudinal vibrations in a rod can be produced by rubbing it along its length with a wet flannel or resin coated cloth. A rod PQ, clamped at the middle, is rubbed at one of its free ends longitudinally, as shown in figure.

Longitudinal waves are set up in the rod and they get reflected from the free ends A and B. By superposition, longitudinal stationary waves are set up in the rod, with free ends acting as antinodes (A,A) and the clamp point acting as a node (N). This mode of vibration of the rod can be compared with an open organ pipe case where it is producing a fundamental note or first harmonic. In that case too, the free ends are two antinodes and the midpoint is a node.



If λ be the wavelength of the wave, we have

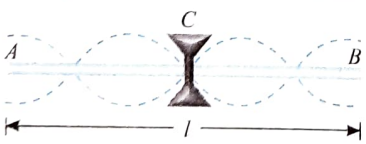
$$l = \frac{\lambda}{2} \Rightarrow \lambda = 2l \qquad \text{...(i)}$$

Thus the frequency of fundamental oscillations of a rod damped at mid point can be given as

$$f_0 = \frac{v}{\lambda} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}} \qquad \text{...(ii)}$$

Where Y is the Young's modulus of the material of rod and ρ is the density of the material of rod.

Next higher frequency at which rod vibrates will be then one when wave length is decreased to a value so that one node is inserted between mid point and an end of rod as shown in figure.



In this case if λ_1 be the wavelength of the waves in rod, we have

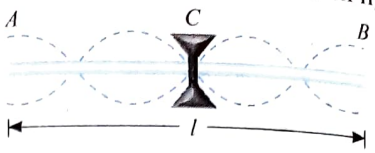
$$l = \frac{3\lambda}{2} \Rightarrow \lambda = \frac{2l}{3} \qquad \text{...(iii)}$$

Thus in this case the oscillation frequency of rod can be given as

$$f_1 = \frac{v}{\lambda} = \frac{3}{2l}\sqrt{\frac{Y}{\rho}} = 3f_0 \qquad \text{...(iv)}$$

This is called first overtone frequency of the damped rod or third harmonic frequency.

Similarly, the next higher frequency of oscillation i.e. second overtone of the oscillating rod can be shown in figure.



Here if λ be the wavelength of the wave then it can be given as

$$l = \frac{5\lambda}{2} \Rightarrow \lambda = \frac{2l}{5}$$

Thus the frequency of oscillation of rod can be given as

$$f_2 = \frac{v}{\lambda} = \frac{5}{2l}\sqrt{\frac{Y}{\rho}} = 5f_0 \qquad \text{...(v)}$$

Thus the second overtone frequency is the fifth harmonic of the fundamental oscillation frequency of rod.

We can also see from the above analysis that the resonant frequencies at which stationary waves are setup in a damped rod are only odd harmonics of fundamental frequency.

Thus when an external source of frequency matching with any of the harmonic of the damped rod then stationary waves are setup in the rod.

ILLUSTRATION 7.52

An aluminium rod 1.60 m long is held at its centre. It is stroked with a rosin-coated cloth to set up a longitudinal vibration. The speed of sound in thin rod of aluminium is 5100 m/s. **(a)** What is the fundamental frequency of the waves established in the rod? **(b)** What harmonics are set up in the rod held in this manner? **(c)** What would be the fundamental frequency if the rod were copper, in which the speed of sound is 3560 m/s?

Sol. Standing waves are important because any wave confined to a restricted region of space will be reflected back onto itself by the boundaries of the region. Then the travelling waves moving in opposite directions constitute a standing wave. The rod can sing at a few hundred hertz and at integer-multiple higher harmonics. The frequency will be proportionately lower with copper.

We must identify where nodes and antinodes are and use the fact that antinodes are separated by half a wavelength with $v = f\lambda$.

(a) Since the central clamp establishes a node at the centre, the fundamental node of vibration will be ANA. Thus the rod length is $L = d_{AA} = \lambda/2$.

Our first harmonic frequency is

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2L} = \frac{5100 \text{ m/s}}{3.20 \text{ m}} = 1.59 \text{ kHz}$$

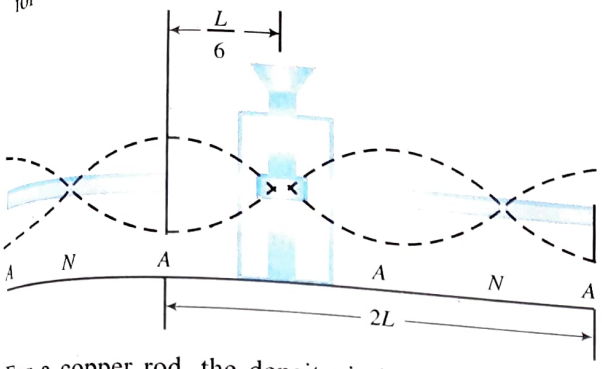
(b) Since the rod is free at each end, the ends will be antinodes. The next vibration state will not have just one more node and one more antinode, reading ANANA, as shown in the diagram, with a wavelength and frequency of

$$\lambda = \frac{2L}{3} \quad \text{and} \quad f = \frac{v}{\lambda} = \frac{3v}{2L} = 3f_1$$

Since $f_2 = 2f_1$ was bypassed as having an antinode at the centre rather than the node required, we know that we get only odd harmonics.

$$f = \frac{nv}{2L} = n(1.59 \text{ kHz})$$

for $n = 1, 3, 5, \ldots$



(c) For a copper rod, the density is higher, so the speed of sound is lower, and the fundamental frequency is lower.

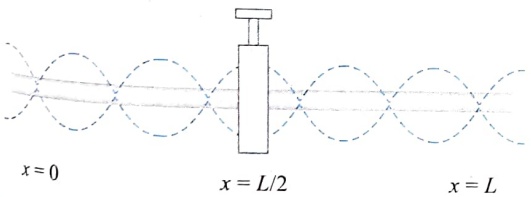
$$f_1 = \frac{v}{2L} = \frac{3560 \text{ m/s}}{3.20 \text{ m}} = 1.11 \text{ kHz}$$

The sound is not at just a few hundred hertz, but is a squeak or a squeal at over a thousand hertz. Only a few higher harmonics are in the audible range. Sound really moves fast in materials that are stiff against compression. For a thin rod, it is Young's modulus that determines the speed of a longitudinal wave.

ILLUSTRATION 7.53

A metallic rod of length 1 m is rigidly clamped at its midpoint. Longitudinal stationary waves are set up in the rod in such a way that there are two nodes on either side of the midpoint. The amplitude of an antinode is 2×10^{-6} m. Write the equation of motion at a point 2 cm from the midpoint and those of constituent waves in the rod. ($Y = 2 \times 10^{11}$ N/m^2 and $\rho = 8 \times 10^3$ kg/m^3)

Sol. As found in case of strings, in case of rods also the clamped point behaves as a node while the free end antinode. The situation is shown in figure.



Because the distance between two consecutive nodes is $(\lambda/2)$ while between a node and antinode is $\lambda/4$, hence

$$4 \times \left[\frac{\lambda}{2}\right] + 2\left[\frac{\lambda}{4}\right] = L \quad \text{or} \quad \lambda = \frac{2 \times 1}{5} = 0.4 \text{ m}$$

Further, it is given that $Y = 2 \times 10^{11}$ N/m^2 and $\rho = 8 \times 10^3$ kg/m^3

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2 \times 10^{11}}{8 \times 10^3}} = 5000 \text{ m/s}$$

Hence, from $v = n\lambda$, $n = \frac{v}{\lambda} = \frac{5000}{0.4} = 12500$ Hz

Now if incident and reflected waves along the rod are

$y_1 = A \sin(\omega t - kx)$ and $y_2 = A \sin(\omega t + kx + \phi)$

The resultant wave will be

$$y = y_1 + y_2 = A[\sin(\omega t - kx) + \sin(\omega t + kx + \phi)]$$

$$= 2A \cos\left(kx + \frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right)$$

Because there is an antinode at the free end of the rod, hence amplitude is maximum at $x = 0$. So

$$\cos\left(k \times 0 + \frac{\phi}{2}\right) = \text{Maximum} = 1 \quad \text{i.e.,} \quad \phi = 0$$

And $A_{max} = 2A = 2 \times 10^{-6}$ m (given)

$$y = 2 \times 10^{-6} \cos kx \sin \omega t$$

$$y = 2 \times 10^{-6} \cos\left[\frac{2\pi x}{\lambda}\right] \sin(2\pi nt)$$

Putting values of λ and n, we get $y = 2 \times 10^{-6} \cos 5\pi x \sin 25000\pi t$

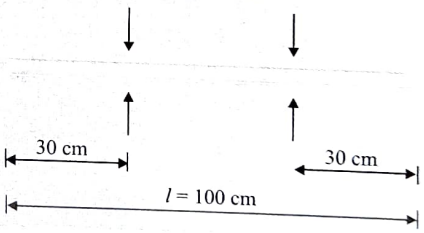
Now, because for a point 2 cm from the midpoint $x = (0.50 \pm 0.02)$, hence

$$y = 2 \times 10^{-6} \cos 5\pi(0.5 \pm 0.02) \sin 25000\pi t$$

ILLUSTRATION 7.54

A metal rod of length $l = 100$ cm is clamped at two points A and B as shown in figure. Distance of each clamp from nearer end is $a = 30$ cm.

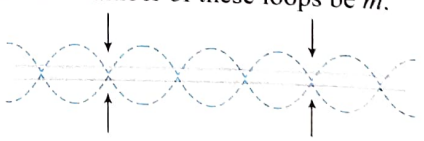
If density and Young's modulus of elasticity of rod material are $\rho = 9000$ kg/m^3 and $Y = 144$ GPa, respectively, calculate minimum and next higher frequency of natural longitudinal oscillations of the rod.



Sol. Speed of longitudinal waves in the rod is

$$v = \sqrt{\frac{Y}{\rho}} = 4000 \text{ m/s}$$

Since points A and B are clamped, therefore, nodes are formed at these points or rod oscillates with integer number of loops in the middle part. Let number of these loops be m.



Since, length of each loop is $\lambda/2$, therefore, $m\lambda/2 = (1 - 2a)$

or $m\lambda = 80$ cm ...(i)

Since the ends of the rod are free, therefore, antinodes are formed at each end of the rod or at one end of each end part is an antinode and at the other end is a node. It means that number of loops in each end part will be an odd multiple of half.

Let these be $(2n - 1)/2$where n is an integer.

Then, $\left(\frac{2n-1}{2}\right)\frac{\lambda}{2} = a$ or $(2n-1)\lambda = 120$ cm ...(ii)

Dividing Eq. (i) by Eq. (ii), $\frac{m}{(2n-1)} = \frac{2}{3}$ ...(iii)

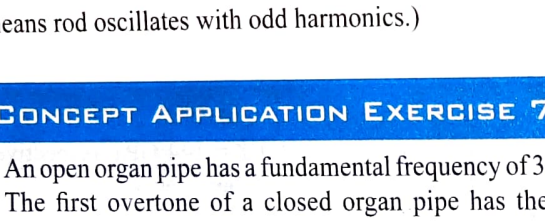
Minimum possible frequency corresponds to maximum possible wavelength, hence, minimum number of loops.

Hence, from Eq. (iii), for minimum frequency m should be equal to 2 and $(2n-1)$ should be equal to 3 or $n = 2$.

Substituting $m = 2$ in Eq. (i), Maximum wavelength $\lambda_0 = 40$ cm

Minimum frequency, $f_0 = \frac{v}{\lambda_0} = 10$ kHz

Next higher frequency corresponds to next higher integer values of m and n which satisfy Eq. (iii). Hence, for this case $m = 6$ and $(2n-1) = 9$ or $n = 5$.



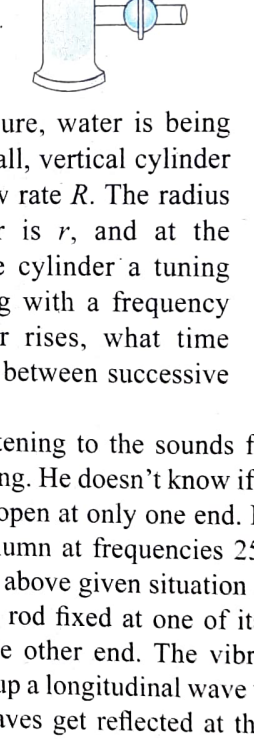
Substituting $m = 6$ in Eq. (i), $\lambda = \frac{80}{6}$ cm or $\frac{40}{3}$ cm

Therefore, next higher frequency, $f_1 = \frac{v}{\lambda} = 30$ kHz

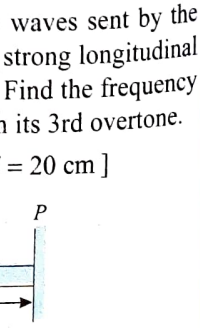
(It means rod oscillates with odd harmonics.)

CONCEPT APPLICATION EXERCISE 7.3

1. An open organ pipe has a fundamental frequency of 300 Hz. The first overtone of a closed organ pipe has the same frequency as the first overtone of the open pipe. Find length of each pipe. The velocity of sound in air = 350 m/s.
2. An open organ pipe of length 11 cm in its fundamental mode vibrates in resonance with the first overtone of a closed organ pipe of length 13.2 cm filled with some gas. If the velocity of sound in the air is 330 m/s, calculate the velocity of sound in the unknown gas.
3. Find the fundamental frequency and the first four overtones of a 15 cm pipe **(a)** if the pipe is closed at one end, and **(b)** if the pipe is open at both ends. **(c)** How many overtones may be heard by a person of normal hearing in each of the above cases? Velocity of sound in air = 330 m/s.
4. A glass tube of length 1.5 m is filled completely with water; the water can be drained out slowly at the bottom of the tube. Find the total number of resonance obtained, when a tuning fork of frequency 606 Hz is put at the upper open end of the tube, $v_{sound} = 340$ m/s.
5. A tube closed at one end has a vibrating diaphragm at the other end, which may be assumed to be a displacement node. It is found that when the frequency of the diaphragm is 2000 Hz, a stationary wave pattern is set up in which the distance between adjacent nodes is 8 cm. When the frequency is gradually reduced, the stationary wave pattern disappears but another stationary wave pattern reappears at a frequency of 1600 Hz. Calculate
   (a) the speed of sound in air,
   (b) the distance between adjacent nodes at a frequency of 1600 Hz,
   (c) the distance between the diaphragm and the closed end,
   (d) the next lower frequencies at which stationary wave patterns will be obtained.
6. The 1st overtone of a pipe closed at one of its end is 900 Hz. Find the fundamental frequency of this pipe open at both ends.
7. Find the ratio of length of a closed pipe to that of the open pipe in order that the second overtone of the former is in unison with fourth overtone of the latter.
8. An organ pipe has two successive harmonics with frequencies 1372 and 1764 Hz. Find these harmonics and the length of the rube. Assume the sound speed $v = 330$ m/s
9. A tuning fork with a frequency of $f = 500$ Hz is placed near the top of the tube shown in figure. The water level is lowered so that the length L slowly increases from an initial value of 20.0 cm. Determine the next two values of L that correspond to resonant modes. Assume the sound speed $v = 320$ m/s



10. In the given figure, water is being pumped into a tall, vertical cylinder at a volume flow rate R. The radius of the cylinder is r, and at the open top of the cylinder a tuning fork is vibrating with a frequency f. As the water rises, what time interval elapses between successive resonances?



11. A student is listening to the sounds from an air column that is 0.70 m long. He doesn't know if the column is open at both ends or open at only one end. He hears resonance from the air column at frequencies 250 Hz and 600 Hz. Justify, why the above given situation impossible.
12. A hard metallic rod fixed at one of its end is fitted with a vibrator at the other end. The vibrator is electrically driven and sets up a longitudinal wave which moves along the rod. The waves get reflected at the fixed end P. The reflected waves superimpose with the waves sent by the vibrator. For the resonance to occur, a strong longitudinal standing wave is generated in the rod. Find the frequency of the vibrator if the rod resonates with its 3rd overtone.
    [Put $Y = 6\times10^8$ N/m^2, $\rho = 24$ gm/cc, $l = 20$ cm]

ANSWERS

1. Open pipe, 0.58 m; Closed pipe, 0.44 m **2.** 264 m/s
3. (a) 550, 1650, 2750, 3850 and 4950 Hz
(b) 1100, 2200, 3300, 4400 and 5500 Hz (c) 17 **4.** 5
5. (a) 320 m/s (b) 10 cm (c) 40 cm (d) 1200 Hz **6.** 600 Hz
7. 1:2 **8.** 0.42 m **9.** 0.48 m and 0.80 m **10.** $\frac{\pi \partial^2 v}{2Rf}$
11. It is impossible because a single column could not produce both frequencies.
12. $\frac{875}{2}\sqrt{10}$ Hz

BEATS: INTERFERENCE IN TIME

If we take two tunning forks of slightly different frequencies f_1 and f_2 (say) are sounded separately, we can not distinguish their tones. However, when they are sounded together, the superposition of two waves produces a resulting sound whose loudness increases and decreases periodically with much lower frequencies compared to their individual frequencies such that we can easily observe the resulting wave . In other words, the loudness (intensity) varies from maximum to minimum (loud and faint) at regular interval of time. This effect is called "beat". The number of variations (loud sound) heard in one-second is called beat frequency

Now let's consider another type of interference, one that results from the superposition of two waves having slightly different frequencies. In this case, when the two waves are observed at a point in space, they are periodically in and out of phase. That is, there is a temporal (time) alternation between constructive and destructive interference. As a consequence, we refer to this phenomenon as interference in time or temporal interference. For example, if two tuning forks of slightly different frequencies are struck, one hears a sound of periodically varying amplitude. This phenomenon is called beating.

Beating is the periodic variation in amplitude at a given point due to the superposition of two waves having slightly different frequencies.

The number of amplitude maxima one hears per second, or the beat frequency, equals the difference in frequency between the two sources as we shall show below.

Consider two sound waves of equal amplitude travelling through a medium with slightly different frequencies f_1 and f_2. We use equations similar to equation $y = A \sin(kx - \omega t)$ to represent the wave functions for these two waves at a point that we choose so that $kx = \pi/2$:

$$y_1 = A \sin\left(\frac{\pi}{2} - \omega_2 t\right) = A \cos(2\pi f_1 t)$$

$$y_2 = A \sin\left(\frac{\pi}{2} - \omega_2 t\right) = A \cos(2\pi f_2 t)$$

Using the superposition principle, we find that the resultant wave function at this point is

$$y = y_1 + y_2 = A(\cos 2\pi f_1 + \cos 2\pi f_2 t)$$

The trigonometric identity

$$\cos a + \cos b = 2 \cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

allows us to write the expression for y as

$$y = \left[2A \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t\right]\cos 2\pi\left(\frac{f_1 + f_2}{2}\right)t \quad \ldots(i)$$

Graphs of the individual waves and the resultant wave are shown in Figs. (a) and (b). From the factors in Eq. (i), we see that the resultant wave has an effective frequency equal to the average frequency $(f_1 + f_2)/2$. This wave is multiplied by an envelope wave given by the expression in the square brackets:

$$y_{\text{envelope}} = 2A \cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t \quad \ldots(ii)$$

That is, the amplitude and therefore the intensity of the resultant sound vary in time. The dashed line in Fig. (b) is a graphical representation of the envelope wave in Eq. (ii) and is a sine wave varying with frequency $(f_1 - f_2)/2$.

A maximum in the amplitude of the resultant sound wave is detected whenever

$$\cos 2\pi\left(\frac{f_1 - f_2}{2}\right)t = \pm 1$$

Hence, there are two maxima in each period of the envelope wave. Because the amplitude varies with frequency as $(f_1 - f_2)/2$, the number of beats per second, or the beat frequency f_{beat}, is twice this value. That is,

$$f_{\text{beat}} = |f_1 - f_2| \quad \ldots(iii)$$

For instance, if one tuning fork vibrates at 438 Hz and a second one vibrates at 442 Hz, the resultant sound wave of the combination has a frequency of 440 Hz (the musical note *A*) and a beat frequency of 4 Hz. A listener would hear a 440 Hz sound wave go through an intensity maximum four times every second.

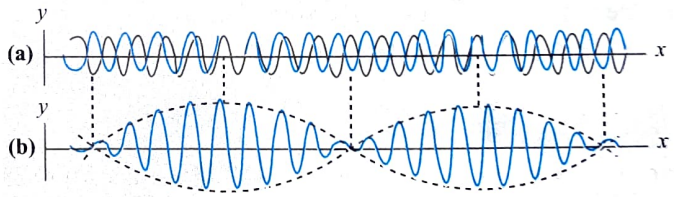


ILLUSTRATION 7.55

Two tuning forks *A* and *B* produce 4 beats per second when sounded simultaneously. The frequency of *A* is known to be 256 Hz. When *B* is loaded with a little wax 4 beats per second are again produced. Find the frequency of *B* before and after loading.

Sol. Since the beat frequency is 4 per second, the possible frequencies of fork *B* before loading are

(256 + 4) Hz and (256 – 4) Hz

or 260 Hz and 252 Hz

After loading *B* with wax, again 4 beats/second are heard. Therefore, after loading, the possible frequencies are

(256 + 4) Hz and (256 – 4) Hz

or 260 Hz and 252 Hz

But, by loading the fork its frequency can only decrease. Hence, before loading fork *B*, its frequency can only be 260 Hz, so that after loading it decreases to 252 Hz.

ILLUSTRATION 7.56

Two identical piano strings of length 0.750 m are each tuned exactly to 440 Hz. The tension in one of the strings is then increased by 1.0%. If they are now struck, what is the beat frequency between the fundamental of the two strings?

Sol. As the tension in one of the strings is changed, its fundamental frequency changes. Therefore, when both strings are played, they will have different frequencies and beats will be heard.

We must combine our understanding of the waves under boundary conditions model for strings with our new knowledge of beats.

Set up a ratio of the fundamental frequencies of the two strings using equation

$$\frac{f_2}{f_1}=\frac{(v_2/2L)}{(v_1/2L)}=\frac{v_2}{v_1}$$

Use equation $v=\sqrt{T/\mu}$ to substitute for the wave speeds on the strings.

$$\frac{f_2}{f_1}=\frac{\sqrt{T_2/\mu}}{\sqrt{T_1/\mu}}=\sqrt{\frac{T_2}{T_1}}$$

Incorporate that the tension in one string is 1.0% larger than the other; that is, $T_2=1.010\,T_1$.

$$\frac{f_2}{f_1}=\sqrt{\frac{1.010T_1}{T_1}}=1.005$$

Solve for the frequency of the tightened string:

$$f_2=1.005\,f_1=1.005\ (440\ \text{Hz})=442\ \text{Hz}$$

Find the beat frequency using Eq. (iii)

$$f_{beat}=442\ \text{Hz}-440\ \text{Hz}=2\ \text{Hz}$$

ILLUSTRATION 7.57

Wavelength of two notes in air is (90/175) m and (90/173) m, respectively. Each of these notes produces 4 beats/s with a third note of a fixed frequency. Calculate the velocity of sound in air.

Sol. Given: $\lambda_1=90/175$ m and $\lambda_2=90/173$ m

If f_1 and f_2 are the corresponding frequencies and v is the velocity of sound in air, we have

$$v=f_1\lambda_1 \quad \text{and} \quad v=f_2\lambda_2$$

$$f_1=\frac{v}{\lambda_1} \quad \text{and} \quad f_2=\frac{v}{\lambda_2}$$

Since, $\lambda_1<\lambda_2$, we must have $f_1>f_2$.

If f is the frequency of the third note, then

$$f_1-f=4 \quad \text{and} \quad f-f_2=4$$

$$\Rightarrow \quad f_1-f_2=8$$

$$\frac{v}{\lambda_1}-\frac{v}{\lambda_2}=8$$

$$v\left[\frac{175}{90}-\frac{173}{90}\right]=8$$

$$v=360\ \text{m/s}$$

ILLUSTRATION 7.58

When the wire of a sonometer is 73 cm long, it is in resonance with a tuning fork. On shortening the wire by 0.5 cm, it makes 3 beats with the same fork. Calculate the frequency of the tuning fork.

Sol. We are given that

L (original length of the wire) = 73 cm

L' (shortened length of the wire) = 73 – 0.5 = 72.5 cm

Let f be the frequency of the tuning fork.

This also is the frequency of the wire when its length is L as it is then in resonance with the tuning fork. Let f' be the frequency of the wire when its length is L'. Since it then produces 3 beats/s with the tuning fork, $f'=f+3$

(On shortening the length, the frequency of the wire increases)

As $f=\frac{p}{2L}\sqrt{\frac{T}{\mu}}\Rightarrow fL=\text{contant}$

$$fL=fL'$$

or $$f\times 73=(f+3)(72.5)=72.5f+217.5$$

or $$73f-72.5f=217.5$$

or $$0.5f=217.5 \quad \text{or} \quad f=435\,\text{Hz}$$

ILLUSTRATION 7.59

The two parts of a sonometer wire divided by a movable knife edge differ by 2 mm and produce one beat per second when sounded together. Find their frequencies if the whole length of the wire is 1 m.

Sol. Let L_1 and L_2 be the lengths of the two parts of a sonometer wire divided by a movable knife edge.

Thus, $L_1+L_2=100$ cm and $l_1-L_2=2$ mm $=0.2$ cm

Adding and subtracting, we get $L_1=50.1$ cm and $L_2=49.9$ cm

Let f_1 and f_2 be the frequencies of the sonometer wire corresponding to the lengths L_1 and L_2 respectively.

Number of the beats produced $f_2-f_1=1$ ...(i)

As, $f=\frac{p}{2L}\sqrt{\frac{T}{\mu}}\Rightarrow fL=\text{contant}$

$$\frac{f_2}{f_1}=\frac{L_1}{L_2}=\frac{50.1}{49.9}$$

or $$f_2=\frac{50.1}{49.9}f_1 \qquad ...(ii)$$

From eqns. (i) and (ii), we have $\frac{50.1}{49.9}f_1-f_1=1$

or $$\frac{50.1f_1-49.9f_1}{49.9}=1$$

or $$0.2f_1=49.9 \text{ or } f_1=\frac{49.9}{0.2}=249.5\,\text{Hz}$$

From eqn. (i), we have $f_2=f_1+1=249.5+1=250.5$ Hz

CONCEPT APPLICATION EXERCISE 7.4

1. Two tuning forks A and B when sounded together produce 3 beats/s. What are the possible frequencies of B, if the frequency of A is 400 cycle/s? how can you verify which of the possible values is correct?
2. Two sound waves travelling in the same direction are superposed. Their frequencies are 300 and 302 Hz and their amplitudes are 0.2 and 0.3 mm, respectively
   (a) What is the number of beats per second?
   (b) What are the maximum and minimum values of resultant amplitude during the formation of beats?
   (c) Calculate the ratio of maximum and minimum intensities of the resultant sound.
3. Calculate the speed of sound in a gas in which two waves of wavelengths 50 cm and 50.5 cm produce 6 beats/s.
4. Two tuning forks A and B produce 4 beats/s when sounded together. A resonates to 32.4 cm of stretched wire and B is in resonance with 32 cm of the same wire. Determine the frequencies of the two tuning forks.
5. A tuning fork A is in resonance with an air column 32 cm long and closed at one end. When the length of this column is increased by 1 cm, it is in resonance with another fork B. When A and B are sounded together, they produce 40 beats in 5 s. Find their frequencies.
6. Two sonometer wires of the same material and cross-section are of lengths 50 cm and 60 cm and are stretched by tensions of 4.5 kg and 5.12 kg, respectively. If the number of beats heard (when the two wires are vibrating) be 2 per second, find the mass per unit length of the wires. Take $g = 10\ m/s^2$.
7. A wire is in unison with a tuning fork when stretched by a weight of density 9000 kg/m^3 in a sonometer experiment. When the weight is immersed in water, the same wire produces 5 beats/s with the same fork. Find the frequency of the fork.

ANSWERS

1. 403 or 397 **2.** (a) 2 (b) 0.5 mm, 0.1 mm (c) 25 : 1
3. 303 m/s **4.** 320 Hz, 324 Hz **5.** 264 Hz, 256 Hz
6. 0.14 kg/m **7.** 87.4 Hz

Solved Examples

EXAMPLE 7.1

A standing wave is set up in a string of variable length and tension by a vibrator of variable frequency. Both ends of the string are fixed. When the vibrator has a frequency f, in a string of length L and under tension T, n antinodes are set up in the string. **(a)** If the length of the string is doubled, by what factor should the frequency be changed so that the same number of antinodes is produced? **(b)** If the frequency and length are held constant, what tension will produce $n + 1$ antinodes? **(c)** If the frequency is tripled and the length of the string is halved, by what factor should the tension be changed so that twice as many antinodes are produced?

Sol. In (a), we expect a lower frequency to go with a longer wavelength. In (b), lower tension should go with lower wave speed for shorter wavelength at constant frequency. In (c), we will just have to divide it out.

(a) We have $f_n = \dfrac{n}{2L}\sqrt{\dfrac{T}{\mu}}$ ...(i)

Keeping n, T and μ constant, we can create two equations.

$$f_n L = \frac{n}{2}\sqrt{\frac{T}{\mu}} \quad \text{and} \quad f_n' L' = \frac{n}{2}\sqrt{\frac{T}{\mu}}$$

Dividing the equations gives $\dfrac{f_n}{f_n'} = \dfrac{L'}{L}$

If $L' = 2L$, then $f_n' = 1/2\, f_n$

Therefore, in order to double the length but keep the same number of antinodes, the frequency should be halved.

(b) From Eq. (i), we can hold L and f_n constant to get

$$\frac{n'}{n} = \sqrt{\frac{T}{T'}}$$

From this relation, we see that the tension must be decreased to

$$T' = T\left(\frac{n}{n+1}\right)^2 \text{ to produce } n+1 \text{ antinodes}$$

(c) The time, we rearrange Eq. (i) to produce

$$\frac{2f_n L}{n} = \sqrt{\frac{T}{\mu}} \quad \text{and} \quad \frac{2f_n' L'}{n'} = \sqrt{\frac{T'}{\mu}}$$

Then dividing gives

$$\frac{T'}{T} = \left(\frac{f_n'}{f_n} \times \frac{n}{n'} \times \frac{L'}{L}\right)^2 = \left(\frac{3f_n}{f_n}\right)^2 \times \left(\frac{n}{2n}\right)^2 \times \left(\frac{L/2}{L}\right)^2 = \frac{9}{16}$$

EXAMPLE 7.2

The water level in a vertical glass tube 1.0 m long can be adjusted to any position in the tube. A tuning fork vibrating at 660 Hz is held just over the open top end of the tube. At what positions of the water level will there be resonance. Speed of sound is 330 m/s.

Sol. Resonance corresponds to a pressure antinode at closed end and pressure node at open end. Further, the distance between a pressure node and a pressure antinode is $\lambda/4$, the condition of resonance would be, length of air column

$$l = n\frac{\lambda}{4} = n\left(\frac{v}{4f}\right)$$

Here, $n = 1, 3, 5, \ldots$

$$l_1 = (1)\left(\frac{330}{4 \times 660}\right) = 0.125 \text{ m}$$

and $l_2 = 3l_1 = 0.375$ m

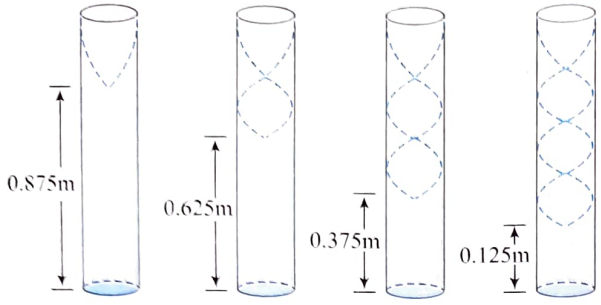
$l_3 = 5l_1 = 0.625$ m

and $l_4 = 7l_1 = 0.875$ m

$l_5 = 9l_1 = 1.125$ m

Since $l_5 > 1$ m (the length of tube), the length of air columns can have the values from l_1 to l_4 only. Therefore, level of water at resonance will be

$(1.0 - 0.125)$ m $= 0.875$ m
$(1.0 - 0.375)$ m $= 0.625$ m
$(1.0 - 0.625)$ m $= 0.375$ m
and $(1.0 - 0.875)$ m $= 0.125$ m



In all the four cases shown in the figure, the resonance frequency is 660 Hz but the first one is the fundamental tone or first harmonic. The second is first overtone or the third harmonic and so on.

EXAMPLE 7.3

Two radio stations broadcast their programmes at the same amplitude A, and at slightly different frequencies ω_1 and ω_2 respectively, where $\omega_2 - \omega_1 = 10^3$ Hz. A detector receives the signals from the two stations simultaneously. It can emit signals only of intensity $\geq 2A^2$.

(a) Find the time intervals between successive maxima of the intensity of the signal received by the detector.

(b) Find the time for which the detector remains idle in each cycle of the intensity of the signal.

Sol.

(a) Let the signal waves be given by

$$y_1 = A \sin 2\pi\omega_1 t, \quad y_2 = A \sin 2\pi\omega_2 t.$$

The resultant disturbance is given by

$$y = y_1 + y_2 = A \sin 2\pi\omega_1 t + A \sin 2\pi\omega_2 t$$

$$= 2A \sin \frac{2\pi(\omega_1 + \omega_2)t}{2} \cos \frac{2\pi(\omega_2 - \omega_1)t}{2}$$

$$= 2A \cos \pi(\omega_2 - \omega_1)t \sin 2\pi \frac{(\omega_1 + \omega_2)t}{2}$$

Let $\omega_1 = \omega$, $\omega_2 = \omega + \omega$

Therefore, $\omega_1 + \omega_2 \approx 2\omega$

$$y = 2A \cos \pi(\omega_2 - \omega_1)t \sin 2\pi\omega t$$

Thus, the resultant disturbance has amplitude

$$2A \cos \pi(\omega_2 - \omega_1)t$$

For maxima: $\cos \pi(\omega_2 - \omega_1)t = \pm 1$

or $\pi(\omega_2 - \omega_1)t = r\pi$, $r = 0, 1, 2, 3$

$$t = \frac{r}{\omega_2 - \omega_1} = 0, \frac{1}{\omega_2 - \omega_1}, \frac{2}{\omega_2 - \omega_1}, \ldots$$

Clearly time interval between successive maxima

$$= \frac{1}{\omega_2 - \omega_1} = \frac{1}{10^3} = 10^{-3} \text{ s}$$

(b) The resultant intensity is given by

$$I = A_1^2 + A_2^2 + 2A_1A_2 \cos \delta$$

When $\delta = 0$, intensity is maximum $I_{max} = 4A^2$
When $\Delta = \pi/2$, intensity $I = 2A^2$
When $\Delta = \pi$, intensity $I_{min} = 0$
When $\Delta = 3\pi/2$, intensity $I = 2A^2$
When $\Delta = 2\pi$, intensity $I_{max} = 4A^2$

The detector remaining idle from $\theta = \pi/2$ to $3\pi/2$ or in each half cycle. Hence the required time

$$t = \frac{T}{2} = \frac{10^{-3}}{2} = 5 \times 10^{-4} \text{ s}$$

EXAMPLE 7.4

A metal wire of diameter 1 mm is held on two knife edges by a distance 50 cm. The tension in the wire is 100 N. The wire vibrating with its fundamental frequency and a vibrating tuning fork together produce 5 beats/s. The tension in the wire is then reduced to 81 N. When the two are excited, beats are heard at the same rate. Calculate

(a) the frequency of the fork and

(b) the density of material of wire

Sol. Let the frequency of the tuning fork be n. When tension in the string is decreased, the number of beats remains unchanged. This means initially the frequency of metal wire is higher than that of fork. As number of beats = 5 per second.

Therefore initial frequency of wire $n_1 = n + 5$

$$\text{So,} \quad n_1 = n + 5 = \frac{1}{2l}\sqrt{\left(\frac{T_1}{m}\right)} = \frac{1}{2 \times 0.5}\sqrt{\left(\frac{100}{m}\right)} \quad \ldots(i)$$

[Since $T = 100$ N, $l = 50$ cm $= 0.50$ m]

When tension is reduced to $T_2 = 81$ N, the frequency of wire becomes $n_2 = n - 5$.

$$\text{Therefore, } n_2 = n - 5 = \frac{1}{2l}\sqrt{\left(\frac{T_2}{m}\right)} = \frac{1}{2 \times 0.5}\sqrt{\left(\frac{81}{m}\right)} \quad \ldots(ii)$$

(i) Dividing Eq. (i) by Eq. (ii), we get $\frac{n+5}{n-5} = \sqrt{\left(\frac{100}{81}\right)} = \frac{10}{9}$

Solving frequency of fork, $n = 95$ cycles/s.

(ii) We have m = mass per unit length $= Ad = \pi r^2 d$.

Substituting this in Eq. (i), we get

$$n + 5 = \frac{1}{2 \times 0.5}\sqrt{\left(\frac{100}{\pi r^2 d}\right)}$$

here $n = 95$, $r = \frac{1 \text{ mm}}{2} = \frac{1}{2} \times 10^{-3}$ m $= 5 \times 10^{-4}$ m

Putting all values $95 + 5 = \frac{1}{1.0} \times \frac{10}{\left[\sqrt{3.14 \times (5 \times 10^{-4})^2 d}\right]}$

$$d = 12.7 \times 10^3 \text{ kg/m}^3$$

EXAMPLE 7.5

An aluminium wire of cross-sectional area 1×10^{-6} m^2 is joined to a steel wire of the same cross-sectional area. This compound wire is stretched on a sonometer, pulled by a weight

of 10 kg. The total length of the compound wire between the bridges is 1.5 m of which the aluminium wire is 0.6 m and the rest is steel wire. Transverse vibrations are set up in the wire by using an external source of variable frequency. Find the lowest frequency of excitation for which standing waves are formed, such that the joint in the wire is a node. What is the total number of nodes observed at this frequency, excluding the two at the ends of the wire? The density of aluminium is 2.6×10^3 kg/m^3 and that of steel is 1.04×10^4 kg/m^3

Sol. Let l_1 and l_2 be the lengths of aluminum and steel wires, respectively.

Given $l_1 = 0.6$ m, $l_1 + l_2 = 1.5$ m

$\therefore \quad l_2 = 1.5 - 0.6 = 0.9$ m

If aluminium wire vibrates in p_1 loops and steel wire in p_2 loops, then frequency n is give by

$$n = n_1 = \frac{p_1}{2l_1}\sqrt{\left(\frac{T}{m_1}\right)} = \frac{p_1}{2l_1}\sqrt{\left(\frac{T}{A\rho_1}\right)}$$

Also, $n = n_2 = \frac{p_2}{2l_2}\sqrt{\left(\frac{T}{m_2}\right)} = \frac{p_2}{2l_2}\sqrt{\left(\frac{T}{A\rho_2}\right)}$

We have $n_1 = n_2 = n$, so $\frac{p_1}{2l_1}\sqrt{\left(\frac{T}{A\rho_1}\right)} = \frac{p_2}{2l_2}\sqrt{\left(\frac{T}{A\rho_2}\right)}$

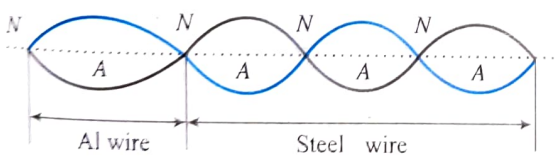
$$\therefore \quad \frac{P_1}{P_2} = \frac{l_1}{l_2}\sqrt{\left(\frac{\rho_1}{\rho_2}\right)} = \frac{0.6}{0.9}\sqrt{\left(\frac{2.6\times10^3}{1.04\times10^4}\right)}$$

$$= \frac{0.6}{0.9}\sqrt{\left(\frac{1}{4}\right)} = \frac{6}{9}\times\frac{1}{2} = \frac{3}{9} = \frac{1}{3}$$

$\therefore \quad p_1 : p_2 \equiv 1 : 3$

The minimum number of loops in aluminium wire = 1 and minimum number of loops in steel wire = 3.

The stationary vibrations in composite wire as shown in the figure. Obviously total number of nodes = 5.



The number of nodes excluding the two at the ends = 5 – 2 = 3. Thus, the lowest frequency

$$n = \frac{1}{2l_1}\sqrt{\left(\frac{T}{m_1}\right)} = \frac{1}{2\times0.6}\sqrt{\left(\frac{10\times9.8}{1\times10^{-6}\times2.6\times10^3}\right)} = 162 \text{ Hz}$$

EXAMPLE 7.6

A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelengths 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope?

Sol. Tension at the lower end of the rope,

$T_1 = 2g = 2 \times 9.8 = 19.6$ N

Tension at the upper end of rope, $T_2 = (2 + 6)\,g = 8 \times 9.8 = 78.4$ N

Let v_1 and v_2 be the speeds of pulse at the lower and upper end, respectively. So

$$v_1 = \sqrt{\left(\frac{T_1}{m}\right)}, \quad v_2 = \sqrt{\left(\frac{T_2}{m}\right)}$$

On dividing, we get $\frac{v_2}{v_1} = \sqrt{\left(\frac{T_2}{T_1}\right)} = \sqrt{\left(\frac{78.4}{19.6}\right)} = \sqrt{4} = 2$

As frequency is independent of medium, therefore if λ_1 and λ_2 are wavelengths at lower and upper ends respectively, Then

$$v_1 = n\lambda_1 \quad \text{and} \quad v_2 = n\lambda_2$$

So, $\frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1} = 2$

Therefore, the wavelength of pulse at upper end $= 2\lambda$,

$= 2 \times 0.06 = 0.12$ m

EXAMPLE 7.7

The vibrations of a string of length 60 cm fixed at both ends are represented by the equation

$$y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t)$$

where x and y are in cm and t in second.

(a) What is the maximum displacement at $x = 5$ cm?

(b) Where are the nodes located along the string?

(c) What is the velocity of the particle at $x = 7.5$ cm at $t = 0.25$ s?

(d) Write down the equations of component waves whose superposition gives the above waves.

Sol. The given equation for standing waves in the string is

$$y = 4\sin\left(\frac{\pi x}{15}\right)\cos(96\pi t) \qquad \ldots(i)$$

(a) The amplitude of the waves is given by

$$A = 4\sin\frac{\pi x}{15} \qquad \ldots(ii)$$

Therefore, the maximum displacement or amplitude at $x = 5$ cm is

$$A = 4\sin\frac{\pi\times5}{15} = 4\sin\frac{\pi}{3}$$

$$= 4\sin60^\circ = 4\times\frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$= 2\times1.732 = 3.464 \text{ cm}$$

(b) The position of zero displacement or nodes are given by

$$\sin\frac{\pi x}{15} = 0 \quad \text{or} \quad \frac{\pi x}{15} = r\pi \quad (\text{where } r = 0, 1, 2, 3, \ldots)$$

$$\Rightarrow \quad x = 15r \quad \Rightarrow \quad x = 0, 0.15 \text{ cm}, 0.30 \text{ cm}, \ldots$$

(c) Differentiating Eq. (i) with respect to t, we get velocity of particle

$$u = \frac{dy}{dt} = -4\times96\pi\sin\left(\frac{\pi x}{15}\right)\sin(96\pi t)$$

Substituting $x = 7.5$ cm and $t = 0.25$ s.

$$u = -384\pi \sin\left(\frac{\pi \times 7.5}{15}\right)\sin(96\pi \times 0.25)$$

$$= -384\sin(\pi/2)\sin(24\pi) = 0$$

(d) Using the relation

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

Equation (i) may be expressed as

$$y = 2\left[\sin\left\{\frac{\pi x}{15} + (96\pi t)\right\} + 2\sin\left\{\frac{\pi x}{15} - (96\pi t)\right\}\right]$$

$$= 2\sin\left\{\frac{\pi x}{15} + 96\pi t\right\} + 2\sin\left\{\frac{\pi x}{15} - 96\pi t\right\}$$

$$= y_1 + y_2$$

Therefore, the component waves are given by

$$y_1 = 2\sin\left(96\pi t + \frac{\pi x}{15}\right)$$

and $$y_2 = -2\sin\left(96\pi t - \frac{\pi x}{15}\right)$$

EXAMPLE 7.8

The first overtone of an open organ pipe beats with the first overtone of a closed organ pipe with a beat frequency 2.2 Hz. The fundamental frequency of the closed organ pipe is 110 Hz, find the lengths of the pipes. Take velocity of sound = 330 m/s.

Sol. Let l_1 and l_2 be lengths of open organ pipe and closed organ pipe respectively.

First overtone of open organ pipe $= 2n_1 = 2 \times \frac{v}{2l_1} = \frac{v}{l_1}$

First overtone of closed organ pipe $= 3n_2 = 3 \times \frac{v}{4l_2}$

According to question, $\frac{v}{l_1} - \frac{3v}{4l_2} = \pm 2.2$ ...(i)

As n_2 is the fundamental frequency of closed organ pipe

$$n_2 = \frac{v}{4l_2}$$

$$l_2 = \frac{v}{4n_2} = \frac{330}{4 \times 110} = 0.75 \text{ m}$$

From Eq. (i), $\frac{v}{l_1} = \frac{3v}{4l_2} \pm 2.2$

$$\frac{v}{l_1} = 3n_2 \pm 2.2$$

Taking positive sign $\frac{v}{l_1} = 3 \times 110 + 2.2 = 332.2$

$$\therefore \quad l_1 = \frac{v}{332.2} = \frac{330}{332.2} \text{ m} = 0.993 \text{ m} = 99.3 \text{ cm}$$

Taking negative sign $\frac{v}{l_1} = 3n_2 - 2.2 = 3 \times 110 - 2.2 = 327.8$

$$\therefore \quad l_1 = \frac{v}{327.8} = \frac{330}{327.8} = 1.006 \text{ m} = 100.6 \text{ cm}$$

EXAMPLE 7.9

The air in a pipe closed at one end is made to vibrate in its second overtone by a tuning fork of frequency 440 Hz. The speed of sound in air 330 m/s. End correction may be neglected. Let P_0 denotes the mean pressure of any point in the pipe and ΔP_0 the maximum amplitudes of pressure variation.

(a) Find the length L of the air column.

(b) What is the amplitude of pressure variation at the middle of the column

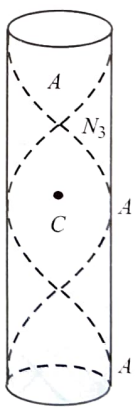
(c) What are maximum and minimum pressures at the open end of the pipe?

Sol.

(a) The fundamental frequency of the closed organ pipe $= v/4l$.
In closed organ pipe only odd harmonics are present.
Second overtone of pipe $= 5v/4L$
Given $5v/4L = 440$
On solving, we get

$$L = \frac{5v}{4 \times 440} = \frac{5 \times 330}{4 \times 440} = \frac{15}{16} \text{ m} = 0.9375 \text{ m} = 93.75 \text{ cm}$$

(b) The equation of variation of pressure amplitude at any distance x from the node is $\Delta P = \Delta P_0 \cos kx$
Pressure variation is maximum at a node and minimum (zero) at antinode.



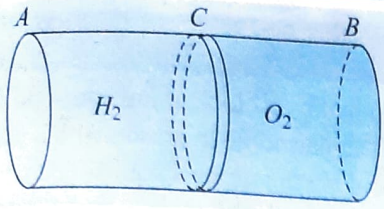
Distance of centre C from N_2 is $\lambda/8$

$$\therefore \quad \Delta P = \Delta P_0 \cos \frac{2\pi}{\lambda} \times \frac{\lambda}{8} = \Delta P_0 \frac{\pi}{4} = \frac{\Delta P_0}{\sqrt{2}}$$

(c) At antinode, the pressure variation is minimum (zero), therefore at antinode pressure remains equal to P_0 (always). Therefore, at antinode $P_{max} = P_{min} = P_0$.

EXAMPLE 7.10

AB is a cylinder of length 1.0 m fitted with a thin flexible diaphragm *C* at the middle and two others thin flexible diaphragms *A* and *B* at the ends. The portions *AC* and *BC* contains hydrogen and oxygen gases, respectively. The diaphragms *A* and *B* are set into vibrations of same frequency. What is the minimum frequency of these vibrations for which the diaphragm *C* is a node? Under the conditions of the experiment, the velocity of sound in hydrogen is 1100 m/s and in oxygen is 300 m/s.



Sol. When diaphragms A and B are set in oscillations, antinodes are formed at A and B while a node is formed at C (given)

Also $AB = 1.0$ m

and $AC = CB = l$ (say) $= 1/2 = 0.5$ m

The portions AC and BC behave as closed pipes.

In a closed pipe, the modes of vibration are given by

$$l = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$$

i.e. $$l = (2r+1)\frac{\lambda}{4}, \quad r = 0,1,2,3,\ldots$$

or $$\lambda = \frac{4l}{2r+1} \quad r = 0,1,2,3,\ldots$$

For hydrogen $\lambda_1 = 4l/(2r_1+1)$

For oxygen $\lambda_2 = 4l/(2r_2+1)$

In both gases the frequency is same

As $n_1 = n_2$

or $$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}$$

$$\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda} = \frac{l_1}{l_2} \times \frac{(2r_2+1)}{(2r_1+1)}$$

As $l_1 = l_2 = 0.5$ m

i.e. $$\frac{v_1}{v_2} = \frac{2r_2+1}{2r_1+1} = \frac{1100}{300} = \frac{2r_2+1}{2r_1+1}$$

i.e. $$\frac{2r_1+1}{2r_2+1} = \frac{3}{11}$$

For minimum frequency, the integers r_1 and r_2 should be least. Therefore by inspection

$r_1 = 1$ and $r_2 = 5$

Therefore, the frequency of oscillations is given by

$$n_{min} = \frac{v_1}{\lambda_1} = (2r_1+1)\frac{v_1}{4l}$$

$$= (2\times1+1)\times\frac{1100}{4\times0.5} = \frac{3\times1100}{2} = 1650 \text{ Hz}$$

EXAMPLE 7.11

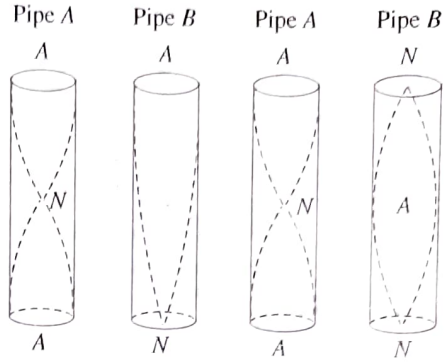
Two narrow cylindrical pipes A and B have the same length. Pipe A is open at both ends and is filled with a monoatomic gas of molar mass M_A. Pipe B is open at one end and closed at the other end and is filled with a diatomic gas of molar mass M_B. Both gases are at the same temperature.

(a) If the frequency to the second harmonic of the fundamental mode in pipe A is equal to the frequency of the third harmonic of the fundamental mode in pipe B; determine the value of M_A/M_B.

(b) Now the open end of pipe B is also closed (so that pipe B is closed at both ends). Find the ratio of the fundamental frequency in pipe A to that in pipe B.

Sol.

(a) If L is the length of each pipe A and B, then fundamental frequency of pipe A (open at both ends)



$$n_A = \frac{v_A}{2L} \quad \ldots(i)$$

Fundamental frequency of pipe B (closed at one end)

$$n_B = \frac{v_B}{4L} \quad \ldots(ii)$$

Given $2n_A = 3n_B$

$$2\left(\frac{v_A}{2L}\right) = 3\left(\frac{v_B}{4L}\right)$$

$$\Rightarrow \quad \frac{v_A}{v_B} = \frac{3}{4} \quad \ldots(iii)$$

But $$v_A = \sqrt{\frac{\gamma_A RT_A}{M_A}}; \; v_B = \sqrt{\frac{\gamma_B RT_B}{M_B}}$$

Given $T_A = T_B$

For monoatomic get $\gamma_A = \frac{5}{3}$

For diatomic get $\gamma_B = \frac{7}{5}$

$$\therefore \quad \frac{v_A}{v_B} = \sqrt{\left\{\frac{\gamma_A}{\gamma_B}\frac{M_B}{M_A}\right\}}$$

$$= \sqrt{\frac{(5/3)}{(7/5)}\frac{M_B}{M_A}} = \sqrt{\frac{25}{21}\frac{M_B}{M_A}} \quad \ldots(iv)$$

From Eqs. (iii) and (iv)

$$\sqrt{\frac{25}{21}\frac{M_B}{M_A}} = \frac{3}{4}$$

$$\frac{M_A}{M_B} = \left(\frac{4}{3}\right)^2 \times \frac{25}{21} = \frac{400}{189}$$

(b) When pipe B is closed at both ends, fundamental frequency of pipe B becomes

$$n_B = \frac{v_B}{2L} \quad \ldots(v)$$

Using Eqs. (i), (iii) and (v), we get $\frac{n_A}{n_B} = \frac{v_A}{v_B} = \frac{3}{4}$

EXAMPLE 7.12

A sonometer wire under tension of 128 N vibrates in resonance with a tuning fork. The vibrating portion of sonometer wire has length of 20 cm and mass 1g. The vibrating tuning fork is now moved away from the vibrating wire at constant speed of 0.75 m/s and an observer standing near the sonometer hears 1 beat/s. Find the speed of sound in air.

Sol. Frequency of string

$$f_s = \frac{1}{2L}\sqrt{\frac{T}{\mu}} = \frac{1}{2}\sqrt{\frac{T}{ML}} = \frac{1}{2}\sqrt{\frac{128}{10^{-3}\times 2\times 10^{-1}}} = 400 \text{ Hz}$$

When tuning fork is moved away from the observer standing near sonometer at constant speed of 0.75 m/s. The apparent frequency of tuning fork.

$$f_1 = f_s\left(\frac{v}{v+0.75}\right) = f_s\left(1+\frac{0.75}{v}\right)^{-1} = f_s\left(1-\frac{0.75}{v}\right)$$

$$\Delta f = f_s - f_1 = f_s\left(\frac{0.75}{v}\right), \quad \Delta f = 1$$

$v = 300$ m/s (approximately)

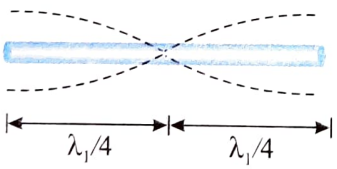
EXAMPLE 7.13

A rod of nickel of length l is clamped at its midpoint. The rod is stuck and vibrations are set up in the rod. Find the general expression for the frequency of the longitudinal vibrations of the rod. Young's modulus and density of the rod is Y and ρ, respectively.

Sol. **Case I:** $l = 2\lambda_1/4$ and $v = f\lambda_1$

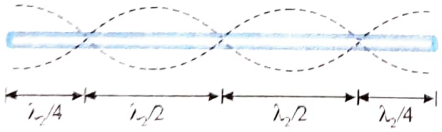
$$V = \sqrt{\frac{Y}{\rho}}, \quad l = \frac{\lambda_1}{2}$$

$$f_1 = \frac{v}{\lambda_1} = \frac{1}{2l}\sqrt{\frac{Y}{\rho}}$$



Case II: $l = 2\left[\frac{\lambda_2}{4}+\frac{\lambda_2}{2}\right]$

$$l = \frac{3\lambda_2}{2}$$



Again $f_2 = \frac{v}{\lambda_2} = \frac{3}{2l}\sqrt{\frac{Y}{\rho}}$

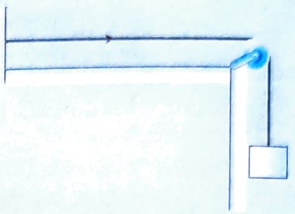
Case III: $l = 2\frac{5\lambda_3}{4}, \quad \lambda_3 = \frac{2l}{5}$

$$f_3 = f_3 = \frac{v}{\lambda_3} = \frac{5}{2l}\sqrt{\frac{Y}{\rho}}$$

Comparing expression f_1, f_2, f_3, we have $f_n = \frac{2n-1}{2l}\sqrt{\frac{Y}{\rho}}$, where $n = 1, 2, 3, \ldots$

EXAMPLE 7.14

A string is stretched by a block going over a pulley. The string vibrates in its fifth harmonic in unison with a particular tuning fork. When a beaker containing a liquid of density ρ is brought under the block so that the block is completely dipped into the beaker, the string vibrates in its seventh harmonic in unison with the same tuning fork. Find the density of the material of the block.



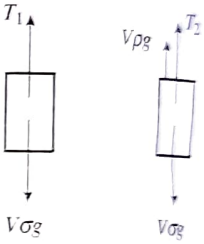
Sol. Let σ be the density of the block

$$n = \frac{5v_1}{2L} = \frac{7v_2}{2L} \quad \text{(i)}$$

$$5v_1 = 7v_2 \quad \text{(ii)}$$

$$5\sqrt{T_1} = 7\sqrt{T_2}$$

where $T_1 = V\sigma g$ and $T_2 = V\sigma g - V\rho g$



squaring both sides of Eq. (ii)

$$25T_1 = 49T_2$$

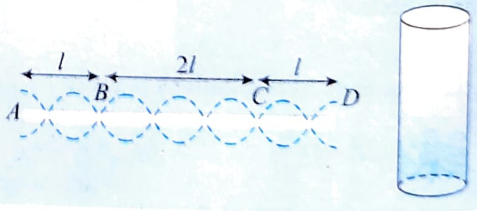
$$25V\sigma\rho = 49\,(V\sigma g - V\rho g)$$

$$(49 - 25)\,V\sigma g = 49\,V\rho g$$

$$\sigma = \frac{49}{24}\rho$$

EXAMPLE 7.15

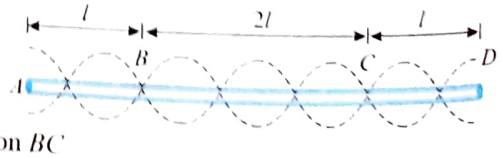
A closed organ pipe of length l_0 is resonating in 5th harmonic mode with rod clamped at two points l and $3l$ from one end. If the length of the rod is $4l$ and it is vibrating in first overtone, find the length of the rod. [Velocity of sound in air = v, Young's modulus for the rod Y and density ρ]



Sol. For section AB

$$l = (2n-1)\frac{\lambda}{4}, \quad \text{where } n = 1, 2, 3, \ldots$$

$$f_1 = \frac{v}{\lambda} = \frac{(2n-1)v}{4l}$$



For section BC

$$2l = m\frac{\lambda}{2}, \quad \text{where } m = 1, 2, 3, \ldots$$

$$f_2 = \frac{vm}{4l}$$

Now, $f_1 = f_2$

$$\frac{(2n-1)v}{4l} = \frac{mv}{4l}$$

$$\therefore \quad n = \frac{m+1}{2}$$

$m = 1, n = 1$ for fundamental; $m = 3, n = 2$

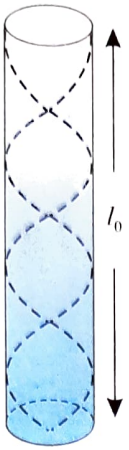
First overtone $f_2 = \frac{mv}{4l} = \frac{3v}{4l}$

Given that $f = f_2$

$$\frac{5v_s}{4l_0} = \frac{3v}{4l}$$

$$l = \frac{3vl_0}{5v_s}$$

Therefore, length of the rod $= 4l = \frac{12}{5}\frac{l_0}{v_s}\sqrt{\frac{Y}{\rho}}$



EXAMPLE 7.16

The following equations represent transverse waves:

$y_1 = A\cos(kx - \omega t)$; $y_2 = A\cos(kx + \omega t)$

Identify the combination of the waves which will produce (1) standing waves and (2) a wave travelling in a direction, making an angle 45° with the positive x and the positive y-axis. In each case, find the positions at which the resultant intensity is always zero.

Sol. The two waves

$y_1 = A\cos(kx - \omega t)$ and $y_2 = A\cos(kx + \omega t)$

have the same amplitude and frequency but travel in opposite directions. So, they produce the standing wave

$$y = y_1 + y_2 = A\{\cos(kx - \omega t) + \cos(kx + \omega t)\}$$

$$= A\{2\cos kx \cos \omega t\} = 2A\cos kx \cos \omega t$$

The amplitude $2A\cos kx$ is 0 when $kx = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots (2m+1)\frac{\pi}{2}$

$$\therefore \quad x = \frac{(2m+1)\pi}{2k}$$

The transverse waves $y_1 = A\cos(kx - \omega t)$ and $y_3 = A\cos(ky - \omega t)$ produce a wave in a direction, making angle of 45° with the positive x and the y-axis. The resultant wave is given by

$$y = y_1 + y_3 = A\{\cos(kx - \omega t) + \cos(ky - \omega t)\}$$

$$= 2A\left\{\cos\left(\frac{kx + ky}{2} - \omega t\right)\cos\frac{k(x-y)}{2}\right\}$$

$$= 2A\cos\cos\frac{k(x-y)}{2}\cos\left(\frac{kx + ky}{2} - \omega t\right)$$

The amplitude is zero when $\cos\frac{k(x-y)}{2} = 0$

or $\frac{k(x-y)}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \frac{(2m+1)\pi}{2}$

or $k(x-y) = (2m+1)\pi \quad \therefore \quad (x-y) = \frac{(2m-1)\pi}{k}$

EXAMPLE 7.17

A stone hangs in air from one end of a wire which is stretched over a sonometer. The wire is in unison with a certain tuning fork when the bridges of the sonometer are 45 cm apart. Now the stone hangs immersed in water at 4°C and the distance between the bridges has to be altered by 9cm to re-establish unison of the wire with the same fork. Calculate the density of the stone.

Sol. Let M be the mass of the stone. Initial tension (T_1) in the wire $= Mg$

On immersing the stone in water, it will lose its weight and the loss in weight will be equal to the weight of the water displaced. Final tension (T_2) in the string $= Mg - (Mg/\rho)$ where ρ is the density of the stone.

Also $L_1 = 45$ cm and $L_2 = 45 - 9 = 36$ cm

(Since the tension has decreased, the length should also decrease to re-establish unison)

Since in both the cases, the wire resonates with the tuning fork, frequency of the wire in both cases should be equal, *i.e.*,

$$\frac{1}{2L_1}\sqrt{\frac{T_1}{\mu}} = \frac{1}{2L_2}\sqrt{\frac{T_2}{\mu}}$$

or $$\sqrt{\frac{T_1}{T_2}} = \frac{L_1}{L_2}$$

or $$\sqrt{\frac{Mg}{Mg - Mg/\rho}} = \frac{45}{36} = \frac{5}{4}$$

or $$\frac{1}{1 - 1/\rho} = \frac{25}{16}$$

or $$25 - \frac{25}{\rho} = 16$$

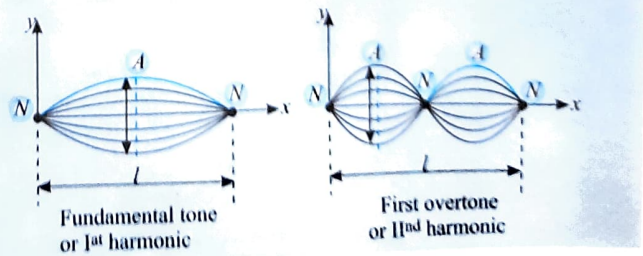
or $$\rho = \frac{25}{9} = 2.8 \text{ g/cm}^3$$

EXAMPLE 7.18

A uniform string of length l is fixed at both ends such that tension T is produced in it. The string is excited to vibrate with maximum displacement amplitude a_0.

Calculate maximum kinetic energy of the string for its:

(i) Fundamental tone, and **(ii)** first overtone



Sol. In case of fundamental tone, it vibrates in single loop. Hence, wavelength of fundamental tone, $\lambda_0 = 2l$.

and $$n_0 = \frac{v}{\lambda_0} = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

In case of first overtone it vibrates in two loops as shown in figure. Hence wavelength of first overtone is $\lambda_1 = l$.

and $n_1 = \frac{v}{\lambda_1} = \frac{1}{l}\sqrt{\frac{T}{\mu}}$

Displacement amplitude at a point distant x from one end $a = a_0 \sin\left(\frac{2\pi x}{\lambda}\right)$, where a_0 is maximum displacement amplitude which occurs at antinode.

Considering an elemental length dx of string at a distance x from left end,

Maximum kinetic energy of string $dK = \frac{1}{2}(\mu dx)a^2(2\pi n_0)^2$

$$= \frac{1}{2}(\mu dx)\left(a_0 \sin\left(\frac{2\pi x}{\lambda}\right)\right)^2\left(2\pi.\frac{1}{2l}\sqrt{\frac{T}{\mu}}\right)^2$$

$$= \frac{a_0^2\pi^2 T}{2l^2}\sin^2\left(\frac{2\pi x}{2l}\right)dx$$

$$\Rightarrow \quad dK = \frac{a_0^2\pi^2 T}{2l^2}\sin^2\left(\frac{2\pi x}{2l}\right)dx$$

Maximum kinetic energy of string, $K = \frac{a_0^2\pi^2 T}{2l^2}\int_0^l \sin^2\left(\frac{\pi x}{l}\right)dx$

$$K = \frac{a_0^2\pi^2 T}{2l^2}\int_0^l\left(\frac{1}{2}\left(1-\cos\frac{2\pi x}{l}\right)\right)dx$$

$$= \frac{a_0^2\pi^2 T}{4l^2}\left[\int_0^l dx - \int_0^l \cos\frac{2\pi x}{l}dx\right]$$

$$= \frac{a_0^2\pi^2 T}{4l^2}\left[l - \left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l\right]$$

$$= \frac{a_0^2\pi^2 T}{4l^2}\left[l - \frac{l}{2\pi}\left(\sin\frac{2\pi l}{l} - \left(\sin\frac{2\pi\times 0}{l}\right)\right)\right] = \frac{a_0^2\pi^2 T}{4l}$$

Considering an elemental length dx of string at a distance from left end, Maximum kinetic energy of string

$$dK = \frac{1}{2}(\mu dx)a^2(2\pi n_1)^2$$

$$= \frac{1}{2}(\mu dx)\left(a_0 \sin\left(\frac{2\pi x}{\lambda}\right)\right)^2\left(2\pi.\frac{1}{l}\sqrt{\frac{T}{\mu}}\right)^2$$

$$= \frac{2a_0^2\pi^2 T}{l^2}\sin^2\left(\frac{2\pi x}{2l}\right)dx$$

$$dK = \frac{2a_0^2\pi^2 T}{l^2}\sin^2\left(\frac{2\pi x}{l}\right)dx$$

Total oscillation energy of the string, $K = \frac{2a_0^2\pi^2 T}{l^2}\int_0^l \sin^2\left(\frac{2\pi x}{l}\right)dx$

$$K = \frac{2a_0^2\pi^2 T}{l^2}\int_0^l\left(\frac{1}{2}\left(1-\cos\frac{4\pi x}{l}\right)\right)dx$$

$$= \frac{a_0^2\pi^2 T}{l^2}\left[\int_0^l dx - \int_0^l \cos\frac{4\pi x}{l}dx\right]$$

$$= \frac{a_0^2\pi^2 T}{l^2}\left[l - \left(\frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}}\right)_0^l\right]$$

$$= \frac{a_0^2\pi^2 T}{l^2}\left[l - \frac{l}{4\pi}\left(\sin\frac{4\pi l}{l} - \left(\sin\frac{4\pi\times 0}{l}\right)\right)\right] = \frac{a_0^2\pi^2 T}{l}$$

Exercises

Single Correct Answer Type

1. The following equations represent progressive transverse waves

$z_1 = A\cos(\omega t - kx)$

$z_2 = A\cos(\omega t + kx)$

$z_3 = A\cos(\omega t + ky)$

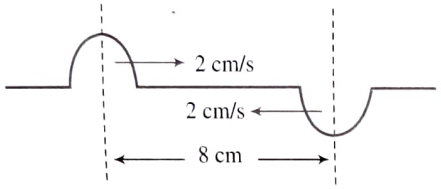
$z_4 = A\cos(2\omega t - 2ky)$

A stationary wave will be formed by superposing

(1) z_1 and z_2 (2) z_1 and z_4

(3) z_2 and z_3 (4) z_3 and z_4

2. Two pulses in a stretched string whose centres are initially 8 cm apart are moving towards each other as shown in figure. The speed of each pulse is 2 cm/s. After 2 s the total energy of the pulses will be



(1) zero

(2) purely kinetic

(3) purely potential

(4) party kinetic and partly potential

3. Which of the following statements is correct for stationary waves

(1) Nodes and antinodes are formed in case of stationary transverse wave only

(2) In case of longitudinal stationary wave, compressions and rarefactions are obtained in place of nodes and antinodes respectively

(3) Suppose two plane waves, one longitudinal and the other transverse having same frequency and amplitude are travelling in a medium in opposite directions with the same speed, by superposition of these waves, stationary waves cannot be obtained

(4) None of the above

4. Mark the correct statement:

(1) In case of stationary waves the maximum pressure change occurs at antinode.

(2) Velocity of longitudinal waves in a medium is its physical characteristic.

(3) Due to propagation of longitudinal wave in air, the maximum pressure change is equal to $2\pi na/\rho v$.

(4) None of the above.

5. Mark out the correct statement(s) regarding standing waves.

(1) Standing waves appear to be stationary but transfer of energy from one particle to another continues to take place.

(2) A standing wave not only appears to be stationary but net transfer of energy from one particle to the other is also equal to zero.

(3) A standing wave does not appear to be stationary and net transfer of energy from one particle to the other is also non-zero.

(4) A standing wave does not appear to be stationary, but net transfer of energy from one particle to the other is zero.

6. On sounding tuning fork A with another tuning fork B of frequency 384 Hz, 6 beats are produced per second. After loading the prongs of A with wax and then sounding it again with B, 4 beats are produced per second. What is the frequency of the tuning fork A.

(1) 388 Hz (2) 80 Hz

(3) 378 Hz (4) 390 Hz

7. Forty-one forks are so arranged that each produces 5 beat/s when sounded with its near fork. If the frequency of last fork is double the frequency of first fork, then the frequencies of the first and last fork, respectively are

(1) 200, 400 (2) 205, 410

(3) 195, 390 (4) 100, 200

8. A sound wave of wavelength λ travels towards the right horizontally with a velocity V. It strikes and reflects from a vertical plane surface, travelling at a speed v towards the left. The number of positive crests striking in a time interval of 3 s on the wall is

(1) $3(V+v)/\lambda$ (2) $3(V-v)/\lambda$

(3) $(V+v)/3\lambda$ (4) $(V-v)/3\lambda$

9. A sonometer wire of length l vibrates in fundamental mode when excited by a tuning fork of frequency 416 Hz. If the length is doubled keeping other things same, the string will

(1) vibrate with a frequency of 416 Hz

(2) vibrate with a frequency of 208 Hz

(3) vibrate with a frequency of 832 Hz

(4) stop vibrating

10. Two closed-end pipes, when sounded together produce 5 beats/s. If their lengths are in the ratio 100:101, then fundamental notes (in Hz) produced by them are

(1) 245, 250 (2) 250, 255

(3) 495, 500 (4) 500, 505.

11. A glass tube of 1.0 m length is filled with water. The water can be drained out slowly at the bottom of the tube. If a vibrating tuning fork of frequency 500 c/s is brought at the upper end of the tube and the velocity of sound is 330 m/s, then the total number of resonances obtained will be

(1) 4 (2) 3

(3) 2 (4) 1

12. An open pipe resonates with a tuning fork of frequency 500 Hz. It is observed that two successive notes are formed at distances 16 and 46 cm from the open end. The speed of sound in air in the pipe is

(1) 230 m/s (2) 300 m/s

(3) 320 m/s (4) 360 m/s

13. If the length of a stretched sting is shortened by 40% and the tension is increased by 44%, then the ratio of the final and initial fundamental frequencies is

(1) 3:4 (2) 4:3

(3) 1:3 (4) 2:1

14. Two uniform strings A and B made of steel are made to vibrate under the same tension. If the first overtone of A is equal to the second overtone of B and if the radius of A is twice that of B, the ratio of the lengths of the strings is

(1) 2:1 (2) 3:2

(3) 3:4 (4) 1:3

15. In a large room, a person receives direct sound waves from a source 120 m away from him. He also receives waves from the same source which reach, being reflected from the 25 m high ceiling at a point halfway between them. The two waves interfere constructively for a wavelength of

(1) $20, \frac{20}{3}, \frac{20}{5}$, etc. (2) 10, 5, 2.5, etc.

(3) 10, 20, 30, etc. (4) 15, 25, 35. etc.

16. Two waves are passing through a region in the same direction at the same time. If the equation of these waves are

$$y_1 = a\sin\frac{2\pi}{\lambda}(vt - x)$$

and $y_2 = b\sin\frac{2\pi}{\lambda}[(vt - x) + x_0]$

then the amplitude of the resulting wave for $x_0 = (\lambda/2)$ is

(1) $|a-b|$ (2) $a+b$

(3) $\sqrt{a^2+b^2}$ (4) $\sqrt{a^2+b^2+2ab\cos x}$

17. The vibrations of string of length 60 cm fixed at both ends are represented by the equations

$y = 4\sin(\pi x/15)\cos(96\pi t)$

where x and y are in cm and t in s. The maximum displacement at $x = 5$ cm is

(1) $2\sqrt{3}$ cm (2) 4 cm

(3) zero (4) $4\sqrt{2}$ cm

18. Two instruments having stretched strings are being played in unison. When the tension in one of the instruments is increases by 1%, 3 beats are produced in 2 s. The initial frequency of vibration of each wire is

(1) 600 Hz (2) 300 Hz

(3) 200 Hz (4) 150 Hz

19. A stretched string of length 1 m fixed at both ends, having a mass of 5×10^{-4} kg is under a tension of 20 N. It is plucked at a point situated at 25 cm from one end. The stretched string would vibrate with a frequency of

(1) 400 Hz (2) 100 Hz

(3) 200 Hz (4) 256 Hz

20. A sonometer wire supports a 4 kg load and vibrates in fundamental mode with a tuning fork of frequency 416 Hz. The length of the wire between the bridges is now doubled. In order to maintain fundamental mode, the load should be changed to

(1) 1 kg (2) 2 kg

(3) 8 kg (4) 16 kg

21. A piano wire having a diameter of 0.90 mm is replaced by another wire of the same material but with a diameter of 0.93 mm. If the tension of the wire is kept the same, then the percentage change in the frequency of the fundamental tone is

(1) +3% (2) +3.2%

(3) –3.2% (4) –3%

22. Two sounding bodies producing progressive waves are given by

$y_1 = 4\sin 400\pi t$ and $y_2 = 3\sin 404\pi t$

One of these bodies situated very near to the ears of a person who will hear:

(1) 2 beats/s with intensity ratio 4/3 between maxima and minima.

(2) 2 beats/s with intensity ratio 49/1 between maxima and minima.

(3) 4 beats/s with intensity ratio 7/2 between maxima and minima.

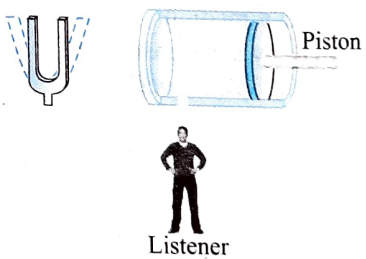
(4) 4 beats/s with intensity ratio 4/3 between maxima and minima

23. Ten tuning forks are arranged in increasing order of frequency in such a way that any two nearest forks produce 4 beats/s. The highest frequency is twice that of the lowest. Possible lowest and highest frequencies are

(1) 40 and 80 (2) 50 and 100

(3) 22 and 44 (4) 36 and 72

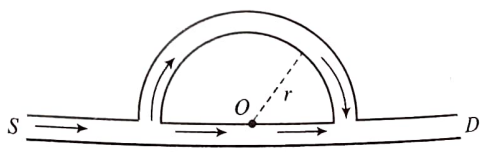
24. A long cylindrical tube carries a highly polished piston and has a side opening. A tuning fork of frequency n is sounded at the open end of the tube. The intensity of the sound heard by the listener changes if the piston is moved in or out. At a particular position of the piston he hears a maximum sound. When the piston is moved through a distance of 9 cm, the intensity of sound becomes minimum. If the speed of sound is 360 m/s, the value of n is



(1) 129.6 Hz (2) 500 Hz

(3) 1000 Hz (4) 2000 Hz

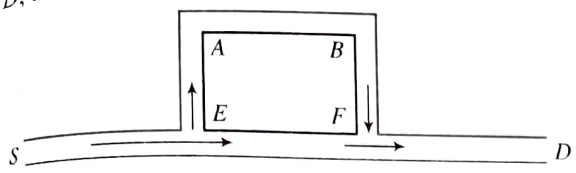
25. A sound wave of wavelength 0.40 m enters the tube at S. The smallest radius r of the circular segment to hear minimum at detector D must be



(1) 1.75 m (2) 0.175 m

(3) 0.93 m (4) 9.3 m

26. A sound wave starting from source S, follows two paths $SEFD$ and $SEABFD$. If $AB = l$, $AE = BF = 0.6\ l$ and wavelength of wave is $\lambda = 11$ m. If maximum sound is heard at D, then minimum value of length l is



(1) 11 m (2) 6 m
(3) 2.5 m (4) 5 m

27. An open pipe of length 2 m is dipped in water. To what depth x is to be immersed in water so that it may resonate with a tuning fork of frequency 170 Hz when vibrating in its first overtone. Speed of sound in air is 340 m/s
(1) 0.5 m (2) 0.75 m
(3) 1 m (4) 1.5 m

28. A stone is hung in air from a wire which is stretched over a sonometer. The bridges of the sonometer are 40 cm apart when the wire is in unison with a tuning fork of frequency 256 Hz. When the stone is completely immersed in water, the length between the bridges is 22 cm for re-establishing unison. The specific gravity of the material of the stone is

(1) $\frac{(40)^2}{(40)^2+(22)^2}$ (2) $\frac{(40)^2}{(40)^2-(22)^2}$

(3) $256\times\frac{22}{40}$ (4) $256\times\frac{40}{22}$

29. A stretched wire of some length under a tension is vibrating with its fundamental frequency. Its length is decreased by 45% and tension is increased by 21%. Now fundamental frequency
(1) increases by 50%
(2) increases by 100%
(3) decreases by 50%
(4) decreases by 25%

30. An open and a closed pipe have same length. The ratio of frequency of their nth overtone is

(1) $\frac{n+1}{2n+1}$ (2) $\frac{2(n+1)}{2n+1}$

(3) $\frac{n}{2n+1}$ (4) $\frac{n+1}{2n}$

31. A string is under tension so that its length is increased by $1/n$ times its original length. The ratio of fundamental frequency of longitudinal vibrations and transverse vibrations will be
(1) $1:n$ (2) $n^2:1$
(3) $\sqrt{n}:1$ (4) $n:1$

32. The frequency of a sonometer wire is 10 Hz. When the weights producing the tension are completely immersed in water the frequency becomes 80 Hz and on immersing the weights in a certain liquid the frequency becomes 60 Hz. The specific gravity of the liquid is
(1) 1.42 (2) 1.77
(3) 1.82 (4) 1.21

33. An open organ pipe of length l is sounded together with another open organ pipe of length $l + x$ in their fundamental tones. Speed of sound in air is v. The beat frequency heard will be ($x \ll l$):

(1) $\frac{vx}{4l^2}$ (2) $\frac{vl^2}{2x}$

(3) $\frac{vx}{2l^2}$ (4) $\frac{vx^2}{2l}$

34. The frequency of B is 3% greater than that of A. The frequency of C is 2% less than that of A. If B and C produce 8 beats/s, then the frequency of A is
(1) 136 Hz (2) 168 Hz
(3) 164 Hz (4) 160 Hz

35. A wire of length 'l' having tension T and radius 'r' vibrates with fundamental frequency 'f'. Another wire of the same metal with length '$2l$' having tension $2T$ and radius $2r$ will vibrate with fundamental frequency:
(1) f (2) $2f$

(3) $\frac{f}{2\sqrt{2}}$ (4) $\frac{f}{2}\sqrt{2}$

36. A string of length 1.5 m with its two ends clamped is vibrating in fundamental mode. Amplitude at the centre of the string is 4 mm. Distance between the two points having amplitude 2 mm is
(1) 1 m (2) 75 cm
(3) 60 cm (4) 50 cm

37. A 75 cm string fixed at both ends produces resonant frequencies 384 Hz and 288 Hz without there being any other resonant frequency between these two. Wave speed for the string is
(1) 144 m/s (2) 216 m/s
(3) 108 m/s (4) 72 m/s

38. A string of length 'L' is fixed at both ends. It is vibrating in its 3rd overtone with maximum amplitude 'a'. The amplitude at a distance $L/3$ from one end is
(1) a (2) 0

(3) $\frac{\sqrt{3}\,a}{2}$ (4) $\frac{a}{2}$

39. A closed organ pipe has length 'l'. The air in it is vibrating in 3rd overtone with maximum amplitude 'a'. The amplitude at a distance of $l/7$ from closed end of the pipe is equal to
(1) a (2) $a/2$

(3) $\frac{a\sqrt{3}}{2}$ (4) zero

40. S_1 and S_2 are two coherent sources of sound separated by 3 m having no initial phase difference. The velocity of sound is 330 m/s. No minima will be formed on the line passing through S_2 and perpendicular to the line joining S_1 and S_2, if the frequency of both the sources is
(1) 50 Hz (2) 60 Hz
(3) 70 Hz (4) 80 Hz

41. When beats are produced by two progressive waves of nearly the same frequency, which one of the following is correct?

(1) The particle vibrate simple harmonically, with the frequency equal to the difference in the component frequencies

(2) The amplitude of vibration at any point changes simple harmonically with a frequency equal to the difference in the frequencies of the two waves

(3) The frequency of beats depends upon the position, where the observer is

(4) The frequency of beats changes as the time progresses

42. Five sinusoidal waves have the same frequency 500 Hz but their amplitudes are in the ratio 2:1/2:1/2:1:1 and their phase angles 0, $\pi/6$, $\pi/3$, $\pi/2$ and π, respectively. The phase angle of resultant wave obtained by the superposition of these five waves is

(1) 30° (2) 45°

(3) 60° (4) 90°

43. The breaking stress of steel is 7.85×10^8 N/m^2 and density of steel is 7.7×10^3 kg/m^3. The maximum frequency to which a string 1 m long can be tuned is

(1) 15.8 Hz (2) 158 Hz

(3) 47.4 Hz (4) 474 Hz

44. There is a set of four tuning forks, one with the lowest frequency vibrating at 550 Hz. By using any two tuning forks at a time, the following beat frequencies are heard: 1, 2, 3, 5, 7, 8. The possible frequencies of the other three forks are

(1) 552, 553, 560 (2) 557, 558, 560

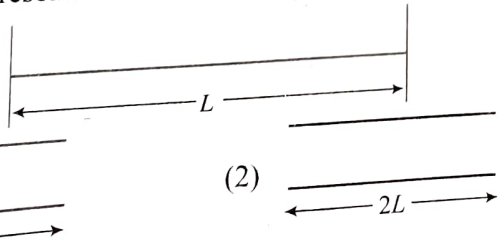
(3) 552, 553, 558 (4) 551, 553, 558

45. A 100-m long rod of density 10.0×10^4 kg/m^3 and having Young's modulus $Y = 10^{11}$ Pa, is clamped at one end. It is hammered at the other free end. The longitudinal pulse goes to right end, gets reflected and again returns to the left end. How much time the pulse take to go back to initial point.

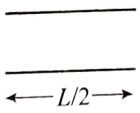
(1) 0.1 s (2) 0.2 s

(3) 0.3 s (4) 2 s

46. Figure shows a stretched string of length L and pipes of length L, $2L$, $L/2$ and $L/2$ in options (a), (b), (c) and (d) respectively. The string's tension is adjusted until the speed of waves on the string equals the speed of sound waves in the air. The fundamental mode of oscillation is then set up on the string. In which pipe will the sound produced by the string cause resonance?



(3)



(4)

47. Equations of a stationary and a travelling waves are as follows $y_1 = \sin kx \cos \omega t$ and $y_2 = a \sin(\omega t - kx)$. The phase difference between two points $x_1 = \pi/3k$ and $x_2 = 3\pi/2k$ is ϕ_1 in the standing wave (y_1) and is ϕ_2 in travelling wave (y_2) then ratio ϕ_1/ϕ_2 is

(1) 1 (2) 5/6

(3) 3/4 (4) 6/7

48. In the resonance tube experiment, the first resonance is heard when length of air column is l_1 and second resonance is heard when length of air column is l_2. What should be the minimum length of the tube so that third resonance can also be heard.

(1) $2l_2 - l_1$ (2) $2l_1$

(3) $5l_1$ (4) $7l_1$

49. Microwaves from a transmitter are directed normally towards a plane reflector. A detector moves along the normal to the reflector. Between positions of 14 successive maxima, the detector travels a distance 0.14 m. If the velocity of light is 3×10^8 m/s, find the frequency of the transmitter.

(1) 1.5×10^{10} Hz (2) 10^{10} Hz

(3) 3×10^{10} Hz (4) 6×10^{10} Hz

50. Let the two waves $y_1 = A \sin(kx - \omega t)$ and $y_2 = A \sin(kx + \omega t)$ form a standing wave on a string. Now if an additional phase difference of ϕ is created between two waves, then

(1) the standing wave will have a different frequency

(2) the standing wave will have a different amplitude for a given point

(3) the spacing between two consecutive nodes will change

(4) none of the above

51. A standing wave on a string is given by $y = (4 \text{ cm}) \cos[\pi t] \sin[50\pi t]$, where x is in metres and t is in seconds. The velocity of the string section at $x = 1/3$ m at $t = 1/5$ s, is

(1) zero (2) π m/s

(3) 840π m/s (4) none of these

52. If the velocity of sound in air is 320 m/s, then the maximum and minimum length of a pipe closed at one end, that would produce a just audible sound would be

(1) 2.6 m and 3.6 mm

(2) 4 m and 4.2 mm

(3) 3 m and 3 mm

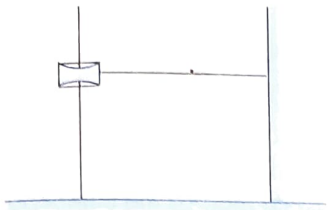
(4) 4 m and 4 mm

53. What percentage change in the tension is necessary in a sonometer of fixed length to produce a note one octave lower (half of original frequency) than before

(1) 25% (2) 50%

(3) 67% (4) 75%

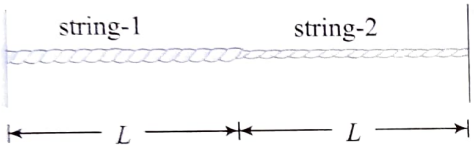
54. One end of a 2.4-m string is held fixed and the other end is attached to a weightless ring that can slide along a frictionless rod as shown in figure. The three longest possible wavelength for standing waves in this string are respectively



(1) 4.8 m, 1.6 m and 0.96 m (2) 9.6 m, 3.2 m and 1.92 m
(3) 2.4 m, 0.8 m and 0.48 m (4) 1.2 m, 0.4 m and 0.24 m

55. A harmonic wave is travelling on a stretched string. At any particular instant, the smallest distance between two particles having same displacement, equal to half of amplitude is 8 cm. Find the smallest separation between two particles which have same values of displacement (magnitude only) equal to half of amplitude.
(1) 8 cm (2) 24 cm
(3) 12 cm (4) 4 cm

56. Two strings, one thick and other thin are connected as shown in figure.



Which of the following statement(s) is correct with regard to above arrangement?
(1) If a wave is travelling from string 1 to string 2, then the joint would be treated as free end.
(2) If a wave is travelling from string 1 to string 2, then the joint would be treated as a fixed end.
(3) If a wave is travelling from string 2 to string 1, then the joint would be treated as a free end.
(4) Both (2) and (3) are correct.

57. A string fixed at both ends whose fundamental frequency is 240 Hz is vibrated with the help of a tuning fork having frequency 480 Hz, then
(1) The string will vibrate with a frequency of 240 Hz
(2) The string will vibrate in resonance with the tuning fork
(3) The string will vibrate with a frequency of 480 Hz, but is not a resonance with the tuning fork
(4) The string is in resonance with the tuning fork and hence vibrate with a frequency of 240 Hz

58. If a string fixed at both ends having fundamental frequency of 240 Hz is vibrated with the help of a tuning fork having frequency 280 Hz, then the
(1) string will vibrate with a frequency of 240 Hz
(2) string will be in resonance with the tuning fork
(3) string will vibrate with the frequency of tuning fork, but resonance condition will not be achieved
(4) string will vibrate with a frequency of 260 Hz

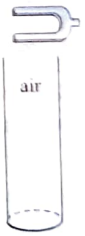
59. A string of length 0.4 m and mass 10^{-2} kg is clamped at one end. The tension in the string is 1.6 N. The identical wave pulses are generated at the free end after regular interval of time, Δt. The minimum value of Δt, so that a constructive interference takes place between successive pulses is
(1) 0.1 s
(2) 0.05 s
(3) 0.2 s
(4) constructive interference cannot take place

60. A train of sound waves is propagated along an organ pipe and gets reflected from an open end. If the displacement amplitude of the waves (incident and reflected) are 0.002 cm, the frequency is 1000 Hz and wavelength is 40 cm. Then, the displacement amplitude of vibration at a point at distance 10 cm from the open end, inside the pipe, is
(1) 0.002 cm (2) 0.003 cm
(3) 0.001 cm (4) 0.000 cm

61. A source of sound attached to the bob of a simple pendulum execute SHM. The difference between the apparent frequency of sound as received by an observer during its approach and recession at the mean position of the SHM motion is 2% of the natural frequency of the source. The velocity of the source at the mean position is (velocity of sound in the air is 340 m/s)
[Assume velocity of sound source << velocity of sound in air]
(1) 1.4 m/s (2) 3.4 m/s
(3) 1.7 m/s (4) 2.1 m/s

62. Two canoes are 10 m apart on a lake. Each bobs up and down with a period of 4.0 s. When one canoe is at its highest point, the other canoe is at its lowest point. Both canoes are always within a single cycle of the waves. The speed of wave is.
(1) 2.5 m/s (2) 5 m/s
(3) 40 m/s (4) 4 m/s

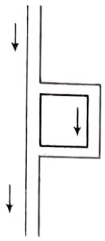
63. A resonance occurs with a tuning fork and an air column of size 12 cm. The next higher resonance occurs with an air column of 38 cm. What is the frequency of the tuning fork? Assume that the speed of sound is 312 m/s.



(1) 500 Hz (2) 550 Hz
(3) 600 Hz (4) 650 Hz

64. In a resonance tube experiment, the first two resonances are observed at length 10.5 cm and 29.5 cm. The third resonance is observed at the length ..... cm
(1) 47.5 (2) 58.5
(3) 48.5 (4) 82.8

65. A sound consists of four frequencies: 300 Hz, 600 Hz, 1200 Hz and 2400 Hz. A sound 'filter' is made by passing this sound through a bifurcate pipe as shown. The sound wave has to travel a distance of 50 cm more in the right branch-pipe than in the straight pipe. The speed of sound in air is 300 m/s. Then, which of the following frequencies will be almost completely muffled or 'silenced' at the outlet?



(1) 300 Hz (2) 600 Hz
(3) 1200 Hz (4) 2400 Hz

66. A glass tube of length 1.5 m is filled completely with water; the water can be drained out slowly at the bottom of the tube. Find the total number of resonance obtained, when a tuning fork of frequency 606 Hz is put at the upper open end of the tube. Take velocity of sound in air = 340 m/s.

(1) 2 (2) 3
(3) 4 (4) 5

67. A wave equation is represented as

$$r = A\sin\left[\alpha\left(\frac{x-y}{2}\right)\right]\cos\left[\omega t - \alpha\left(\frac{x+y}{2}\right)\right],$$

where x and y are in metres and t is in seconds. Then,

(1) the wave is a stationary wave.
(2) the wave is a progressive wave propagating along $+x$-axis.
(3) the wave is a progressive wave propagating at right angle to the $+x$-axis
(4) all points lying on line $y = x + (4\pi/\alpha)$ are always at rest.

68. A wave represented by the equation $y = a\cos(kx - \omega t)$ is superposed with another wave to form a stationary wave such that the point $x = 0$ is a node. The equation for the other wave is

(1) $a\sin(kx+\omega t)$ (2) $-a\cos(kx-\omega t)$
(3) $-a\cos(kx+\omega t)$ (4) $-a\sin(kx-\omega t)$

69. A tuning fork A of frequency as given by the manufacturer is 512 Hz is being tested using an accurate oscillator. It is found that they produce 2 beats/s when the oscillator reads 514 Hz and 6 beats/s when it reads 510 Hz. The actual frequency of the fork in Hz is

(1) 508 (2) 512
(3) 516 (4) 518

70. A metal bar clamped at its centre resonates in its fundamental mode to produce longitudinal waves of frequency 4 kHz. Now the clamp is moved to one end. If f_1 and f_2 be the frequencies of first overtone and second overtone respectively then,

(1) $3f_2 = 5f_1$ (2) $3f_1 = 5f_2$
(3) $f_2 = 2f_1$ (4) $2f_2 = f_1$

71. A stiff wire is bent into a cylinder loop of diameter D. It is clamped by knife edges at two points opposite to each other. A transverse wave is sent around the loop by means of a small vibrator which acts close to one clamp. The resonance frequency (fundamental mode) of the loop in terms of wave speed v and diameter D is

(1) $\frac{v}{D}$ (2) $\frac{2v}{\pi D}$
(3) $\frac{v}{\pi D}$ (4) $\frac{v}{2\pi D}$

72. Two wires of radii r and $2r$ are welded together end to end. The combination is used as a sonometer wire and is kept under a tension T. The welded point lies midway between the bridges. The ratio of the number of loops formed in the wires, such that the joint is a node when the stationary waves are set up in the wire is

(1) 2/3 (2) 1/3
(3) 1/4 (4) 1/2

73. An air column closed at one end and opened at the other end, resonates with a tuning fork of frequency v when its length is 45 cm and 99 cm and at two other lengths in between these values. The wavelength of sound in air column is

(1) 180 cm (2) 108 cm
(3) 54 cm (4) 36 cm

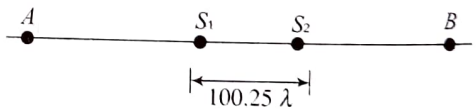
74. Two identical sonometer wires have a fundamental frequency of 500 Hz when kept under the same tension. The percentage change in tension of one of the wires that would cause an occurrence of 5 beats/s, when both wires vibrate together is

(1) 0.5% (2) 1%
(3) 2% (4) 4%

75. A long tube open at the top is fixed vertically and water level inside the tube can be moved up or down. A vibrating tuning fork is held above the open end and the water level is pushed down gradually so as to get first and second resonance at 24.1 cm and 74.1 cm, respectively below the open end. The diameter of the tube is

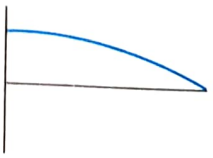
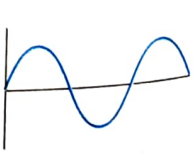
(1) 5 cm (2) 4 cm
(3) 3 cm (4) 2 cm

76. S_1 and S_2 are two coherent current sources of radiations separated by distance 100.25λ where λ is the wavelength of radiation. S_1 leads S_2 in phase by $\pi/2$. A and B are two points on the line joining S_1 and S_2. The ratio of amplitude of sources S_1 and S_2 is in ratio 1:2. The ratio of intensity at A to that at B (I_A/I_B) is



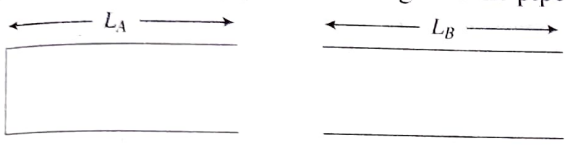
(1) ∞ (2) 1/9
(3) 0 (4) 9

77. From the figures below, which shows the pressure difference from regular atmospheric pressure for an organ pipe of length L closed at one end, corresponds to the 1st overtone for the pipe?

(1)
(2)
(3) 
(4) 

18. An ideal organ pipe resonates at successive frequencies of 50 Hz, 150 Hz, 250 Hz, etc.(speed of sound = 340 m/s) The pipe is
(1) open at both ends and of length 3.4 m
(2) open at both ends and of length 6.8 m
(3) closed at one end, open at the other, and of length 1.7 m
(4) closed at one end, open at the other, and of length 3.4 m

19. Two pipes are submerged in sea water, arranged as shown in figure. Pipe A with length $L_A = 1.5$ m and one open end, contains a small sound source that sets up the standing wave with the second lowest resonant frequency of that pipe. Sound from pipe A sets up resonance in pipe B, which has both ends open. The resonance is at the second lowest resonant frequency of pipe B. The length of the pipe B is:



(a) (b)

(1) 1 m (2) 1.5 m
(3) 2 m (4) 3 m

Multiple Correct Answers Type

1. Mark the correct statements
(1) If all the particles of a string are oscillating in same phase, the string is resonating in its fundamental tone
(2) To observe interference, two sources of same frequency must be placed some distance apart from each other
(3) To observe beats, two sources of same amplitude must be placed some distance apart from each other
(4) None of the above

2. Which of the following functions represent a stationary wave? Here a, b and c are constants:
(1) $y = a\cos(bx)\sin(ct)$
(2) $y = a\sin(bx)\cos(ct)$
(3) $y = a\sin(bx + ct)$
(4) $y = a\sin(bx + ct) + a\sin(bx - ct)$

3 The stationary waves set up on a string has the equation:

$$y = (2\text{ mm})\sin\left[(6.28\text{ m}^{-1})x\right]\cos\omega t$$

The stationary wave is created by two identical waves, of amplitude A each, moving in opposite directions along the string. Then:
(1) $A = 2$ mm
(2) $A = 1$ mm
(3) the smallest length of the string is 50 cm
(4) the smallest length of the string is 2 m

4. A sinusoidal wave $y_1 = a\sin(\omega t - kx)$ is reflected from a rigid support and the reflected wave superpose with the incident wave y_1. Assume the rigid support to be at $x = 0$.
(1) Stationary waves are obtained with antinodes at the rigid support.
(2) Stationary waves are obtained with nodes at the rigid support.
(3) Stationary waves are obtained with intensity varying periodically with distance.
(4) Stationary waves are obtained with intensity varying periodically with time.

5. If the tension in a stretched string fixed at both ends is changed by 21%, the fundamental frequency is found to increase by 15 Hz then the
(1) original frequency is 150 Hz
(2) velocity of propagation of the transverse wave along the string changes by 5%
(3) velocity of propagation of the transverse wave along the string changes by 10%.
(4) fundamental wave length on the string does not change.

6. Velocity of sound in air is 320 m/s. A pipe closed at one end has a length of 1 m. Neglecting the corrections, the air column in the pipe can resonate for sound of frequency
(1) 80 Hz (2) 240 Hz
(3) 320 Hz (4) 400 Hz

7. Two identical straight wires are stretched so as to produce 6 beats s^{-1}, when vibrating simultaneously. On changing the tension slightly in one of them, the beat frequency remains unchanged. Denoting by T_1 and T_2 the higher and lower initial tensions in the string, then it could be said that while making the above changes in tension
(1) T_2 was decreased (2) T_2 was increased
(3) T_1 was increased (4) T_1 was decreased

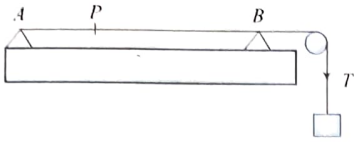
8. A loudspeaker that produces signals from 50 to 500 Hz is placed at the open end of a closed tube of length 1.1 m. The lowest and the highest frequency that excites resonance in the tube are f_l and f_h respectively. The velocity of sound is 330 m/s. Then
(1) $f_l = 50$ Hz (2) $f_h = 500$ Hz
(3) $f_l = 75$ Hz (4) $f_h = 450$ Hz

9. Three simple harmonic waves, identical in frequency n and amplitude A moving in the same direction are superimposed in air in such a way, that the first, second and the third wave have the phase angles ϕ, $\phi + (\pi/2)$ and $(\phi + \pi)$, respectively at a given point P in the superposition Then as the waves progress, the superposition will result in
(1) a periodic, non-simple harmonic wave of amplitude $3A$
(2) a stationary simple harmonic wave of amplitude $3A$
(3) a simple harmonic progressive wave of amplitude A
(4) the velocity of the superposed resultant wave will be the same as the velocity of each wave

10. A sound wave passes from a medium A to a medium B. The velocity of sound in B is greater than that in A. Assume that there is no absorption or reflection at the boundary. As the wave moves across the boundary:
(1) the frequency of sound will not change
(2) the wavelength will increase
(3) the wavelength will decrease
(4) the intensity of sound will not change

11. A sonometer string AB of length 1m is stretched by a load and the tension T is adjusted so that the string resonates to a frequency of 1 kHz. Any point P of the wire may be held

fixed by use of a movable bridge that can slide along the base of sonometer.



(1) If point P is fixed so that $AP:PB::1:4$, then the smallest frequency for which the sonometer wire resonates is 5 kHz.

(2) If P be taken at midpoint of AB and fixed, then when the wire vibrates in the third harmonic of its fundamental, the number of nodes in the wire (including A and B) will be totally seven.

(3) If the fixed point P divides AB in the ratio 1:2, then the tension needed to make the string vibrate at 1 kHz will be $3T$. (neglecting the terminal effects)

(4) The fundamental frequency of the sonometer wire when P divides AB in the ratio $a:b$ will be the same as the fundamental frequency when P divides AB in the ratio $b:a$.

12. A wire of density 9×10^3 kg/m^3 is stretched between two clamps 1 m apart and is stretched to an extension of 4.9×10^{-4} m. Young's modulus of material is 9×10^{10} N/m^2. Then:

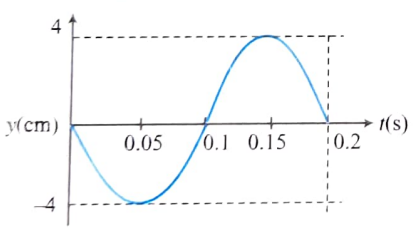
(1) The lowest frequency of standing wave is 35 Hz

(2) The frequency of 1st overtone is 70 Hz

(3) The frequency of 1st overtone is 105 Hz

(4) The stress in the wire is 4.41×10^7 N/m^2

13. For a certain transverse standing wave on a long string, an antinode is formed at $x = 0$ and next to it, a node is formed at $x = 0.10$ m, the displacement $y(t)$ of the string particle at $x = 0$ is shown in figure.



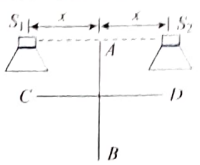
(1) Transverse displacement of the particle at $x = 0.05$ m and $t = 0.05$ s is $-2\sqrt{2}$ cm

(2) Transverse displacement of the particle at $x = 0.04$ m and $t = 0.025$ s is $-2\sqrt{2}$ cm

(3) Speed of the travelling waves that interface to produce this standing wave is 2 m/s

(4) The transverse velocity of the string particle at $x = 1/15$ m and $t = 0.1$ s is 20π cm/s

14. Two speakers are placed as shown in figure.



Mark out the correct statement(s)

(1) If a person is moving along AB, he will hear the sound as loud, faint, loud and so on

(2) If a person moves along CD, he will hear loud, faint, loud and so on

(3) If a person moves along AB, he will hear uniform intense sound

(4) If a person moves along CD, he will hear uniform intense sound

15. Two coherent waves represented by

$$y_1 = A \sin\left(\frac{2\pi}{\lambda}x_1 - \omega t + \frac{\pi}{4}\right) \text{ and}$$

$$y_2 = A \sin\left(\frac{2\pi}{\lambda}x_2 - \omega t + \frac{\pi}{6}\right)$$

are superposed. The two waves will produce

(1) constructive interference at $(x_1 - x_2) = 2\lambda$

(2) constructive interference at $(x_1 - x_2) = 23/24\lambda$

(3) destructive interference at $(x_1 - x_2) = 1.5\lambda$

(4) destructive interference at $(x_1 - x_2) = 11/24\lambda$

16. Two waves travel down the same string. These waves have the same velocity, frequency f and wavelength but having different phase constants ϕ_1 and $\phi_2 (< \phi_1)$ and amplitudes A_1 and A_2 $(< A_1)$. Mark the correct statement(s) for the resultant wave which is produced due to superposition of these two waves.

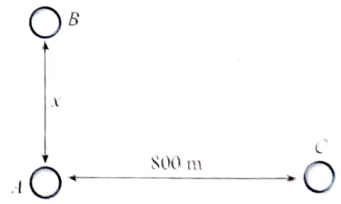
(1) The amplitude of the resultant waves is $A = A_1 + A_2$

(2) The amplitude of the resultant wave lies between $A_1 - A_2$ to $A_1 + A_2$

(3) The frequency of the resultant wave is f

(4) The frequency of the resultant wave is $f/2$

17. A radio transmitter at position A operates at a wavelength of 20 m. A second, identical transmitter is located at a distance x from the first transmitter, at position B. The transmitters are phase locked together such that the second transmitter is lagging $\pi/2$ out of phase with the first. For which of the following values of $BC - CA$ will the intensity at C be maximum.



(1) $BC - CA = 60$ m (2) $BC - CA = 65$ m

(3) $BC - CA = 55$ m (4) $BC - CA = 75$m

18. Following are equations of four waves:

(i) $y_1 = a \sin\omega\left(t - \frac{x}{v}\right)$ **(ii)** $y_2 = a \cos\omega\left(t + \frac{x}{v}\right)$

(iii) $z_1 = a \sin\omega\left(t - \frac{x}{v}\right)$ **(iv)** $z_2 = a \cos\omega\left(t + \frac{x}{v}\right)$

Which of the following statements are correct?

(1) On superposition of waves (i) and (iii), a travelling wave having amplitude $a\sqrt{2}$ will be formed

(2) Superposition of waves (ii) and (iii) is not possible

(3) On superposition of (i) and (ii), a stationary wave having amplitude $a\sqrt{2}$ will be formed

(4) On superposition of (iii) and (iv), a transverse stationary wave will be formed

10. Two waves of equal frequency f and velocity v travel in opposite directions along the same path. The waves have amplitudes A and $3A$. Then:

(1) the amplitude of the resulting wave varies with position between maxima of amplitude $4A$ and minima of zero amplitude

(2) the distance between a maxima and adjacent minima of amplitude is $v/2f$

(3) maximum amplitude is $4A$ and minimum amplitude is $2A$

(4) the position of a maxima or minima of amplitude does not change with time

11. A plane wave $y = a \sin(kx + ct)$ is incident on a surface. Equation of the reflected wave is: $y' = a' \sin(ct - bx)$. Then which of the following statements are correct?

(1) The wave is incident normally on the surface

(2) Reflecting surface is y–z plane

(3) Medium, in which incident wave is travelling, is denser than the other medium

(4) a' cannot be greater than a

12. A string is fixed at both ends and transverse oscillations with amplitude a_0 are excited. Which of the following statements are correct?

(1) Energy of oscillations in the string is directly proportional to tension in the string

(2) Energy of oscillations in nth overtone will be equal to n^2 times of that in first overtone

(3) Average kinetic energy of string (over an oscillation period) is half of the oscillation energy

(4) None of the above

13. Two sine waves of slightly different frequencies f_1 and f_2 ($f_1 > f_2$) with zero phase difference, same amplitudes, travelling in the same direction superimpose.

(1) Phenomenon of beats is always observed by human ear.

(2) Intensity of resultant wave is a constant.

(3) Intensity of resultant wave varies periodically with time with maximum intensity $4a^2$ and minimum intensity zero.

(4) A maxima appears at a time $1/[2(f_1 - f_2)]$ later (or earlier) than a minima appears.

14. Two waves of nearly same amplitude, same frequency travelling with same velocity are superimposing to give phenomenon of interference. If a_1 and a_2 be their respectively amplitudes, ω be the frequency for both, v be the velocity for both and $\Delta\phi$ is the phase difference between the two waves then,

(1) the resultant intensity varies periodically with time and distance.

(2) the resulting intensity with $\frac{I_{min}}{I_{max}} = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2$ is obtained.

(3) both the waves must have been travelling in the same direction and must be coherent.

(4) $I_R = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\Delta\phi)$, where constructive interference is obtained for path difference that are odd multiple of $1/2\lambda$ and destructive interference is obtained for path difference that are even multiple of $1/2\lambda$.

Linked Comprehension Type

For Problems 1–4

Consider a standing wave formed on a string. It results due to the superposition of two waves travelling in opposite directions. The waves are travelling along the length of the string in the x-direction and displacements of elements on the string are along the y-direction. Individual equations of the two waves can be expressed as

$$Y_1 = 6(\text{cm}) \sin [5 (\text{rad/cm})\, x - 4 (\text{rad/s})\, t]$$

$$Y_2 = 6(\text{cm}) \sin [5 (\text{rad/cm})\, x + 4 (\text{rad/s})\, t]$$

Here x and y are is cm.

Answer the following questions.

1. Maximum value of the y-positions coordinate in the simple harmonic motion of an element of the string that is located at an antinode will be

(1) ± 6 cm (2) ± 8 cm

(3) ± 12 cm (4) ± 3 cm

2. If one end of the string is at $x = 0$, positions of the nodes can be described as

(1) $x = n\,\pi/5$ cm, where $n = 0, 1, 2, \ldots$

(2) $x = n\,2\pi/5$ cm, where $n = 0, 1, 2, \ldots$

(3) $x = n\,\pi/5$ cm, where $n = 1, 3, 5, \ldots$

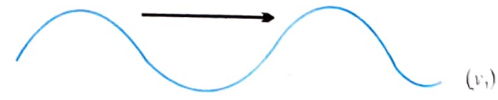
(4) $x = n\,\pi/10$ cm, where $n = 1, 3, 5, \ldots$

3. Amplitude of simple harmonic motion of a point on the string that is located at $x = 1.8$ cm will be

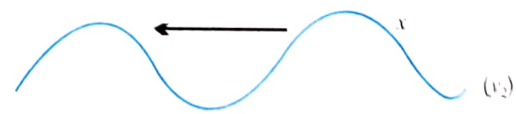
(1) 3.3 cm (2) 6.7 cm

(3) 4.9 cm (4) 2.6 cm

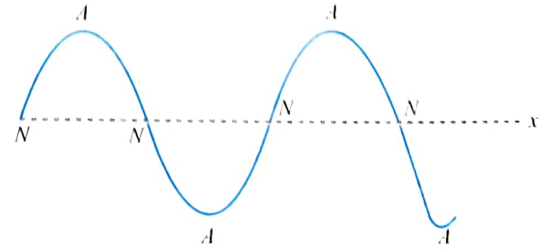
4. Figure (c) shows the standing wave pattern at $t = 0$ due to superposition of waves given by y_1 and y_2 in Figs. (a) and (b). In Fig. (c), N is a node and A an antinode. At this instant say $t = 0$, instantaneous velocity of points on the string



(a)



(b)



(c)

(1) is different for different points
(2) is zero for all points
(3) changes with position of the point
(4) is constant but not equal to zero for all points

For Problems 5–7

Two waves $y_1 = A\cos(0.5\pi x - 100\pi t)$ and $y_2 = A\cos(0.46\pi x - 92\pi t)$ are travelling in a pipe placed along the x-axis.

5. Find the number of times intensity is maximum in time interval of 1 s

(1) 4 (2) 6
(3) 8 (4) 10

6. Find wave velocity of louder sound

(1) 100 m/s (2) 192 m/s
(3) 200 m/s (4) 96 m/s

7. Find the number of times $y_1 + y_2 = 0$ at $x = 0$ in 1 s

(1) 100 (2) 46
(3) 192 (4) 96

For Problems 8–10

Estimation of frequency of a wave forming a standing wave represented by $y = A \sin kx \cos t$ can be done if the speed and wavelength are known using speed = Frequency × wavelength. Speed of motion depends on the medium properties namely tension in string and mass per unit length of string. A string may vibrate with different frequencies. The corresponding wavelength should be related to the length of the string by a whole number for a string fixed at both ends. Answer the following questions:

8. Speed of a wave in a string forming a stationary wave does not depend on

(1) Tension
(2) Mass of wire for a given length
(3) Length of the wire for a given mass
(4) Harmonics of string

9. If $y = 10\sin 5x \cos 2t$ *m* represents a stationary wave then, the possible one of the travelling waves causing this is

(1) $y = 10\sin(5x - 2t)$ (2) $y = 5\sin(2t - 5x)$
(3) $y = 10\sin 2t$ (4) $y = 5\cos 5x$

10. A string fixed at both ends having a third overtone frequency of 200 Hz while carrying a wave at a speed of 30 ms^{-1} has a length of

(1) 30 m (2) 22.5 cm
(3) 30 cm (4) 10.25 cm

For Problems 11–13

In a standing wave experiment, a 1.2-kg horizontal rope is fixed in place at its two ends ($x = 0$ and $x = 2.0$ m) and made to oscillate up and down in the fundamental mode, at frequency of 5.0 Hz. At $t = 0$, the point at $x = 1.0$ m has zero displacement and is moving upward in the positive direction of y-axis with a transverse velocity 3.14 m/s.

11. Tension in the rope is

(1) 60 N (2) 100 N
(3) 120 N (4) 240 N

12. Speed of the participating travelling wave on the rope is

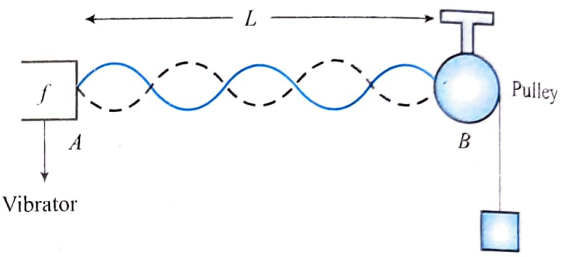
(1) 6 m/s (2) 15 m/s
(3) 20 m/s (4) 24 m/s

13. What is the correct expression of the standing wave equation?

(1) $(0.1)\sin(\pi/2)\,x \sin(10\pi)\,t$
(2) $(0.1)\sin(\pi)\,x \sin(10\pi)\,t$
(3) $(0.05)\sin(\pi/2)\,x \cos(10\pi)\,t$
(4) $(0.04)\sin(\pi/2)\,x \sin(10\pi)\,t$

For Problems 14–16

In the arrangement shown in figure, a mass can be hung from a string with a linear mass density of 2×10^{-3} kg/m that passes over a light pulley. The string is connected to a vibrator of frequency 700 Hz and the length of the string between the vibrator and the pulley is 1 m.



14. If the standing waves are observed, the largest mass to be hung is

(1) 16 kg (2) 25 kg
(3) 32 kg (4) 400 kg

15. If the mass suspended is 16 kg, then the number of loops formed in the string is

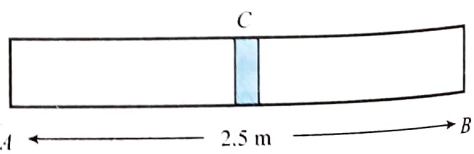
(1) 1 (2) 3
(3) 5 (4) 8

16. The string is set into vibrations and represented by the equation $y = 6\sin\left(\frac{\pi x}{10}\right)\cos(14\times10^3\pi t)$ where x and y are in cm, and t in s, the maximum displacement at $x = 5$ m from the vibrator is

(1) 6 cm (2) 3 cm
(3) 5 cm (4) 2 cm

For Problems 17–19

A steel rod 2.5 m long is rigidly clamped at its centre C and longitudinal waves are set up on both sides of C by rubbing along the rod. Young's modulus for steel = 2×10^{11} N/m^2, density of steel = 8000 kg/m^3



17. If two antinodes are observed on either side of C, the frequency of the mode in which the rod is vibrating will be

(1) 1000 Hz (2) 3000 Hz
(3) 7000 Hz (4) 1500 Hz

If the amplitude of the wave at the antinode, when it is vibrating in its fundamental mode is 2×10^{-6} m, the maximum velocity of a steel particle in its vibration is

(1) 1.25×10^{-2} m/s (2) 1.25×10^{-3} m/s

(3) 1 m/s (4) 0.12 m/s

If the clamp of the rod be shifted to its end A and totally four antinodes are observed in the rod when longitudinal waves are set up in it, the frequency of vibration of the rod in this mode is

(1) 500 Hz (2) 2500 Hz

(3) 3500 Hz (4) 1500 Hz

For Problems 20–24

A vertical pipe open at both ends is partially submerged in water. A tuning fork of unknown frequency is placed near the top of the pipe and made to vibrate. The pipe can be moved up and down and thus length of air column in the pipe can be adjusted. For definite lengths of air column in the pipe, standing waves will be set up as a result of superposition of sound waves travelling in opposite directions. Smallest value of length of air column, for which sound intensity is maximum is 10 cm [take speed of sound, $v = 344$ m/s].

Answer the following questions.

The air column here is closed at one end because the surface of water acts as a wall. Which of the following is correct?

(1) At the closed end of the air column, there is a displacement node and also a pressure node

(2) At the closed end of the air column, there is a displacement node and a pressure antinode

(3) At the closed end of the air column, there is a displacement antinode and a pressure node

(4) At the closed end of the air column, there is a displacement antinode and also a pressure antinode

Frequency of the tuning fork is

(1) 1072 Hz (2) 940 Hz

(3) 860 Hz (4) 533 Hz

Length of air column for second resonance will be

(1) 30 cm (2) 45 cm

(3) 20 cm (4) 50 cm

Length of air column for third resonance will be

(1) 30 cm (2) 45 cm

(3) 20 cm (4) 50 cm

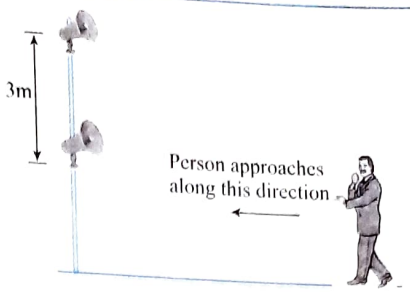
Frequency of the second overtone is

(1) 3400 Hz (2) 2500 Hz

(3) 4300 Hz (4) 1720 Hz

For Problems 25–28

An oscillator of frequency 680 Hz drives two speakers. The speakers are fixed on a vertical pole at a distance 3 m from each other as shown in figure. A person whose height is almost the same as that of the lower speaker walks towards the lower speaker in a direction perpendicular to the pole. Assuming that there is no reflection of sound from the ground and speed of sound is $v = 340$ m/s, answer the following questions.



25. As the person walks toward the pole, his distance from the pole when he first hears a minimum in sound intensity is nearly

(1) 14.6 m (2) 17.9 m

(3) 10.1 m (4) 22.4 m

26. How far is the person from the pole when he hears a minimum in sound intensity a second time?

(1) 5.6 m (2) 7.8 m

(3) 12.4 m (4) 17.6 m

27. As the person walks toward the pole, the total number of times that the person hears a minimum in sound intensity will be

(1) 2 (2) 8

(3) 4 (4) 6

28. At some instant, when the person is at a distance 4 m from the pole, the wave function (at the person's location) that describes the waves coming from the lower speaker is $y = A \cos(kx - \omega t)$, where A is the amplitude, $\omega = 2\pi\nu$ with $\nu = 680$ Hz (given) and $k = 2\pi/\lambda$

Wave function (at the person's location) that describes waves coming from the upper speaker can be expressed as:

(1) $y = A \cos(kx - \omega t + \pi)$

(2) $y = A \cos(kx - \omega t + \pi/2)$

(3) $y = A \cos(kx - \omega t + 2\pi)$

(4) $y = A \cos\left(kx - \omega t + \frac{3\pi}{2}\right)$

For Problems 29–32

In an organ pipe (may be closed or open) of 99 cm length standing wave is set up, whose equation is given by longitudinal displacement.

$$\xi = (0.1\text{ mm})\cos\frac{2\pi}{0.8}(y + 1\text{ cm})\cos(400)\,t$$

where y is measured from the top of the tube in metres and t in seconds. Here 1 cm is the end correction.

29. The upper end and the lower end of the tube are respectively.

(1) open–closed (2) closed–open

(3) open–open (4) closed–closed

30. The air column is vibrating in

(1) first overtone (2) second overtone

(3) third overtone (4) fundamental mode

31. Equation of the standing wave in terms of excess pressure is (take bulk modulus = 5×10^5 N/m^2)

(1) $P_{ex} = (125\pi \text{ N/m}^2)\sin\frac{2\pi}{0.8}(y+1\text{cm})\cos(400t)$

(2) $P_{ex} = (125\pi \text{ N/m}^2)\cos\frac{2\pi}{0.8}(y+1\text{cm})\sin(400t)$

(3) $P_{ex} = (225\pi \text{ N/m}^2)\sin\frac{2\pi}{0.8}(y+1\text{cm})\cos(200t)$

(4) $P_{ex} = (225\pi \text{ N/m}^2)\cos\frac{2\pi}{0.8}(y+1\text{cm})\sin(200t)$

32. Assume the end correction approximately equals to (0.3) × (diameter of tube), estimate the moles of air present inside the tube (Assume tube is at NTP, and at NTP, 22.4 litre contains 1 mole).

(1) $\frac{10\pi}{36\times22.4}$ (2) $\frac{10\pi}{18\times22.4}$

(3) $\frac{10\pi}{72\times22.4}$ (4) $\frac{10\pi}{60\times22.4}$

Matrix Match Type

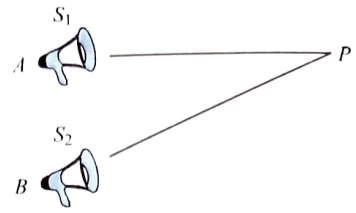
1. A closed organ pipe of length L vibrating in second overtone, then match the following:

Column I	Column II
i. Displacement node	**a.** Closed end
ii. Displacement antinode	**b.** open end
iii. Pressure node	**c.** $4L/5$ from closed end
iv. Pressure antinode	**d.** $L/5$ from closed end

2. A string fixed at both ends is vibrating in resonance. In Column I some statement(s) are given which can match with one or more entries of Column II. Match these entries.

Column I	Column II
i. All the particles of the string are vibrating in phase.	**a.** Fundamental tone
ii. The particles near both the ends of the string are vibrating in phase.	**b.** Ist harmonic
iii. The particles near the ends of the string are vibrating in opposite phase.	**c.** Even harmonic
iv. All the particles of string cross mean and extreme positions simultaneously twice in one cycle.	**d.** Odd harmonic

3. Two identical speakers emit sound waves of frequency 10^3 Hz uniformly in all directions. The audio output of each speaker is $9\pi/10$ mW. A point 'P' is at a distance 3 m from the speaker S_1 and 5m from speaker S_2. Resultant intensity at P is I_R. Match the items in Column I with the items in Column II:



Column I	Column II
i. If the speakers are incoherent, then	**a.** $I_R = 64\ \mu\text{W/m}^2$
ii. If the speakers are driven coherently and in phase at P	**b.** $I_R = 25\ \mu\text{W/m}^2$
iii. If the speakers are driven coherently and out of phase by 180° at P, then	**c.** $I_R = 34\ \mu\text{W/m}^2$
iv. If the speaker S_2 is switched off, then	**d.** $I_R = 4\ \mu\text{W/m}^2$

4. Match the statements in Column I with the statements in Column II:

Column I	Column II
i. A tight string is fixed at both ends and sustaining standing wave.	**a.** At the middle, antinode is formed in odd harmonic.
ii. A tight string is fixed at one end and free at the other end.	**b.** At the middle, node is formed in even harmonic.
iii. Standing wave is formed in an open organ pipe. End correction is not negligible.	**c.** At the middle, neither node nor antinode is formed.
iv. Standing wave is formed in a closed organ pipe. End correction is not negligible.	**d.** Phase difference between SHMs of any two particles will be either π or zero.

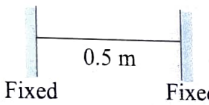
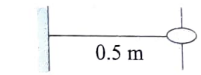
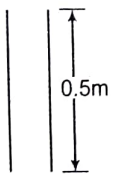
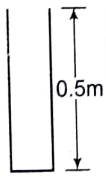
5. Three successive resonance frequencies in an organ pipe are 1310, 1834 and 2358 Hz. Velocity of sound in air is 340 m/s, then match the items given in Column I with that in Column II:

Column I	Column II
i. Length of the pipe in cm	**a.** 262
ii. Fundamental frequency (Hz)	**b.** 786
iii. Frequency of fifth harmonic (Hz)	**c.** 32.4
iv. Frequency of 1 overtone (Hz)	**d.** 1310

6. Column-I shows different sets of standing waves in a string of length L whose ends are fixed or free according to respective figure and Column-II shows possible equations for them where symbols have usual meaning.

Column I	Column II
i. $x=0$ $x=L$	a. $y = A\sin\frac{\pi x}{L}\cos\omega t$
ii. $x=0$ $x=L$	b. $y = A\sin\frac{2\pi x}{L}\cos\omega t$
iii. $x=0$ $x=L$	c. $y = A\cos\frac{\pi x}{2L}\cos\omega t$
iv. $x=0$ $x=L$	d. $y = A\cos\frac{\pi x}{L}\cos\omega t$

7. In each of the four situations of column I a stretched string or an organ pipe is given along with the required data. In case of strings the tension in string is $T = 102.4$ N and the mass per unit length of string is 1g/m. Speed of sound in air is 320 m/s. Neglect end corrections. The frequencies of resonance are given in Column II. Match each situation in Column I with the possible resonance frequencies given in Column II.

Column I	Column II
i. String fixed at both ends 	a. 320 Hz
ii. String fixed at one end and free at other end 	b. 480 Hz
iii. Open organ pipe 	c. 640 Hz
iv. Closed organ pipe 	d. 800 Hz

8 Match the Columns I and II.

Column I	Column II
i. $y = 4\sin(5x-4t) + 3\cos(4t-5x+\pi/6)$	a. Particles at every position are performing SHM
ii. $y = 10\cos\left(t-\frac{x}{330}\right)\sin(100)\left(t-\frac{x}{330}\right)$	b. Equation of travelling wave
iii. $y = 10\sin(2\pi x-120t) + 10\cos(120t+2\pi x)$	c. Equation of standing wave
iv. $y = 10\sin(2\pi x-120t) + 8\cos(118t-59/30\pi x)$	d. Equation of beats

9. With respect to various types of strings on piano, match the entries of Column I with that of Column II.

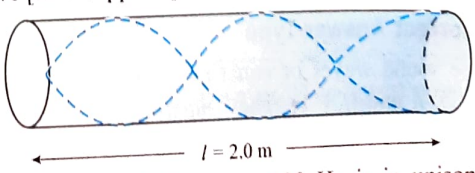
Column I	Column II
i. Bass strings (low frequency)	a. Thick
ii. Treble strings (high frequency and small wavelength)	b. Thin
iii. To have larger wavelengths, string should be	c. Long
iv. To have shorter wavelengths string should be	d. Short

10. Two strings are joined as shown in figure (assume the strings under tension).

Column I	Column II
i. Wave speed is	a. same on both the strings
ii. Wavelength is	b. different on both the strings
iii. Frequency is	c. more on the Ist string
iv. Power, assuming same amplitude, is	d. less on the Ist string

Numerical Value Type

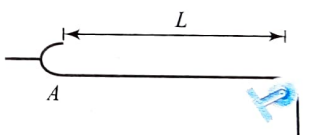
1. Two identical sinusoidal waves travel in opposite direction in a wire 15 m long and produce a standing wave in the wire. If the speed of the waves is 12 ms^{-1} and there are 6 nodes in the standing wave. Find the frequency.
2. For a certain organ pipe, three successive resonance observed are 425, 595 and 765 Hz. Taking the speed of sound to be 340 ms^{-1}, find the length of the pipe, in meter.
3. The standing wave pattern shown in the tube has a wave speed of 5.0 ms^{-1}. What is the frequency of the standing wave [in Hz approx.]?



4. A tuning fork of frequency 200 Hz is in unison with a sonometer wire. How many beats are heard in 30 s if the tension is increased by 1% (in terms of × 10)

5. A closed and an open organ pipe of same length are set into vibrations simultaneously in their fundamental mode to produce 2 beats. The length of open organ pipe is now halved and of closed organ pipe is doubled. Now find the number of beats produced .
6. A tube, opened from both ends is vibrated in its second overtone. At how many points inside the tube maximum pressure variation is observed?
7. nth harmonic of a closed organ pipe is equal to mth harmonic of an open pipe. First overtone frequency of the closed organ pipe is also equal to first overtone frequency of the open organ pipe. Find the value of n, if $m = 6$.
8. The length, radius, tension and density of string A are twice the same parameters of string B. Find the ratio of fundamental frequency of B to the fundamental frequency of A.
9. A glass tube of 1.0 m length is filled with water. The water can be drained out slowly at the bottom of the tube. A vibrating tuning fork of frequency 500 Hz is brought at the upper end of the tube and the velocity of sound is300 ms^{-1}. Find the number of resonances that can be obtained.
10. A 40 cm long wire having a mass 3.2 g and area of cross-section 1 mm^2 is stretched between the support 40.05 cm apart. In its fundamental mode, it vibrates with a frequency 1000/64 Hz. The Young's modulus of the wire is (in $\times 10^9$ N/m^2)
11. A bat emits ultrasonic sound of frequency 1000 kHz in air. If the sound meets a water surface, it gets partially reflected back and partially refracted (transmitted) in water. What Would be the difference of wavelength transmitted to wavelength reflected (speed of sound in air = 330 m/s, Bulk modulus of water = 2.25×10^9, $\rho_{water} = 1000$ kg/m^2).
12. A source emitting sound of frequency 180 Hz is placed in front of a wall at distance of 2 m from it. A detector is also placed in front of the wall at the same distance from it. Find the minimum distance (in m) between the source and the detector for which the detector detects a maximum of sound. Speed of sound in air = 360 m/s.
13. A set of 56 tuning forks is arranged in a sequence of increasing frequencies. If each fork gives 4 beats/s with the preceding one and the last fork is found to be an octave higher of the first, find the frequency of the first fork.
14. Two tuning forks A and B are sounded together and 8 beats/s are heard. A is in resonance with a column of air 32 cm long in a pipe closed at one end and B is similarly in resonance when the length of the column is increased by one cm. Calculate the frequency of forks.
15. A certain fork is found to give 2 beats/s when sounded in conjunction with a stretched string vibrating transversely under a tension of either 10.2 or 9.9 kg weight. Calculate the frequency of fork.
16. Two tuning forks A and B give 18 beats in 2 s. A resonates with one end closed air column of 15 cm long and B with both ends open column of 30.5 long. Calculate their frequencies.
17. Six antinodes are observed in the air column when a standing wave forms in a Kundt's tube. What is the length of the air column if steel bar of 1m length is clamped at the middle.

    The velocity of sound in steel is 5250 m/s and in air 343 m/s.
18. A column of air at 51°C and a tuning fork produce 4 beats/s when sounded together. As the temperature of air column is decreased the number of beats per second tends to decrease and when temperature is 16°C the two produce 1 beat/second. Find the frequency of tuning fork.
19. The linear mass density of the string shown in the figure is $\mu = 1$ g/m. One end (A) of the string is tied to a prong of a tuning fork and the other end carries a block of mass M. The length of the string between the tuning fork and the pulley is $L = 2.0$ m. When the tuning fork vibrates, the string resonates with it when mass M is either 16 kg or 25 kg. However, standing waves are not observed for any other value of M lying between 16 kg and 25 kg. Assume that end A of the string is practically at rest and calculate the frequency of the fork in Hz.



20. Fundamental frequency of a stretched sonometer wire is f_0. When its tension is increased by 96% and length decreased by 35%, its fundamental frequency becomes $\eta_1 f_0$. When its tension is decreased by 36% and its length is increased by 30%, its fundamental frequency becomes $\eta_2 f_0$.

    Find $\frac{\eta_1}{\eta_2}$.

Archives

JEE Main

Single Correct Answer Type

1. Three sound waves of equal amplitudes have frequencies $(f - 1)$, f, and $(f + 1)$. They superpose to give beats. The number of beats produced per second will be

   (1) 4 (2) 3

   (3) 2 (4) 1 **(AIEEE 2009)**

2. A cylindrical tube, open at both ends, has a fundamental frequency, f, in air. The tube is dipped vertically in water so that half of it is in water. The fundamental frequency of the air-column is now

   (1) f (2) $f/2$

   (3) $3f/4$ (4) $2f$

   (AIEEE 2012)

3. A sonometer wire of length 1.5 m is made of steel. The tension in it produces an elastic strain of 1%. What is the fundamental frequency of steel if density and elasticity of steel are 7.7×10^3 and 2.2×10^{11} N/m^2 respectively?
(1) 178.2 Hz (2) 200.5 Hz
(3) 770 Hz (4) 188.5 Hz
(JEE Main 2013)

4. A pipe of length 85 cm is closed from one end. Find the number of possible natural oscillations of air column in the pipe whose frequencies lie below 1250 Hz. The velocity of sound in air is 340 m/s.
(1) 6 (2) 4
(3) 12 (4) 8 **(JEE Main 2014)**

5. A pipe open at both ends has a fundamental frequency f in air. The pipe is dipped vertically in water so that half of it is in water. The fundamental frequency of the air column is now:
(1) $\frac{f}{2}$ (2) $\frac{3f}{4}$
(3) $2f$ (4) f **(JEE Main 2016)**

6. A granite rod of 60 cm length is clamped at its middle point and is set into longitudinal vibrations. The density of granite is 2.7×10^3 kg/m^3 and its Young's modulus is 9.27×10^{10} Pa. What will be the fundamental frequency of the longitudinal vibrations?
(1) 7.5 kHz (2) 5 kHz
(3) 2.5 kHz (4) 10 kHz
(JEE Main 2018)

JEE ADVANCED

Single Correct Answer Type

1. A hollow pipe of length 0.8 m is closed at one end. At its open end a 0.5 m long uniform string is vibrating in its second harmonic and it resonates with the fundamental frequency of the pipe. If the tension in the wire is 50 N and the speed of sound is 320 ms^{-1}, the mass of the string is
(1) 5 g (2) 10 g
(3) 20 g (4) 40 g **(IIT-JEE 2010)**

2. A student is performing the experiment of resonance column. The diameter of the column tube is 4 cm. The frequency of the tuning fork is 512 Hz. The air temperature is 38°C in which the speed of sound is 336 m/s. The zero of the meter scale coincides with the top end of the resonance column tube. When the first resonance occurs, the reading of the water level in the column is
(1) 14.0 cm (2) 15.2 cm
(3) 16.4 cm (4) 17.6 cm
(IIT-JEE 2012)

3. A student is performing an experiment using a resonance column and a tuning fork of frequency 244 s^{-1}. He is told that the air in the tube has been replaced by another gas (assume that the column remains filled with the gas). If the minimum height at which resonance occurs is (0.350 ± 0.005)m, the gas in the tube is
(Useful information: $\sqrt{167RT} = 640$ J$^{1/2}$ mole$^{-1/2}$; $\sqrt{140RT} = 590$ J$^{1/2}$ mole$^{-1/2}$. The molar masses M in grams are given in the options. Take the values of $\sqrt{\frac{10}{M}}$ for each gas as given there.)
(1) Neon $\left(M = 20, \sqrt{\frac{10}{20}} = \frac{7}{10}\right)$
(2) Nitrogen $\left(M = 28, \sqrt{\frac{10}{28}} = \frac{3}{5}\right)$
(3) Oxygen $\left(M = 32, \sqrt{\frac{10}{32}} = \frac{9}{16}\right)$
(4) Argon $\left(M = 36, \sqrt{\frac{10}{36}} = \frac{17}{32}\right)$ **(JEE Advanced 2014)**

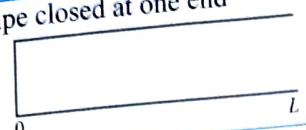
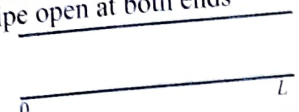
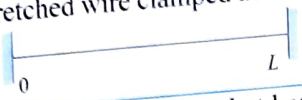
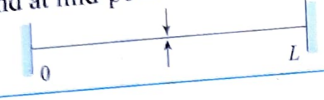
Multiple Correct Answers Type

1. One end of a taut string of length 3 m along the x axis is fixed at $x = 0$. The speed of the waves in the string is 100 ms^{-1}. The other end of the string is vibrating in the y direction so that stationary waves are set up in the string. The possible waveform(s) of these stationary waves is (are)
(1) $y(t) = A \sin\frac{\pi x}{6}\cos\frac{50\pi t}{3}$
(2) $y(t) = A \sin\frac{\pi x}{3}\cos\frac{100\pi t}{3}$
(3) $y(t) = A \sin\frac{5\pi x}{6}\cos\frac{250\pi t}{3}$
(4) $y(t) = A \sin\frac{5\pi x}{2}\cos 250\pi t$ **(JEE Advanced 2014)**

2. In an experiment to measure the speed of sound by a resonating air column, a tuning fork of frequency 500 Hz is used. The length of the air column is varied by changing the level of water in the resonance tube. Two successive resonances are heard at air columns of length 50.7 cm and 83.9 cm. Which of the following statements is (are) true?
(1) The speed of sound determined from this experiment is 332 ms^{-1}
(2) The end correction in this experiment is 0.9 cm
(3) The wavelength of the sound wave is 66.4 cm
(4) The resonance at 50.7 cm corresponds to the fundamental harmonic
(JEE Advanced 2018)

Matrix Match Type

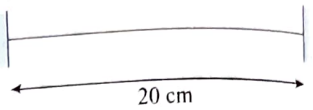
1. Column I shows four systems, each of the same length L, for producing standing waves. The lowest possible natural frequency of a system is called its fundamental frequency, whose wavelength is denoted as λ_f. Match each system with statements given in Column II describing the nature and wavelength of the standing waves.

Column I	Column II
i. Pipe closed at one end 	a. Longitudinal waves
ii. Pipe open at both ends 	b. Transverse waves
iii. Stretched wire clamped at both ends 	c. $\lambda_f = L$
iv. Stretched wire clamped at both ends and at mid-point 	d. $\lambda_f = 2L$
	e. $\lambda_f = 4L$

(IIT-JEE 2011)

Numerical Value Types

1. A 20 cm long string, having a mass of 1.0 g, is fixed at both the ends. The tension in the string is 0.5 N. The string is set into vibrations using an external vibrator of frequency 100 Hz. Find the separation (in cm) between the successive nodes on the string.



(IIT-JEE 2009)

2. When two progressive waves $y_1 = 4\sin(2x - 6t)$ and $y_2 = 3\sin\left(2x - 6t - \frac{\pi}{2}\right)$ superimposed, the amplitude of the resultant wave is **(IIT-JEE 2010)**

3. Four harmonic waves of equal frequencies and equal intensities I_0 have phase angles 0, $\frac{\pi}{3}, \frac{2\pi}{3}$ and π. When they are superposed, the intensity of the resulting wave is nI_0. The value of n is: **(JEE Advanced 2015)**

Answers Key

EXERCISES

Single Correct Answer Type

1. (1)	2. (2)	3. (3)	4. (2)	5. (2)
6. (4)	7. (1)	8. (1)	9. (1)	10. (4)
11. (2)	12. (2)	13. (4)	14. (4)	15. (2)
16. (1)	17. (1)	18. (2)	19. (3)	20. (4)
21. (3)	22. (2)	23. (4)	24. (3)	25. (2)
26. (4)	27. (1)	28. (2)	29. (2)	30. (2)
31. (3)	32. (2)	33. (3)	34. (4)	35. (3)
36. (1)	37. (1)	38. (3)	39. (1)	40. (1)
41. (2)	42. (2)	43. (2)	44. (4)	45. (2)
46. (2)	47. (4)	48. (1)	49. (1)	50. (2)
51. (2)	52. (4)	53. (4)	54. (2)	55. (4)
56. (1)	57. (2)	58. (3)	59. (3)	60. (4)
61. (2)	62. (2)	63. (3)	64. (3)	65. (1)
66. (4)	67. (4)	68. (3)	69. (3)	70. (1)
71. (3)	72. (4)	73. (4)	74. (3)	75. (3)
76. (2)	77. (1)	78. (3)	79. (3)	

Multiple Correct Answers Type

1. (1),(2)	2. (1),(2),(4)	3. (2),(3)
4. (2),(3)	5. (1),(3),(4)	6. (1),(2)
7. (2),(4)	8. (3),(4)	9. (3),(4)
10. (1),(2),(4)	11. (1),(2),(4)	12. (1),(2),(4)
13. (1),(3),(4)	14. (2),(3)	15. (2),(4)
16. (2),(3)	17. (3),(4)	18. (1),(4)
19. (3),(4)	20. (1),(2),(3),(4)	21. (1),(3)
22. (3),(4)	23. (2),(3)	

Linked Comprehension Type

1. (3)	2. (1)	3. (3)	4. (2)	5. (1)
6. (3)	7. (1)	8. (4)	9. (2)	10. (3)
11. (4)	12. (3)	13. (1)	14. (4)	15. (3)
16. (1)	17. (2)	18. (1)	19. (3)	20. (2)
21. (3)	22. (1)	23. (4)	24. (3)	25. (2)
26. (1)	27. (4)	28. (3)	29. (1)	30. (2)
31. (1)	32. (1)			

Matrix Match Type

1. i. → a., c.; ii. → b., d.; iii. → b., d.; iv. → a., c.
2. i. → a.,d.; ii. → a., b., d.; iii. → c.; iv. → a., b., c., d.
3. i. → c.; ii. → a.; iii. → d.; iv. → b.
4. i. → a., b., d.; ii. → c., d.; iii. → d.; iv. → c., d.
5. i. → c.; ii. → a.; iii. → d.; iv. → b.
6. i. → b.; ii. → a.; iii. → c.; iv. → d.
7. i. → a., c.; ii. → b., d.; iii. → a., c.; iv. → b., d.
8. i. → a., b.; ii. → d.; iii. → a., c.; iv. → d.
9. i. → a., c.; ii. → b., d.; iii. → c.; iv. → d.
10. i. → b., d.; ii. → b., d.; iii. → a. iv. → b., c.

Numerical Value Type

1. (2)	2. (1)	3. (3)	4. (3)	5. (7)
6. (3)	7. (9)	8. (4)	9. (2)	10. (1)
11. (1.17)	12. (3)	13. (220)	14. (256)	15. (532)
16. (540)	17. (39.2)	18. (50)	19. (500)	20. (3.5)

ARCHIVES

JEE Main

Single Correct Answer Type

1. (3)	2. (1)	3. (1)	4. (1)	5. (4)
6. (2)				

JEE Advanced

Single Correct Answer Type

1. (2) 2. (2) 3. (4)

Multiple Correct Answers Type

1. (1),(3),(4) 2. (1), (2), (3)

Matrix Match Type

1. i. → a., e.; ii. → a., d.; iii. → b., d.; iv. → b., c.

Numerical Value Type

1. (5) 2. (5) 3. (3)

8 Sound Waves and Doppler Effect

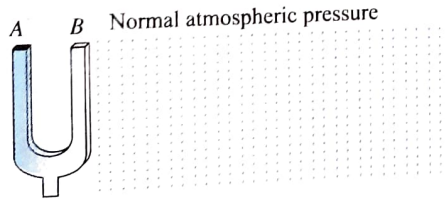
INTRODUCTION

Sound is a pressure variation that propagates through some medium. As sound waves travel through air, the elements of air vibrate to produce changes in density and pressure along the direction of motion of the wave. This pressure variation causes abnormal motion of air molecules in the direction of propagation; thus, a sound wave is longitudinal. When we hear a sound, our eardrums are set in vibration by the air next to them. If this air has a pressure variation that repeats with a certain frequency, the eardrums vibrate at this frequency. If the source of the sound waves vibrates sinusoidally, the pressure variations are also sinusoidal. The mathematical description of sinusoidal sound waves is very similar to that of sinusoidal waves on strings. In this chapter we will discuss the mathematical description of this pressure variation due to travelling of sound wave, speed of sound wave and various other properties of sound wave.

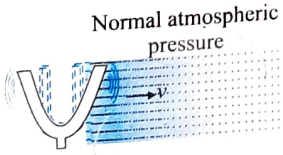
PROPAGATION OF SOUND WAVES

Sound is a mechanical three-dimensional and longitudinal wave that is created by a vibrating source such as a guitar string, the human vocal cords, the prongs of a tuning fork or the diaphragm of a loudspeaker. Being a mechanical wave, sound needs a medium having properties of inertia and elasticity for its propagation. Sound waves propagate in any medium through a series of periodic compressions and rarefactions of pressure, which is produced by the vibrating source.

Consider a tuning fork producing sound waves. When prong B moves outward towards right it compresses the air in front of it, causing the pressure to rise slightly. The region of increased pressure is called a compression pulse and it travels away from the prong with the speed of sound.



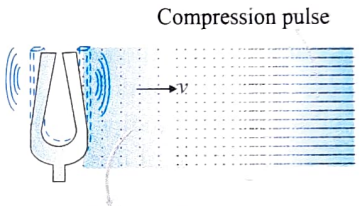
Undisturbed tunning fork



compression pulse

After producing the compression pulse, prong B reverses its motion and moves inward. This drags away some air from the region in front of it, causing the pressure to dip slightly below the normal pressure. This region of decreased pressure is called a rarefaction pulse. Following immediately behind the compression pulse, the rarefaction pulse also travels away from the prong with the speed of sound.

If the prongs vibrate in SHM, the pressure variation in the layer close to the prong also varies simple harmonically and hence increase in pressure above normal value can be written as



$$\delta P = \delta P_0 \sin \omega t$$

where δP_0 is the maximum increase in pressure above normal value.

As this disturbance travels towards right with wave velocity v, the excess pressure at any position x at time t will be given by

$$\delta P = \delta P_0 \sin [\omega(t - x/v)]$$

Using $\Delta p = \delta P$, $\Delta p_{max} = \delta P_0$, the above equation of sound wave can be written as

$$\Delta p = \Delta p_{max} \sin [\omega(t - x/v)]$$

ILLUSTRATION 8.1

The equation of a sound wave in air is given by $\Delta p = (0.02) \sin [(3000)\, t - (9.0)x]$, where all variables are in SI units. **(a)** Find the frequency, wavelength and the speed of sound wave in air. **(b)** If the equilibrium pressure of air is 1.01×10^5 N/m^2, what are the maximum and minimum pressures at a point as the wave passes through that point?

Sol.

(a) Comparing with the standard form of a travelling wave,

$$\Delta p = \Delta p_{max} \sin [\omega (t - x/v)]$$

we see that $\omega = 3000$ s^{-1}. The frequency is

$$f = \frac{\omega}{2\pi} = \frac{3000}{2\pi} \text{ Hz}$$

or $\frac{P}{\rho} = \text{constant}$

Therefore, with the change in pressure, the density also changes in such proportion, so that p/ρ remains constant. Hence pressure has no effect on the speed of sound in a gas.

Effect of Density

For two gases of densities ρ_1 and ρ_2 at same pressure with ratios of specific heats γ_1 and γ_2,

$$\frac{v_1}{v_2} = \sqrt{\frac{\gamma_1}{\gamma_2} \times \frac{\rho_2}{\rho_1}}$$

Effect of Temperature

We have, $\frac{P}{\rho} = \frac{RT}{M}$

$$\therefore \quad v = \sqrt{\frac{\gamma RT}{M}}$$

Clearly, $v \propto \sqrt{T}$

Hence the speed of sound in a gas is proportional to the square root of its absolute temperature. If v_0 and v_t are the velocities of sound in gas at 0°C and t°C, respectively, then

$$v_0 = \sqrt{\frac{\gamma R(273+0)}{M}} \text{ and } v_t = \sqrt{\frac{\gamma R(273+t)}{M}}$$

$$\therefore \quad \frac{v_t}{v_0} = \left(\frac{273+t}{273}\right)^{1/2} = \left(1+\frac{t}{273}\right)^{1/2}$$

For small value of t, $v_t = v_0\left(1+\frac{1}{2}\frac{t}{273}\right)$

or $v_t = v_0 + \frac{v_0 t}{546}$

But $v_0 = 332$ m/s.

$$\therefore \quad v_t - v_0 = \frac{332t}{546} = 0.61t$$

When $t = 1$°C, $v_t - v_0 = 0.61$ m/s

Hence the velocity of sound in air increases by 0.61 m/s for every 1°C rise in temperature.

Effect of Humidity

With the increase in humidity, the density of air decreases. Now, the speed of sound in air is given by

$$v \propto \frac{1}{\sqrt{\rho}}$$

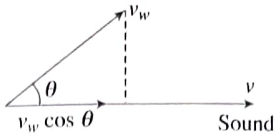
Hence, the speed of sound will increase.

Effect of Frequency

With the change in frequency of the sound wave, wavelength also changes, so that $f\lambda = v$ (constant)
Thus, the speed of sound is independent of its frequency.

Effect of Wind

As the sound is carried by air, so its speed is affected by the wind velocity. Suppose the wind is blowing with a velocity v_w at an angle θ with the direction of propagation of the sound. Clearly, the component of wind velocity in the direction of sound is $v_w \cos\theta$. Therefore, the Resultant velocity of sound is $v + v_w \cos\theta$



When wind blows in the direction of sound ($\theta = 0$), the resultant velocity is $v + v_w$.

When wind blows in the opposite direction of sound ($\theta = 180°$), the resultant velocity is $= v - v_w$.

Illustration 8.3

Calculate the velocity of sound in air at NTP. The density of air at NTP is 1.29 g/L. Assume air to be diatomic with $\gamma = 1.4$. Also calculate the velocity of sound in air at 27°C.

Sol. Velocity of sound in air is given by

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{1.4 \times 1.013 \times 10^5 \text{ N/m}^2}{1.29 \text{ kg/m}^3}} = 331.6 \text{ m/s}$$

$$\left(\text{using } \frac{P}{\rho} = \frac{R}{M}T\right)$$

We can see that the velocity of sound is proportional to the square root of absolute temperature. Hence,

$$\frac{v_2}{v_1} = \sqrt{\frac{T_2}{T_1}}$$

$$\Rightarrow \quad v_2 = v_1\sqrt{\frac{T_2}{T_1}} = 331.6\sqrt{\frac{273+27}{273}} = 347.6 \text{ m/s}$$

Illustration 8.4

Calculate the stress in a tight wire of a material whose Young's modulus is 19.6×10^{11} dyne/cm² so that the speed of the longitudinal waves is 10 times the speed of transverse waves.

Sol. The speed of transverse waves in string is

$$c_1 = \sqrt{\frac{T}{m}} = \sqrt{\frac{T}{Ad}}$$

where A is cross-sectional area and d is the density.
The speed of longitudinal waves in string is $c_2 = \sqrt{Y/d}$. We have,

$$c_2 = 10c_1$$

$$\Rightarrow \quad \sqrt{\frac{Y}{d}} = 10\sqrt{\frac{T}{Ad}}$$

$$\Rightarrow \quad \text{Stress} = \frac{T}{A} = \frac{Y}{100} = 19.6 \times 10^9 \text{ dyne/cm}^2$$

Illustration 8.5

Taking the composition of air to be 75% of nitrogen and 25% of oxygen by weight, calculate the velocity of sound through air.

Sol. The molecular weight of a mixture is given by

$$\frac{m_1+m_2+m_3+\cdots}{M}=\frac{m_1}{M_1}+\frac{m_2}{M_2}+\frac{m_3}{M_3}+\cdots$$

$$\frac{75+25}{M}=\frac{75}{28}+\frac{25}{32}$$

$$M=28.9 \text{ g}$$

$$c=\sqrt{\gamma\frac{RT}{M}}=\sqrt{\frac{1.4\times8.3\times1000\times273}{28.9}}=331.3 \text{ m/s}$$

Here $\gamma=1.4 \rightarrow$ as both the gases are diatomic.

ILLUSTRATION 8.6

If the speed of sound through hydrogen is 1280 m/s, what will be the speed of sound through a mixture of 4 parts by volume of hydrogen and 1 part of oxygen? Given density of oxygen is 16 times that of hydrogen.

Sol. If V is the volume of the mixture,

Volume of hydrogen $=\frac{4}{5}V$

And volume of oxygen $=\frac{1}{5}V$

Mass of oxygen $=\frac{1}{5}V\rho_O=\frac{1}{5}V\rho_O=\frac{V}{5}(16\rho_H)$ (as $\rho_O=16\rho_H$)

Total mass of the mixture, $m_{mix}=\frac{4}{5}V\rho_H+\frac{V}{5}(16\rho_H)=4V\rho_H$

Density of mixture, $\rho_m=\frac{4V\rho_H}{V}=4\rho_H$

Thus, $\frac{v_m}{v_H}=\sqrt{\frac{\rho_H}{\rho_m}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$

or $v_m=\frac{1}{2}v_H=\frac{1}{2}\times1280=640$ m/s

ILLUSTRATION 8.7

A certain gas mixture is composed of two diatomic gases (molecular weights M_1 and M_2). The ratio of the masses of the two gases in a given volume is $m_2/m_1=r$. Find the speed of sound in the gas mixture if the gases are ideal.

Sol. The speed of sound $v=\sqrt{\frac{\gamma P}{\rho}}$ ...(i)

The density of the gas mixture, $\rho=\frac{m}{V}=\frac{m_1+m_2}{V}$

$$P=P_1+P_2=n_1\left(\frac{RT}{V}\right)+n_2\left(\frac{RT}{V}\right)$$

$$=(n_1+n_2)\left(\frac{RT}{V}\right)=\left(\frac{m_1}{M_1}+\frac{m_2}{M_2}\right)\left(\frac{RT}{V}\right)$$

or $$\frac{P}{\rho}=\frac{\left(\frac{m_1}{M_1}+\frac{m_2}{M_2}\right)\left(\frac{RT}{V}\right)}{(m_1+m_2)/V}$$

$$\Rightarrow \frac{P}{\rho}=\frac{m_1\left(\frac{1}{M_1}+\frac{r}{M_2}\right)RT}{m_1(1+r)}=\frac{(M_2+rM_1)RT}{M_1M_2(1+r)} \quad ...(ii)$$

From eqns. (i) and (ii) and putting $\gamma=1.40$, we get

$$v=\sqrt{\frac{1.40RT(M_2+rM_1)}{M_1M_2(1+r)}}$$

CONCEPT APPLICATION EXERCISE 8.1

1. The excess pressure in a travelling sound wave is given by $\Delta p=1.5\sin\left[\frac{2\pi}{\lambda}(x-330t)\right]$, where x and λ, are in meters and t is in seconds and p is in pascals.
- **(a)** What is the velocity of the wave?
- **(b)** If $\lambda = 2$ m, what is the frequency of the wave?
- **(c)** What is the maximum increase in pressure (pressure amplitude)?
- **(d)** If the equilibrium pressure of air is 1.01×10^5 Pa, what is the pressure at $x = (1/6)$ m and $t = 0$?

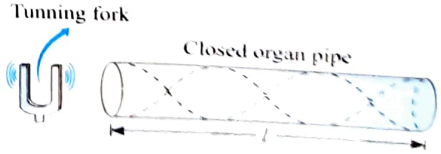
2. The excess pressure in a travelling sound wave in air is given by $\Delta p=(0.01 \text{ N/m}^2)\sin\pi[(600\text{s}^{-1})t-(2\text{m}^{-1})x]$
- **(a)** Find the frequency, wavelength and the speed of sound wave in air.
- **(b)** If the equilibrium pressure of air is 1.01×10^5 N/m^2, what are the maximum and minimum pressures at a point as the wave passes through that point?

3. By how much must the temperature of air near 0°C be changed to cause the speed in it to change by 1%.

4. Determine the percentage change in the speed of sound in air when the temperature changes from 10°C to 20°C.

5. Speed of sound in hydrogen at 0°C is 1200 m/s. When some amount of oxygen is mixed with hydrogen, the speed decreases to 500 m/s. Determine the ratio of hydrogen and oxygen by volume in the mixture. Given density of oxygen is 16 times that of hydrogen.

6. A tunning fork force generates a sound of frequency f and resonates with the air column closed at one end by a tube of length l. If air column vibrates with its fundamental mode at temperature T, assuming $\gamma=\frac{C_p}{C_v}$ of air, find its molar mass.



ANSWERS

1. (a) 330 m/s (b) 165 Hz (c) 1.5 Pa (d) 1.76×10^5 Pa
2. (a) 300 Hz, 1.0 m, 300 m/s (b) $(1.01\times10^5\pm0.01)$ N/m^2
3. 5.5 °C **4.** 1.77%
5. 256/119 **6.** $M=\frac{\gamma RT}{16l^2f^2}$

Displacement Wave, Pressure Wave and Density Wave

In last two chapters we have described mechanical waves primarily in terms of displacement waves and we describe a sinusoidal wave equation $y(x, t)$ travelling in positive x-direction as

$$y(x,t) = A\sin(kx - \omega t)$$

Here y is the instantaneous displacement of a particle in the medium at position x at time t.

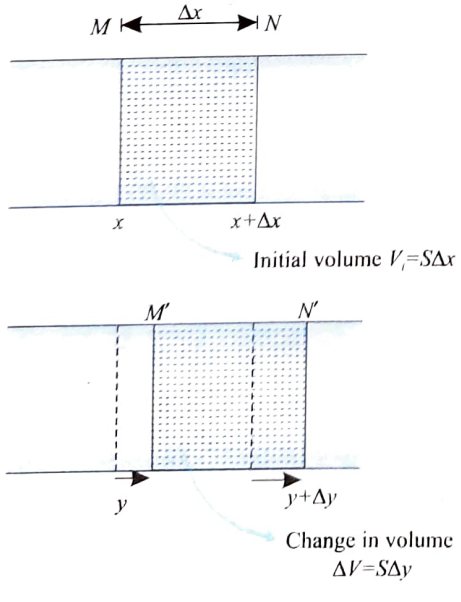
In case of a longitudinal wave the displacements of medium particles are parallel to the direction of motion of the wave hence, x and y are measured parallel to each other, not perpendicular as in a transverse wave. Here A is the the maximum displacement of a particle in a medium from its equilibrium positions.

These displacements are along the direction of the motion of the wave lead to variations in the density and pressure of air. It means a sound wave can also be described in terms of variations of pressure or density at various points. In a sinusoidal sound wave in air, the pressure fluctuates above and below atmospheric pressure with the same frequency as the motions of the air particles. The variation in pressure is of the order of 1 Pa (= 1 N/m^2), whereas atmospheric pressure is about 10^5 Pa. Similarly, the density of air also vibrates sinusoidally above and below its normal level. So, a sound wave can be expressed either in terms of $y(x,t)$ or $\Delta p(x,t)$ or $\Delta\rho(x,t)$. All the three equations are related to one another.

Relation Between Displacement Wave and Pressure Wave

Let us discuss the propagation of excess pressure in a medium in longitudinal wave analytically. Consider a longitudinal wave propagating through air contained in a long tube of cross-sectional area S, in positive x direction as shown in figure. The figure shows a segment of the medium of width Δx. In this medium let a longitudinal wave is propagating whose displacement equation is given as

$$y = A\cos(kx - \omega t) \qquad \ldots(i)$$



The initial volume of gas in this segment is, $V_i = S\Delta x$.

The change in volume ΔV is $S\Delta y$; where Δy is the difference between the value of displacements; at $x+\Delta x$ and x. From the definition of bulk modulus, the pressure variation in the gas is

$$\Delta p = -B\frac{\Delta V}{V_i} = -B\left(\frac{S\Delta y}{S\Delta x}\right) = -B\left(\frac{\Delta y}{\Delta x}\right) \qquad \ldots(ii)$$

As Δx approaches zero, the ratio $\frac{\Delta y}{\Delta x}$ becomes $\frac{\partial y}{\partial x}$. (The partial derivative indicates that we are interested in the variation of y with position at a fixed time).

Therefore equation (ii) becomes, $\Delta p = B\frac{\partial y}{\partial x}$ ....(iii)

So, this is the equation which relates the displacement equation with the pressure equation. Partial differentiating equation (i) w.r.t. x

Then, $\frac{\partial y}{\partial x} = -kA\sin(kx - \omega t)$ ...(iv)

and from eqs. (i) and (iv), we get

$$\Delta p = BAk\sin(kx - \omega t) = (\Delta p)_m \sin(kx - \omega t) \qquad \ldots(v)$$

Here, $(\Delta p)_m = BAk$

where Δp_m is the amplitude of pressure variation. From Eqs. (i) and (v), we see that pressure equation is $\pi/2$ out of phase with displacement equation. When the displacement is zero, the pressure variation is either maximum or minimum and vice-versa.

The Fig. (a) shows displacement from equilibrium of air molecules when a harmonic sound wave versus position at some instants. The Fig. (b) shows the variation of pressure at different points in the medium at the same time.

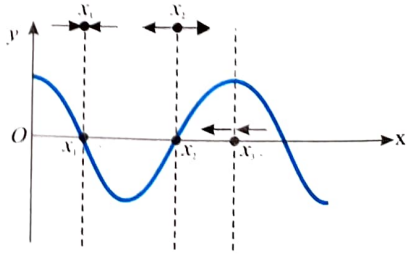
At point x_1: the pressure of the gas is maximum because the molecules on both sides of that point are displaced toward point x_1. So, if p_0 is the atmospheric pressure (normal pressure), the pressure at x_1 will be

$$p(x_1) = p_0 + (\Delta p)_m = p_0 + BAk$$

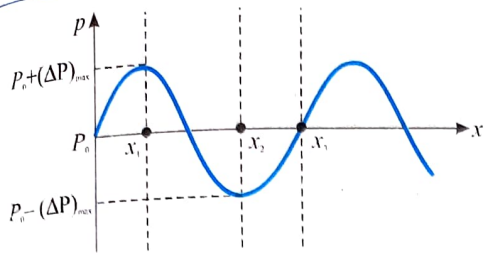
At point x_2: the pressure (and hence the density also) is minimum because the molecules on both sides of that point are displaced away from point x_2.

Hence, $p(x_2) = p_0 - (\Delta p)_m = p_0 - BAk$

At point x_3: the pressure (and hence the density) does not change because the gas molecules on both sides of that point have equal displacements in the same direction or $p(x_3) = p_0$



(a) A plot of displacement function for t = 0



(b) A plot of pressure variation

From Fig. (a) and (b), we see that pressure change and displacement are 90° out of phase.

RELATION BETWEEN PRESSURE WAVE AND DENSITY WAVE

In the compression zone, more particles stay in a unit volume of the medium. Hence, density and pressure of the region will be more. In other words, the change in density ($\Delta\rho$) and change in pressure (Δp) is positive. In dilated (rarefacted) zone, lesser particles stay in any unit volume. Hence, $\Delta\rho$ and Δp will be negative. Let a sound wave is propagating in a medium of bulk modulus B and density ρ.

According to definition of bulk modulus (B),

$$B=\left(-\frac{dp}{dV/V}\right) \qquad \ldots(i)$$

Further, $\text{Volume}=\dfrac{\text{mass}}{\text{density}}$

$$V=\frac{m}{\rho} \quad \text{or} \quad dV=-\frac{m}{\rho^2}d\rho=-\frac{V}{\rho}\cdot d\rho$$

$$\frac{dV}{V}=-\frac{d\rho}{\rho} \qquad \ldots(ii)$$

From Eq. (i) and (ii), we get, $d\rho=\dfrac{\rho}{B}dp$ $\qquad \ldots(iii)$

The speed of sound is given by, $v=\sqrt{\dfrac{B}{\rho}} \Rightarrow \dfrac{\rho}{B}=\dfrac{1}{v^2}$ $\qquad \ldots(iv)$

Hence, $d\rho=\dfrac{\rho}{B}dp=\dfrac{dp}{v^2}$

Or, this can be written as $\Delta\rho=\dfrac{\rho}{B}\cdot\Delta p=\dfrac{1}{v^2}\cdot\Delta p$

So, this relation relates the pressure equation with the density equation. For example, if we express variation in pressure as $\Delta p=(\Delta p)_m \sin(kx-\omega t)$ then the variation in density can be expressed as

$$\Delta\rho=(\Delta\rho)_m \sin(kx-\omega t)$$

where, $(\Delta\rho)_m=\dfrac{\rho}{B}(\Delta p)_m=\dfrac{(\Delta p)_m}{v^2}$

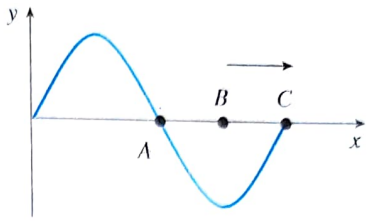
Thus, density equation is in phase with the pressure equation and this is $\pi/2$ out of phase with the displacement equation.

Important Point:

The pressure or density wave is 90° out of phase with the displacement wave, i.e. when the displacement is zero, the pressure and density change a either maximum or minimum and when the displacement is a maximum or minimum the pressure and density changes are zero.

ILLUSTRATION 8.8

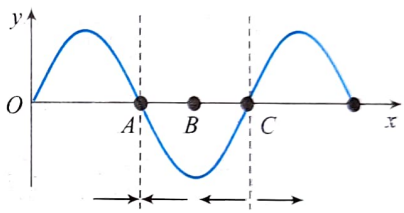
The figure shows an instantaneous displacement–position graph of a sound wave travelling along the positive x-axis. Identify the points of



(a) maximum pressure;
(b) minimum pressure and
(c) atmospheric pressure (or normal pressure).

Sol. **Method 1:**

(a) Maximum pressure occurs at A because air particles are displaced towards it.
(b) Minimum pressure occurs at C because air particles are displaced away from it.
(c) Pressure remains atmospheric at B.



Compression Rarefaction

Method 2: Pressure change in the medium is given by

$$\Delta p=B\left(-\frac{dy}{dx}\right)$$

The pressure change will be positive where dy/dx is negative and pressure change will be negative where dy/dx is positive.

(a) Maximum pressure occurs at the points of maximum negative slope at point A.
(b) Minimum pressure occurs at the points of maximum positive slope at point C.
(c) Atmospheric pressure remains at the positions of zero slope at point B.

ILLUSTRATION 8.9

Corresponding to displacement equation, $y=A\sin(kx+\omega t)$ of a longitudinal wave make its pressure and density wave also. Bulk modulus of the medium is B and density is ρ.

Sol. Given, $y(x,t)=A\sin(\omega t+kx)$

$$\frac{\partial y}{\partial x}=kA\cos(kx+\omega t)$$

Now, pressure equation is given by $\Delta p=-B\left(\dfrac{\partial y}{\partial x}\right)$

Substituting the value of $\dfrac{\partial y}{\partial x}$ from Eq. (i), we have

$$\Delta p=-BAk\cos(kx+\omega t)$$

Similarly, density equation is given by

$$(\Delta p) = \frac{\rho}{B}(\Delta p) = \frac{\rho}{B}\left(-B\frac{\partial y}{\partial x}\right) = -\rho\left(\frac{\partial y}{\partial x}\right)$$

Substituting the values of $\frac{\partial y}{\partial x}$ from Eq. (i), we have

$$\Delta\rho = -\rho Ak\cos(kx+\omega t)$$

ILLUSTRATION 8.10

If the maximum tolerable pressure variation in sound is 28 Pa at 1000 Hz. Find the amplitude of density and displacement. Given: Bulk modulus of medium $B = 1.4\times10^5$ Pa, density of the medium $\rho = 1.20$ kg/m^3 and speed of sound = 341 m/s

Sol. The density amplitude is

$$\Delta\rho_m = \frac{\rho}{B}\Delta p_m = \frac{1.20}{1.4\times10^5}28 = 2.4\times10^{-4}\ kg/m^3$$

The displacement amplitude is $A = \frac{\Delta p_m}{KB}$ where $K = \frac{2\pi}{\lambda} = \frac{2\pi f}{v}$

or $$A = \frac{v}{2\pi f}\cdot\frac{\Delta p_m}{B} = \frac{341}{2\times(22/7)\times1000}\times\frac{28}{1.4\times10^5}$$

$$= 1.1\times10^{-5}\text{ m}$$

INTENSITY OF PERIODIC SOUND WAVES

We define the intensity I of a wave, or the power per unit area, as the rate at which the energy transported by the wave transfers through a unit area A perpendicular to the direction of travel of the wave.

$$I = \frac{P}{A} \qquad ...(i)$$

In this case, the intensity is therefore $I = \frac{1}{2}\rho v(\omega A)^2$

Hence, the intensity of a periodic sound wave is proportional to the square of the displacement amplitude and to the square of the angular frequency. For sound wave,

$$\Delta P_m = ABk$$

$$\therefore \quad A = \frac{\Delta P_m}{Bk}$$

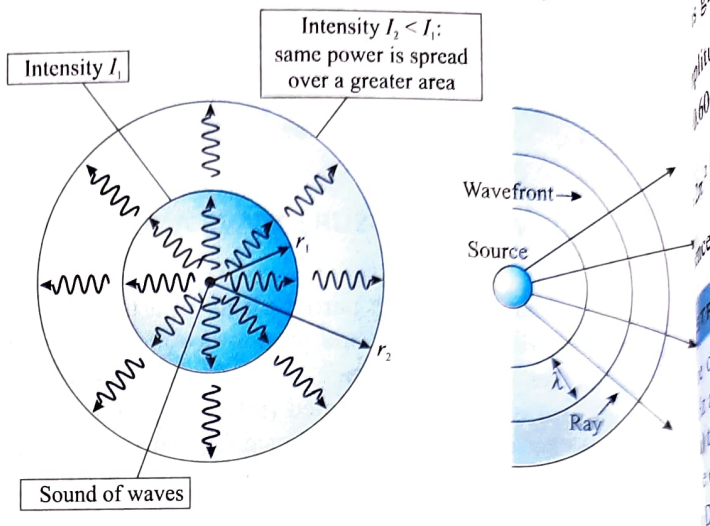
Substituting this value in Eq. (i), we get

$$I = \frac{1}{2}\rho v\omega^2\left(\frac{\Delta P_m}{Bk}\right)^2 = \frac{1}{2}\rho v\omega^2\frac{\Delta P_m^2}{B^2k^2}$$

As $k = \omega/v$ and $B = v^2\rho$,

$$\therefore \quad I = \frac{1}{2}\rho v\omega^2\frac{\Delta P_m^2}{B^2\frac{\omega^2}{v^2}} = \frac{v\Delta P_m^2}{2B} = \frac{\Delta P_m^2}{2\rho v}$$

Now consider a point source emitting sound wave equally in all directions. From day-to-day experience, we know that the intensity of sound decreases as we move farther from the source. When a source emits sound equally in all directions, the result is a spherical wave. Figure shows these spherical waves as a series of circular arcs concentric with the source. Ea arc represents a surface over which the phase of the wave constant. We call such a surface of constant phase a wavefro The distance between adjacent wavefronts that have the san phase is called the wavelength λ of the wave. The radial lin pointing outward from the source are called rays.



The average power P_{avg} emitted by the source must be distributed uniformly over each spherical wavefront of area $4\pi r^2$. Hence, the wave intensity at a distance r from the source is

$$I = \frac{P_{avg}}{A} = \frac{P_{avg}}{4\pi r^2} \qquad ...(ii)$$

The inverse square law, which is reminiscent of the behaviour of gravity, states that the intensity decreases in proportion to the square of the distance from the source.

ILLUSTRATION 8.11

If the area of a typical eardrum is about 5.00×10^{-5} m^2. **(a)** Find the average sound power incident on an eardrum at the threshold of pain, which corresponds to an intensity of 1.00 W/m^2. **(b)** How much energy is transferred to the eardrum exposed to this sound for 1.00 min?

Sol. The sound power incident on the eardrum is $P = IA$, where I is the intensity of the sound and $A = 5.00\times10^{-5}$ m^2 is the area of the eardrum.

(a) At the threshold of pain, $I = 1.00$ W/m^2.

Thus, $P = IA = (5.00\times10^{-5}\text{ m}^2)(1.00\text{ W/m}^2) = 5.00\times10^{-5}$ W

(b) Energy transfer can be obtained from power by

$$P = \frac{E}{\Delta t} \Rightarrow E = P\Delta t.$$

Thus, $E = P\Delta t = (5.00\times10^{-5}\text{ J/s})(6.00\text{ s}) = 3.00\times10^{-3}$ J

ILLUSTRATION 8.12

The intensity of a sound wave at a fixed distance from a speaker vibrating at 1.00 kHz is 0.60 W/m^2. **(a)** Determine the intensity that results if the frequency is increased to 2.50 kHz while a constant displacement amplitude is maintained. **(b)** Calculate the intensity if the frequency is reduced to 0.500 kHz and the displacement amplitude is doubled.

The intensity of sound wave is given by, $I = 2\pi^2 f^2 A^2 \rho v$

(a) For constant A, $I \propto f^2$

At $f = 2500$ Hz, the frequency is increased by a factor of 2.50, so the intensity increases by $(2.50)^2 = 6.25$.

Therefore, final intensity $I_{final} = 6.25(0.60) = 3.75$ W/m^2

(b) It is given new frequency $f_{new} = \frac{f}{2}$ and displacement amplitude $A_{new} = 2A$ original intensity $I = 2\pi^2 f^2 A^2 \rho v = 0.60$ W/m^2 new intensity: $I_{new} = 2\pi^2 f_{new}^2 A_{new}^2 \rho v$

$= 2\pi^2 \left(\frac{f}{2}\right)^2 (2A)^2 \rho v = 2\pi^2 f^2 A^2 \rho v$

Hence new intensity, $I_{new} = 0.60$ W/m^2

ILLUSTRATION 8.13

In one observation, the sound wave had a frequency of 1665 Hz and a pressure amplitude of 1.13×10^{-3} Pa. What were **(a)** the displacement amplitude and **(b)** the intensity of the wave emitted by the ear?

(Given: Density of air = $\rho_{air} = 1.21 \text{kg/m}^3$)

Sol.

(a) The relation between pressure amplitude and displacement amplitude is given as,

$$\Delta p_{max} = BAk = (v^2 \rho) A \left(\frac{\omega}{v}\right) = v^2 \rho A \left(\frac{2\pi f}{v}\right)$$

Hence displacement amplitude, $A = \frac{\Delta p_{max}}{2\pi f \rho v}$

$$\Rightarrow \quad A = \frac{1.13 \times 10^{-3} \text{ Pa}}{2\pi (1665 \text{Hz})(1.21 \text{kg/m}^3)(343 \text{m/s})}$$

$$= 0.26 \times 10^{-9} \text{ m} = 0.26 \text{ nm}$$

(b) The intensity of the wave in terms of the pressure amplitude:

$$I = \frac{(\Delta p_m)^2}{2\rho v} = \frac{(1.13 \times 10^{-3} \text{ Pa})^2}{2(1.21 \text{kg/m}^3)(343 \text{m/s})} = 1.5 \text{nW/m}^2.$$

ILLUSTRATION 8.14

A certain loudspeaker system emits sound isotropically with a frequency of 2000 Hz and an intensity of 1.0 mW/m^2 at a distance of 6.0 m. Assume that there are no reflections, (a) What is the intensity at 30.0 m? At 6.10 m, what are (b) the displacement amplitude and (c) the pressure amplitude?

Given: Density of air, $\rho = 1.21$ kg/m^3

Sol. The source being isotropic then, $\frac{I_2}{I_1} = \frac{P/4\pi r_2^2}{P/4\pi r_1^2} = \left(\frac{r_1}{r_2}\right)^2$.

(a) With $I_1 = 1.0 \times 10^{-4}$ W/m^2, $r_1 = 6.0$ m and $r_2 = 30.0$ m, we find

$$I_2 = (1.0 \times 10^{-4} \text{ W/m}^2)\left(\frac{6.0}{30.0}\right)^2 = 4.0 \times 10^{-6} \text{ W/m}^2$$

(b) The intensity of sound wave is given by,

$$I = \frac{1}{2}\omega^2 A^2 \rho v \Rightarrow A = \sqrt{\frac{2I}{\rho v \omega^2}}$$

Substituting $I_1 = 1.0 \times 10^{-4}$ W/m^2, $\omega = 2\pi(2000 \text{ Hz})$ $v = 343$ m/s, and $\rho = 1.21$ kg/m^3,

we obtain, $A = \sqrt{\frac{2I}{\rho v \omega^2}} = 1.71 \times 10^{-7}$ m.

(c) The relation between pressure amplitude and displacement amplitude is given as,

$$\Delta p_{max} = BAk = (v^2 \rho) A \left(\frac{\omega}{v}\right) = \rho v (2\pi f) A$$

Substituting $\omega = 2\pi(2000 \text{ Hz})$, $v = 343$ m/s, and $\rho = 1.21$ kg/m^3,

we obtain, $\Delta p_{max} \approx 0.9$ Pa

ILLUSTRATION 8.15

A point sound source is situated in a medium of bulk modulus 1.6×10^5 N/m^2. An observer standing at a distance 10 m from the source writes down the equation for the wave as $y = A\sin(15\pi x - 6000\pi t)$. Here y and x are in metres and t is in second. The maximum pressure amplitude received to the observer's ear is 24 π Pa, then find.

(a) the density of the medium,

(b) the displacement amplitude A of the waves received by the observer and

(c) the power of the sound source.

Sol.

(a) Comparing with equation of a travelling wave,

$$y = a\sin(kx - \omega t)$$

We get, $k = 15\pi$ and $\omega = 6000\pi$

$\therefore$ Velocity of the sound, $v = \frac{\omega}{k} = \frac{6000}{15\pi} = 400 \text{ ms}^{-1}$

As $v = \sqrt{\frac{B}{\rho}}$, hence, $\rho = \frac{B}{v^2} = \frac{1.6 \times 10^5}{(400)^2} = 1 \text{kg/m}^3$

(b) Pressure amplitude, $\Delta p_{max} = BAk$

Hence, $A = \frac{\Delta p_{max}}{Bk} = \frac{24\pi}{1.6 \times 10^5 \times 15\pi} = 10^{-5} \text{ m} = 10\,\mu\text{m}$

(c) Intensity received by the person, $I = \frac{W}{4\pi R^2} = \frac{W}{4\pi(10)^2}$

Also, $I = \frac{(\Delta p_{max})^2}{2\rho v} \Rightarrow \frac{W}{4\pi(10)^2} = \frac{(24\pi)^2}{2 \times 1 \times 400} = 288\pi^3$ W

ILLUSTRATION 8.16

(a) If two sound waves, one in air and one in (fresh) water, are equal in intensity and angular frequency, what is the ratio of the pressure amplitude of the wave in water to that of the wave in air? Assume the water and the air are at 20°C.

(b) If the pressure amplitudes are equal instead, what is the ratio of the intensities of the waves?

(Given: Density of air $\rho_{air} = 1.20 \text{kg/m}^3$ Density of water $\rho_{water} = 1000 \text{kg/m}^3$, velocity of sound in air $v_{air} = 350$ m/s and velocity of sound in water $v_{water} = 1512$ m/s **)**

Similarly, density equation is given by

$$(\Delta p) = \frac{\rho}{B}(\Delta p) = \frac{\rho}{B}\left(-B\frac{\partial y}{\partial x}\right) = -\rho\left(\frac{\partial y}{\partial x}\right)$$

Substituting the values of $\frac{\partial y}{\partial x}$ from Eq. (i), we have

$$\Delta\rho = -\rho Ak\cos(kx+\omega t)$$

ILLUSTRATION 8.10

If the maximum tolerable pressure variation in sound is 28 Pa at 1000 Hz. Find the amplitude of density and displacement. Given: Bulk modulus of medium $B = 1.4\times10^5$ Pa, density of the medium $\rho = 1.20$ kg/m^3 and speed of sound = 341 m/s

Sol. The density amplitude is

$$\Delta\rho_m = \frac{\rho}{B}\Delta p_m = \frac{1.20}{1.4\times10^5}28 = 2.4\times10^{-4}\ kg/m^3$$

The displacement amplitude is $A = \frac{\Delta p_m}{KB}$ where $K = \frac{2\pi}{\lambda} = \frac{2\pi f}{v}$

or $$A = \frac{v}{2\pi f}\cdot\frac{\Delta p_m}{B} = \frac{341}{2\times(22/7)\times1000}\times\frac{28}{1.4\times10^5}$$

$$= 1.1\times10^{-5}\text{ m}$$

INTENSITY OF PERIODIC SOUND WAVES

We define the intensity I of a wave, or the power per unit area, as the rate at which the energy transported by the wave transfers through a unit area A perpendicular to the direction of travel of the wave.

$$I = \frac{p}{A} \qquad ...(i)$$

In this case, the intensity is therefore $I = \frac{1}{2}\rho v(\omega A)^2$

Hence, the intensity of a periodic sound wave is proportional to the square of the displacement amplitude and to the square of the angular frequency. For sound wave,

$$\Delta P_m = ABk$$

$$\therefore\quad A = \frac{\Delta P_m}{Bk}$$

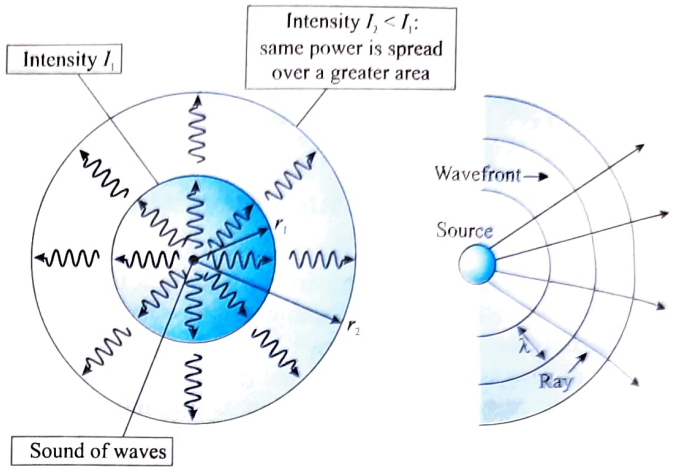
Substituting this value in Eq. (i), we get

$$I = \frac{1}{2}\rho v\omega^2\left(\frac{\Delta P_m}{Bk}\right)^2 = \frac{1}{2}\rho v\omega^2\frac{\Delta P_m^2}{B^2k^2}$$

As $k = \omega/v$ and $B = v^2\rho$,

$$\therefore\quad I = \frac{1}{2}\rho v\omega^2\frac{\Delta P_m^2}{B^2\frac{\omega^2}{v^2}} = \frac{v\Delta P_m^2}{2B} = \frac{\Delta P_m^2}{2\rho v}$$

Now consider a point source emitting sound wave equally in all directions. From day-to-day experience, we know that the intensity of sound decreases as we move farther from the source. When a source emits sound equally in all directions, the result is a spherical wave. Figure shows these spherical waves as a series of circular arcs concentric with the source. Each arc represents a surface over which the phase of the wave is constant. We call such a surface of constant phase a wavefront. The distance between adjacent wavefronts that have the same phase is called the wavelength λ of the wave. The radial lines pointing outward from the source are called rays.



The average power P_{avg} emitted by the source must be distributed uniformly over each spherical wavefront of area $4\pi r^2$. Hence, the wave intensity at a distance r from the source is

$$I = \frac{P_{avg}}{A} = \frac{P_{avg}}{4\pi r^2} \qquad ...(ii)$$

The inverse square law, which is reminiscent of the behaviour of gravity, states that the intensity decreases in proportion to the square of the distance from the source.

ILLUSTRATION 8.11

If the area of a typical eardrum is about 5.00×10^{-5} m^2. **(a)** Find the average sound power incident on an eardrum at the threshold of pain, which corresponds to an intensity of 1.00 W/m^2. **(b)** How much energy is transferred to the eardrum exposed to this sound for 1.00 min?

Sol. The sound power incident on the eardrum is $P = IA$, where I is the intensity of the sound and $A = 5.00\times10^{-5}$ m^2 is the area of the eardrum.

(a) At the threshold of pain, $I = 1.00$ W/m^2.

Thus, $P = IA = (5.00\times10^{-5}\text{ m}^2)(1.00\text{ W/m}^2) = 5.00\times10^{-5}$ W

(b) Energy transfer can be obtained from power by

$$P = \frac{E}{\Delta t} \Rightarrow E = P\Delta t.$$

Thus, $E = P\Delta t = (5.00\times10^{-5}\text{ J/s})(6.00\text{ s}) = 3.00\times10^{-3}$ J

ILLUSTRATION 8.12

The intensity of a sound wave at a fixed distance from a speaker vibrating at 1.00 kHz is 0.60 W/m^2. **(a)** Determine the intensity that results if the frequency is increased to 2.50 kHz while a constant displacement amplitude is maintained. **(b)** Calculate the intensity if the frequency is reduced to 0.500 kHz and the displacement amplitude is doubled.

Sol. The intensity of sound wave is given by, $I = 2\pi^2 f^2 A^2 \rho v$

(a) For constant A, $I \propto f^2$

At $f = 2500$ Hz, the frequency is increased by a factor of 2.50, so the intensity increases by $(2.50)^2 = 6.25$.

Therefore, final intensity $I_{final} = 6.25(0.60) = 3.75\ W/m^2$

(b) It is given new frequency $f_{new} = \frac{f}{2}$ and displacement amplitude $A_{new} = 2A$ original intensity $I = 2\pi^2 f^2 A^2 \rho v$ $= 0.60\ W/m^2$ new intensity: $I_{new} = 2\pi^2 f_{new}^2 A_{new}^2 \rho v$

$$= 2\pi^2 \left(\frac{f}{2}\right)^2 (2A)^2 \rho v = 2\pi^2 f^2 A^2 \rho v$$

Hence new intensity, $I_{new} = 0.60\ W/m^2$

ILLUSTRATION 8.13

In one observation, the sound wave had a frequency of 1665 Hz and a pressure amplitude of 1.13×10^{-3} Pa. What were **(a)** the displacement amplitude and **(b)** the intensity of the wave emitted by the ear?

(Given: Density of air = $\rho_{air} = 1.21 kg/m^3$)

Sol.

(a) The relation between pressure amplitude and displacement amplitude is given as,

$$\Delta p_{max} = BAk = (v^2\rho)A\left(\frac{\omega}{v}\right) = v^2 \rho A\left(\frac{2\pi f}{v}\right)$$

Hence displacement amplitude, $A = \frac{\Delta p_{max}}{2\pi f \rho v}$

$$\Rightarrow \quad A = \frac{1.13\times10^{-3}\ Pa}{2\pi(1665\ Hz)(1.21 kg/m^3)(343 m/s)}$$

$$= 0.26\times10^{-9} m = 0.26\ nm$$

(b) The intensity of the wave in terms of the pressure amplitude:

$$I = \frac{(\Delta p_m)^2}{2\rho v} = \frac{(1.13\times10^{-3}\ Pa)^2}{2(1.21 kg/m^3)(343 m/s)} = 1.5\ nW/m^2.$$

ILLUSTRATION 8.14

A certain loudspeaker system emits sound isotropically with a frequency of 2000 Hz and an intensity of 1.0 mW/m^2 at a distance of 6.0 m. Assume that there are no reflections, **(a)** What is the intensity at 30.0 m? At 6.10 m, what are **(b)** the displacement amplitude and **(c)** the pressure amplitude?

Given: Density of air, $\rho = 1.21\ kg/m^3$

Sol. The source being isotropic then, $\frac{I_2}{I_1} = \frac{P/4\pi r_2^2}{P/4\pi r_1^2} = \left(\frac{r_1}{r_2}\right)^2$.

(a) With $I_1 = 1.0\times10^{-4}\ W/m^2$, $r_1 = 6.0 m$ and $r_2 = 30.0$ m, we find

$$I_2 = (1.0\times10^{-4}\ W/m^2)\left(\frac{6.0}{30.0}\right)^2 = 4.0\times10^{-5}\ W/m^2$$

(b) The intensity of sound wave is given by,

$$I = \frac{1}{2}\omega^2 A^2 \rho v \Rightarrow A = \sqrt{\frac{2I}{\rho v \omega^2}}$$

Substituting $I_1 = 1.0\times10^{-4}\ W/m^2$, $\omega = 2\pi(2000\ Hz)$ $v = 343$ m/s, and $\rho = 1.21\ kg/m^3$,

we obtain, $A = \sqrt{\frac{2I}{\rho v \omega^2}} = 1.71\times10^{-7} m$.

(c) The relation between pressure amplitude and displacement amplitude is given as,

$$\Delta p_{max} = BAk = (v^2\rho)A\left(\frac{\omega}{v}\right) = \rho v(2\pi f)A$$

Substituting $\omega = 2\pi(2000\ Hz)$, $v = 343$ m/s, and $\rho = 1.21\ kg/m^3$,

we obtain, $\Delta p_{max} \approx 0.9\ Pa$

ILLUSTRATION 8.15

A point sound source is situated in a medium of bulk modulus $1.6 \times 10^5\ N/m^2$. An observer standing at a distance 10 m from the source writes down the equation for the wave as $y = A\sin(15\pi x - 6000\pi t)$. Here y and x are in metres and t is in second. The maximum pressure amplitude received to the observer's ear is 24 π Pa, then find.

(a) the density of the medium,

(b) the displacement amplitude A of the waves received by the observer and

(c) the power of the sound source.

Sol.

(a) Comparing with equation of a travelling wave,

$$y = a\sin(kx - \omega t)$$

We get, $k = 15\pi$ and $\omega = 6000\pi$

$\therefore$ Velocity of the sound, $v = \frac{\omega}{k} = \frac{6000}{15\pi} = 400\ ms^{-1}$

As $v = \sqrt{\frac{B}{\rho}}$, hence, $\rho = \frac{B}{v^2} = \frac{1.6\times10^5}{(400)^2} = 1 kg/m^3$

(b) Pressure amplitude, $\Delta p_{max} = BAk$

Hence, $A = \frac{\Delta p_{max}}{Bk} = \frac{24\pi}{1.6\times10^5 \times 15\pi} = 10^{-5} m = 10\ \mu m$

(c) Intensity received by the person, $I = \frac{W}{4\pi R^2} = \frac{W}{4\pi(10)^2}$

Also, $I = \frac{(\Delta p_{max})^2}{2\rho v} \Rightarrow \frac{W}{4\pi(10)^2} = \frac{(24\pi)^2}{2\times1\times400} = 288\pi^3\ W$

ILLUSTRATION 8.16

(a) If two sound waves, one in air and one in (fresh) water, are equal in intensity and angular frequency, what is the ratio of the pressure amplitude of the wave in water to that of the wave in air? Assume the water and the air are at 20°C.

(b) If the pressure amplitudes are equal instead, what is the ratio of the intensities of the waves?

(Given: Density of air $\rho_{air} = 1.20 kg/m^3$ Density of water $\rho_{water} = 1000 kg/m^3$, velocity of sound in air $v_{air} = 350$ m/s and velocity of sound in water $v_{water} = 1512$ m/s)

Sol.

(a) The intensity of sound wave is given by, $I = \frac{1}{2}\omega^2 A^2 \rho v$, where ρ is the density of the medium, v is the speed of sound, ω is the angular frequency, and A is the displacement amplitude.

The displacement and pressure amplitudes are related by

$$\Delta p_{max} = BAk = (v^2\rho)A\left(\frac{\omega}{v}\right) = v\rho A\omega$$

The intensity of the wave in terms of the pressure amplitude:

$$I = \frac{(\Delta p_m)^2}{2\rho v}$$

For waves of the same frequency, the ratio of the intensity for propagation in water to the intensity for propagation in air is $\frac{I_w}{I_a} = \left(\frac{\Delta p_{mw}}{\Delta p_{ma}}\right)^2 \frac{\rho_a v_a}{\rho_w v_w}$, where the subscript a denotes air and the subscript w denotes water.

Since $I_a = I_w$, $\frac{\Delta p_{mw}}{\Delta p_{ma}} = \sqrt{\frac{\rho_w v_w}{\rho_a v_a}}$

$$= \sqrt{\frac{(1000\,\text{kg/m}^3)(1512\,\text{m/s})}{(1.20\,\text{kg/m}^3)(350\,\text{m/s})}} = 60.$$

(b) If the pressure amplitudes are equal Now,

$$\Delta p_{mw} = \Delta p_{ma} \Rightarrow I \propto \frac{1}{\rho v}$$

$$\Rightarrow \quad \frac{I_w}{I_a} = \frac{\rho_a v_a}{\rho_w v_w} = \frac{1.20\times 350}{1000\times 1512} = \frac{1}{3600}.$$

FREQUENCY AND PITCH OF SOUND WAVES

Each cycle of a sound wave includes one compression and one rarefaction, and frequency is the number of cycles per second that pass through a given location. This is normally equal to the frequency of vibration of the (tuning fork) source producing sound. If the source vibrates in SHM of a single frequency, sound produced has a single frequency and is called a pure tone.

However, a sound source may not always vibrate in SHM (this is the case with most of the common sound sources, e.g. guitar string, human vocal card, surface of drum, etc.), and hence the pulse generated by it may not have the shape of a sine wave. But even such a pulse can be obtained by superposition of a large number of sine waves of different frequency and amplitudes. (This is possible for a pulse of any shape, according to an important theorem of mathematics, the Fourier theorem, the description of which is outside the scope of this book.) We say that the pulse contains all these frequencies.

A normal person hears all frequencies between 20 Hz and 20 kHz. This is a subjective range (obtained experimentally) which may vary slightly from person to person. The ability to hear the high frequencies decreases with age and a middle-age person can hear only up to 12 Hz to 14 kHz.

Sound can be generated with frequency below 20 Hz called infrasonic sound and above 20 kHz called ultrasonic sound. Even through human cannot hear these frequencies, other animals may For instance, rhinos communicate through infrasonic frequencies as low as 5 Hz, and bats use ultrasonic frequencies as high as 10(kHz for navigating.

Frequency as we have discussed till now is an objective property measured in units of Hz and which can be assigned a unique value. However, a person's perception of frequency is subjective. The brain interprets frequency primarily in terms of a subjective quality called pitch. A pure note of high frequency is interpreted as high-pitched sound and a pure note of low frequency as low-pitched sound.

MEASUREMENTS OF SOUND LEVELS

The intensity of a wave was defined as the power per unit area. We saw that for spherical waves the intensity falls as the second power of the distance to the source, $I \propto r^{-2}$, giving the ratio

$$\frac{I_1}{I_2} = \left(\frac{r_2}{r_1}\right)^2$$

This relationship also holds for sound waves. Since the intensity is the power per unit area, its physical units are watts per square meter (W/m^2).

The human ear is an extremely sensitive detector, capable of hearing sounds over an extremely large range of intensities. Sound waves that can be detected by the human ear have a very large range of intensities, from whispers as low as 10^{-12} W/m^2 to the output of a jet engine or a rock band at close distance, which can reach 1 W/m^2. The oscillations in pressure for even the loudest sounds, at the pain threshold of 10 W/m^2, are on the order of only tens of micro pascals (μPa). Normal atmospheric air pressure, by comparison, is 10^5 Pa. Thus, you can see that the air pressure varies by only 1 part in 10 billion, even for the loudest sounds. And the variation is several orders of magnitude less for the quietest sounds you can hear. This might give you a new appreciation for the capabilities of your ears.

It is been observed that the people perceive a sound to be about twice as loud as a reference sound when its intensity is ten times as large as the reference sound. A sound perceived to be four times as loud as a reference requires an increase in sound intensity by a factor of 100. This relationship is approximately logarithmic, or we can state that loudness is proportional to the logarithm of the sound intensity. Since human ears can register sounds over many orders of magnitude of intensity, a logarithmic scale is used to measure sound intensities. The unit of this scale is the bel (B), named for Alexander Graham Bell, but much more commonly used is the decibel (dB): 1 dB = 0.1 bel. The sound level measured on this decibel scale and is defined as

$$L(in\,dB) = 10\log_{10}\frac{I}{I_0}$$

The notation "log" refers to the base-10 logarithm. The standard reference level is taken $I_0 = 10^{-12}$ W/m^2, which is the sound intensity that can just be heard by a person with good hearing. This intensity 10^{-12} W/m^2 is called the threshold of hearing.

RELATIVE INTENSITY AND DYNAMIC RANGE

The relative sound level, ΔL, is the difference between two sound levels:

$$\Delta L(\text{in dB}) = L_2 - L_1 = 10\log_{10}\frac{I_2}{I_0} - 10\log_{10}\frac{I_1}{I_0}$$

$$= 10(\log_{10} I_2 - \log_{10} I_0) - 10(\log_{10} I_1 - \log_{10} I_0)$$

$$= 10(\log_{10} I_2 - \log_{10} I_1)$$

$$\Delta L(\text{in dB}) = 10\log_{10}\frac{I_2}{I_1}$$

This dynamic range is a measure of the relative sound levels of the loudest and the quietest. Table shows the approximate sound levels of some of the sounds commonly encountered.

Sound levels	
Minimum audible sound	≈ 0 dB
Whispering (at 1 m)	10 dB
Normal talk (at 1 m)	60 dB
Maximum tolerable sound	120 dB

ILLUSTRATION 8.17

Two loud speakers are being compared, and one is perceived to be 32 times louder than the other. what will be the difference in intensity levels between the two when measured in decibels?

Sol. In general for a change from I_1 to I_2, $\Delta L = 10\log_{10}\left(\frac{I_2}{I_1}\right)$ dB

Increasing the intensity by a factor of 10 changes the sound level by 1 bel or 10 dB; increasing it by 100 changes β by 2 bels or 20 dB. A doubling in loudness changes intensity by 10 dB. Since $32 = 2^5$, $\Delta\beta = 5(10\,\text{dB}) = 50\,\text{dB}$

ILLUSTRATION 8.18

Consider that 10 identical sitars are playing a note at 70 dB. If they are in a chorus, what will be the sound level as each contributes?

Sol. Let's imagine that sitars join one by one in chorus, thus the intensity goes from I to 2I to 3I upto 10 I. For 2 sitars,

$$10\log_{10}\frac{2I}{I_0} = 10\log_{10} 2 + 10\log_{10}\frac{I}{I_0} = 3\,\text{dB} + 70\,\text{dB}$$

Doubling the intensity produces a 3-dB sound level increase, with 3 sitars playing, the sound level rises to

$$10\log_{10}\frac{3I}{I_0} = 10\log_{10} 3 + 10\log_{10}\frac{I}{I_0} = 75\,\text{dB}$$

For four sitars the sound level rises to 76 dB, and so one when it reaches to 80 dB for 10 sitars.

ILLUSTRATION 8.19

As the number of people at a party increases, you must raise your voice for a listener to hear you against the background noise of the other partygoers. However, once you reach the level of yelling, the only way you can be heard is if you move closer to your listener, into the listener's "personal space." Model the situation by replacing you with an isotropic point source of fixed power P and replacing your listener with a point that absorbs part of your sound waves. These points are initially separated by $r_i = 1.20$ m. If the background noise increases by $\Delta L = 5$ dB, the sound level at your listener must also increase. What separation r_f is then required?

Sol. The difference in sound level is given by,

$$\Delta L = L_f - L_i = (10\,\text{dB})\log\left(\frac{I_f}{I_i}\right).$$

Thus, if $\Delta L = 5.0$ dB, then $\log\left(\frac{I_f}{I_i}\right) = \frac{1}{2} \Rightarrow I_f = \sqrt{10} I_i$

On the other hand, the intensity at a distance r from the source is $I = \frac{P}{4\pi r^2}$, where P is the power of the source. A fixed P implies that $I_i r_i^2 = I_f r_f^2$. Thus, with $r_i = 1.2$ m, we obtain

$$r_f = \left(\frac{I_i}{I_f}\right)^{1/2} r_i = \left(\frac{1}{10}\right)^{1/4}(1.2\,\text{m}) = (0.67)\,\text{m}.$$

ILLUSTRATION 8.20

A train sounds its horn as it approaches an intersection. The horn can just be heard at a level of 50 dB by an observer 10 km away. **(a)** What is the average power generated by the horn? **(b)** What intensity level of the horn's sound is observed by someone waiting at an intersection 50 m from the train? Treat the horn as a point source and neglect any absorption of sound by the air.

Sol.

(a) As, $L = (10\,\text{dB})\log_{10}\frac{I}{I_0} \Rightarrow \log_{10}\frac{I}{I_0} = \frac{L}{10}$

or $\frac{I}{I_0} = 10^{\frac{L}{10}} \Rightarrow I = I_0 10^{\frac{L}{10}}$

The horn can just be heard at a level of 50 dB by an observer 10 km away.

$$\Rightarrow \quad I = I_0 10^{\frac{L}{10}} = 1.0\times10^{-12}\times10^{5.0} = 1.0\times10^{-7}\ \text{W/m}^2$$

Thus, from $I = \frac{P}{4\pi r^2}$, the power emitted by the source is

$$P = 4\pi r^2 I = 4\pi(10.0\times10^3)^2(1.0\times10^{-7}) = 126\,\text{W}$$

(b) At $r = 50$ m, the intensity of the sound will be

$$I = \frac{P}{4\pi r^2} = \frac{1.3\times10^2}{4\pi(50)^2} = 4.0\times10^{-3}\ \text{W/m}^2$$

and the sound level is, $L = 10\log\left(\frac{I}{I_0}\right)$

$$= (10)\log_{10}\left(\frac{4.0\times10^{-3}}{1.0\times10^{-12}}\right) = 96\,\text{dB}$$

ILLUSTRATION 8.21

Find the ratios (greater to smaller) of the **(a)** intensities, **(b)** pressure amplitudes, and particle displacement amplitudes for two sounds whose sound levels differ by 40 dB.

Sol. The difference in sound level is given by,

$$\Delta L = L_f - L_i = (10\,\text{dB})\log\left(\frac{I_f}{I_i}\right).$$

(a) As, $\Delta L = (10\,\text{db})\log\left(\frac{I_f}{I_i}\right) = 40\,\text{dB}$

we get, $\left(\frac{I_f}{I_i}\right) = 10^{\frac{40}{10}} = 10^4$

(b) Since $I = \frac{(\Delta p_m)^2}{2\rho v} \Rightarrow \Delta p_m \propto \sqrt{I}$ and

$\Delta p_m = BAk \Rightarrow \Delta p_m \propto A$

It means $\Delta p_m \propto A \propto \sqrt{I}$

we have $\frac{\Delta p_{m_1}}{\Delta p_{m_2}} = \sqrt{\frac{I_1}{I_2}} = \frac{A_1}{A_2} = \sqrt{10^{-4}} = 10^{-2}$

ILLUSTRATION 8.22

Calculate the sound level (in decibels) of a sound wave that has an intensity of 4.00×10^{-6} W/m^2.

Sol. We expect about 60 dB. We use the definition of the decibel scale. We use the equation $\beta = 10$ dB log (I/I_0)

where $I_0 = 10^{-12}$ W/m^2

$$\therefore \quad \beta = 10 \log\left(\frac{4.00\times10^{-6}\ \text{W/m}^2}{10^{-12}\ \text{W/m}^2}\right) = 66.0\ \text{dB}$$

This could be the sound level of a chamber music concert, heard from a few rows back in the audience.

ILLUSTRATION 8.23

A family ice show is held at an enclosed arena. The skaters perform to music with level 80.0 dB. This level is too loud for your baby, who yells at 75.0 dB. **(a)** What total sound intensity engulfs you? **(b)** What is the combined sound level?

Sol. Resist the temptation to say 155 dB. The decibel scale is logarithmic, so addition does not work that way. We must figure out the intensity of each sound, add the intensities, and then translate back to a level. We have,

$$\beta = 10 \log\left(\frac{I}{10^{-12}\ \text{W/m}^2}\right)$$

$$\therefore \quad I = \left[10^{\beta/10}\right]10^{-12}\ \text{W/m}^2$$

(a) For your baby, $I_b = (10^{75.0/10})\,(10^{-12}\ \text{W/m}^2)$

$$= \sqrt{10} \times 10^{-5}\ \text{W/m}^2$$

For the music, $I_m = (10^{80.0/10})\,(10^{-12}\ \text{W/m}^2) = 10.0 \times 10^{-5}\ \text{W/m}^2$

The combined intensity is

$$I_{\text{total}} = I_m + I_b = 10.0 \times 10^{-5}\ \text{W/m}^2 + 3.16 \times 10^{-5}\ \text{W/m}^2$$
$$= 13.2 \times 10^{-5}\ \text{W/m}^2$$

(b) The combined sound level is then

$$\beta_{\text{total}} = 10 \log\left(\frac{I_{\text{total}}}{10^{-12}\ \text{W/m}^2}\right)$$

$$= 10 \log\left(\frac{1.32\times10^{-4}\ \text{W/m}^2}{10^{-12}\ \text{W/m}^2}\right) = 81.2\ \text{dB}$$

The result is only a little louder than 80 dB.

ILLUSTRATION 8.24

A firework charge is detonated many metres above the ground. At a distance of 400 m from the explosion, the acoustic pressure reaches a maximum of 10.0 N/m^2. Assume that the speed of sound is constant at 343 m/s throughout the atmosphere over the region considered, the ground absorbs all the sound falling on it, and the air absorbs sound energy at the rate of 7.00 dB/km. What is the sound level (in decibels) at 4.00 km from the explosion?

Sol. At a distance of 4 km, an explosion should be audible, but probably not extremely loud. So based on the data given, we might expect the sound level to be somewhere between 40 and 80 dB. From the sound pressure data given in the problem, we can find the intensity, which is used to find the sound level in dB. The sound intensity will decrease with increased distance from the source and from the absorption of the sound by the air.

At a distance of 400 m from the explosion, $\Delta P_{\text{max}} = 10.0$ Pa. At this point the intensity is

$$I = \frac{\Delta P_{\text{max}}^2}{2\rho v} = \frac{(10.0\ \text{N/m})^2}{2(1.20\ \text{kg/m}^3)(343\ \text{m/s})} = 0.121\ \text{W/m}^2$$

From the inverse square law, we can calculate the intensity and decibel level (due to distance alone) 4 km away

$$I' = I(400\ \text{m})^2/(4000\ \text{m})^2 = 1.21 \times 10^{-3}\ \text{W/m}^2$$

and $\beta = 10 \log\left(\frac{I'}{I_0}\right) = 10 \log\left(\frac{1.21\times10^{-3}\ \text{W/m}^2}{1.00\times10^{-12}\ \text{W/m}^2}\right) = 90.8\ \text{dB}$

At a distance of 4 km from the explosion, absorption from the air will decrease the sound level by an additional amount,

$$\Delta\beta = (7.00\ \text{dB/km})\,(3.60\ \text{km}) = 25.2\ \text{dB}$$

So at 4 km, the sound level will be

$$\beta_f = \beta - \Delta\beta = 90.8\ \text{dB} - 25.2\ \text{dB} = 65.6\ \text{dB}$$

This sound level falls within our expected range. Evidently, this explosion is rather loud (about the same as a vacuum cleaner) even at a distance of 4 km from the source. It is interesting to note that the distance and absorption effect each reduces the sound level by about the same amount (~25 dB). If the explosion were at ground level, the sound level would be further reduced by reflection and absorption form obstacles between the source and observer, and the calculation could be much more complicated.

CONCEPT APPLICATION EXERCISE 8.2

1. If the intensity is increased by a factor of 20, by how many decibels is the sound level increased?
2. A person wears a hearing aid that uniformly increases the sound level of all audible frequencies of sound by 30.0 dB. The hearing aid picks up sound having a frequency of 250 Hz at an intensity of 3.0×10^{-11} W/m^2. What is the intensity delivered to the eardrum?
3. The explosion of a fire cracker in the air at a height of 40 m produces a 100 dB sound level at ground below. What is the instantaneous total radiated power? Assuming that it radiates as a point source.
4. For a person with normal hearing, the faintest sound that can be heard at a frequency of 400 Hz has a pressure amplitude of about $\sqrt{33} \times 10^{-5}$ Pa. Calculate the corresponding intensity and sound intensity level at 20° C. (Given: $v = 330$ m s and $\rho_{air} = 1.25$ kg/m^3 and $\log_{10} 2 = 0.3$)
5. A point A is located at a distance $r = 1.5$ m from a point source of sound of frequency 500 Hz. The power of the source is 0.9 W. Speed of sound in air is 360 m/s and density of air is $\pi/2$ kg/m^3. Find at the point A,
   (a) the pressure oscillation amplitude $(\Delta p)_m$.
   (b) the displacement oscillation amplitude A.
6. Two sound waves from two different sources interfere at a point to yield a sound of varying intensity. The intensity level between the maximum and minimum is 20 dB. What is the ratio of the intensities of the individual waves?
7. A sound differs by 6 dB from a sound of intensity equal to 10 nW/cm^2. Find the absolute value of intensity of the sound.
8. If the sound level in a room is increased from 50 dB to 60 dB, by which factor is the pressure amplitude increased?
9. A window whose area is 2 m^2 opens on street where the street noise result in an intensity level at the window of 60 dB. How much 'acoustic power' enters the window via sound waves. Now if an acoustic absorber is fitted at the window, how much energy from street will it collect in 5 h?
10. **(a)** The power of sound from the speaker of a radio is 20 mW. By turning the knob of volume control the power of sound is increased to 400 mW. What is the power increase in dB as compared to original power? **(b)** How much more intense is an 80 dB sound than a 20 dB whisper?
11. What is the maximum possible sound level in dB of sound waves in air? Given that density of air is 1.3 kg/m^3, $v = 332$ m/s and atmospheric pressure $P = 1.01 \times 10^5$ N/m^2.

ANSWERS

1. 13 dB **2.** 3.0×10^{-8} W/m^2 **3.** 64π W
4. 4.0×10^{-12} W/m^2, 6.02 dB **5.** (a) 6.0 N/m^2 (b) 3.3×10^{-6} m
6 121/81 **7.** 4.0×10^{-4} W/m^2 **8.** 3.16
9. 2 μW, 36×10^{-3} J **10.** 13 dB, 10^6 **11.** 190 dB

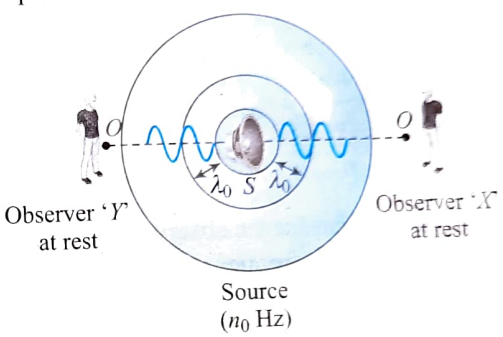
DOPPLER EFFECT

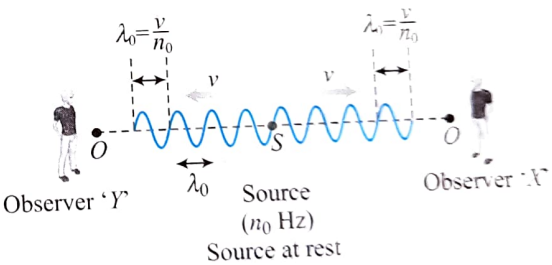
When a car at rest on a road sounds its high frequency horn and you are also standing on the road nearby, you will hear the sound of same frequency it is sounding but when the car approaches you with its horn sounding, the pitch (frequency) of its sound seams to drop as the car passes. This phenomenon was first described by the Austrian scientist Christien Doppler and is called the Doppler effect. He explained that when a source of sound and a listener are in motion relative to each other, the frequency of the sound heard by the listener is not the same as the source frequency. Let us discuss the Doppler effect in detail for different cases.

Figure shows a stationary source of frequency n_0 which produces sound waves in air of wavelength λ_0 given as

$$\lambda_0 = \frac{v}{n_0}$$

where v is speed of sound in air.





Although sound waves are longitudinal, here we represent sound waves by the transverse displacement curve as shown in figure to understand the concept in a better way. As source produces waves, these waves travel towards the stationary observers 'X' and 'Y' in the medium (air) with speed v and wavelength λ_0, so the observers will hear the frequency n given by

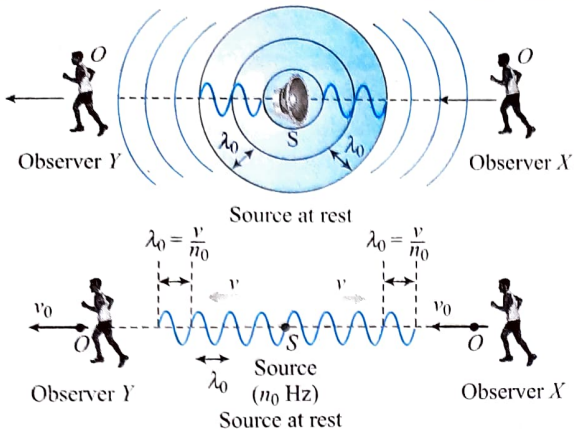
$$n = \frac{v}{\lambda_0} = n_0 \text{ (same as that of source)} \qquad \ldots(i)$$

This is why when a stationary observer listens the sound from a stationary source of sound, it detects the same frequency of sound which the source is producing. Thus, no Doppler effect takes place if there is no relative motion between source and observer.

STATIONARY SOURCE AND MOVING OBSERVER

Figure shows the case when a stationary source of frequency n_0 produces sound waves which have wavelength in air given as

$$\lambda_0 = \frac{v}{n_0}$$



These waves travel towards a moving observer 'X' which moves with velocity v_o towards the source. When sound waves approach the observer, he will receive the waves of wavelength λ_0 with speed $v + v_o$ (relative speed). Thus the frequency of sound heard by the observer 'X' (apparent frequency) can be given as

$$n_{ap} = \frac{v+v_o}{\lambda_0} = \frac{v+v_0}{\left(\frac{v}{n_0}\right)} = n_0\left(\frac{v+v_0}{v}\right) \quad ...(ii)$$

Similarly, we can say that for the observer 'Y' is receding away from the source the apparent frequency heard by the observer 'Y' will be given as

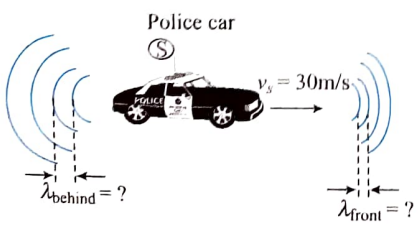
$$n_{ap} = n_0\left(\frac{v-v_0}{v}\right) \quad ...(iii)$$

ILLUSTRATION 8.25

A police siren emits a sinusoidal wave with frequency f_s = 300 Hz. The speed of sound is 340 m/s. **(a)** Find the wavelength of waves if the siren is at rest in the air. **(b)** If the siren is moving at 30 m/s, then find the wavelengths of the waves in front of and behind the source.

Sol. The Doppler effect is not involved in part (a), since neither the source nor the listener is moving. In part (b), the source is in motion and we must involve the Doppler effect. Figure shows the situation. We use the relationship $v = \lambda f$ to determine the wavelength when the police siren is at rest. When it is in motion, we find the wavelength on either side of the siren.

(a) When the source is at rest, $\lambda = \frac{v}{f_s} = \frac{340 \text{ m/s}}{300 \text{ Hz}} = 1.13 \text{ m}$



(b) The situation is shown in figure. Now, in front of the siren,

$$\lambda_{in\ front} = \frac{v-v_s}{f_s} = \frac{340 \text{ m/s} - 30 \text{ m/s}}{300 \text{ Hz}} = 1.03 \text{ m}$$

Behind the siren,

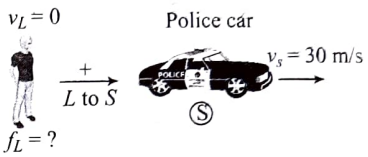
$$\lambda_{behind} = \frac{v+v_s}{f_s} = \frac{340 \text{ m/s} + 30 \text{ m/s}}{300 \text{ Hz}} = 1.23 \text{ m}$$

The wavelength is less in front of the siren and greater behind the siren, as it should be.

ILLUSTRATION 8.26

If a listener L is at rest and the siren in previous illustration is moving away from L at a speed of 30 m/s, what frequency does the listener hear?

Sol. Our target variable is the frequency f_L heard by the listener, who is behind the moving source. Figure shows the situation. We know that v_L = 0 and v_s = 30 m/s. (The source velocity v_s is positive because the siren is moving in the same direction as the direction from listener to source.)



Now, $f_L = \frac{v}{v+v_s} f_s = \frac{340 \text{ m/s}}{340 \text{ m/s} + 30 \text{ m/s}} (300 \text{ Hz}) = 276 \text{ Hz}$

The source and listener are moving apart, so the frequency f_L heard by the listener is less than the frequency f_S emitted by the source.

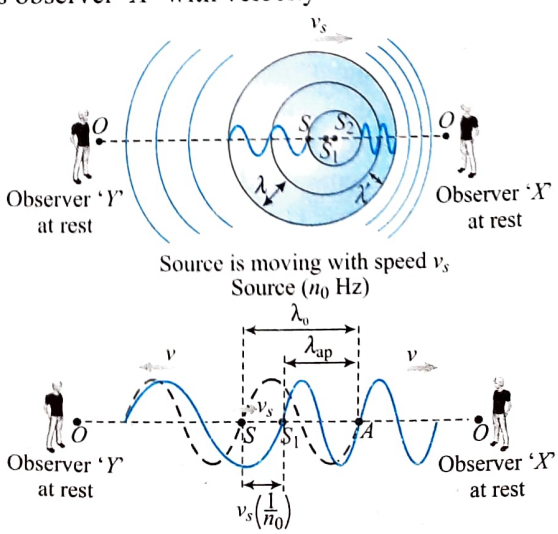
Here is an alternative approach we can use to check our result. From previous illustration, the wavelength behind the source (which is where the listener in figure is located) is 1.23 m, so

$$f_L = \frac{v}{\lambda} = \frac{340 \text{ m/s}}{1.23 \text{ m}} = 276 \text{ m}$$

Even though the source is moving, the wave speed v relative to the stationary listener is unchanged.

MOVING SOURCE AND STATIONARY OBSERVER

Figure shows the situation when a moving source S of frequency n_0 produces sound waves in medium (air) and the waves travel towards observer 'X' with velocity v.



Here let us carefully look at the initial situation when the source starts moving with velocity v_s as well as it starts producing

waves. The period of one oscillation is $(1/n_0)$ seconds and in this duration the source emits one wavelength λ_0 in the direction of propagation of waves with speed v, but in this duration the source will also move forward by a distance $v_s\,(1/n_0)$. Thus the effective wavelength of emitted sound in air is slightly compressed by this distance as shown in figure. This is known as the apparent wavelength of sound in medium (air) by the moving source. This is given as

$$\lambda_{ap} = \lambda_0 - v_s\left(\frac{1}{n_0}\right) \qquad \text{...(iv)}$$

$$= \frac{\lambda_0 n_0 - v_s}{n_0} = \frac{v - v_s}{n_0}$$

Now, this wavelength will approach observer 'X' with speed v (as O is at rest). Thus, the frequency of sound heard by the observer 'X' can be given as

$$n_{ap} = \frac{v}{\lambda_{ap}} = \frac{v}{(v - v_s)/n_0} = n_0\left(\frac{v}{v - v_s}\right) \qquad \text{...(v)}$$

Similarly, if the source is receding away from observer 'Y' the apparent wavelength emitted by source in air towards the observer 'Y' will be slightly expanded and the apparent frequency heard by the stationary observer 'Y' can be given as

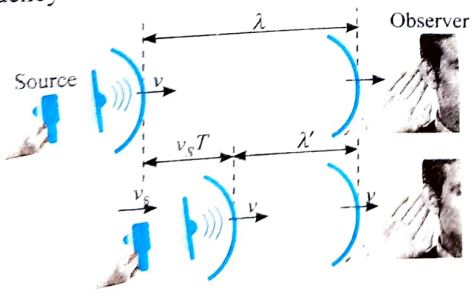
$$n_{ap} = n_0\left(\frac{v}{v + v_s}\right) \qquad \text{...(vi)}$$

Important Points:

- The motion of the source affects the apparent wavelength and the motion of the observer affects the velocity of the sound waves received. In given figure the apparent wavelength observed by observer will be

$$\lambda' = \lambda - v_s T = vT - v_s T = \frac{(v - v_s)}{n_o}$$

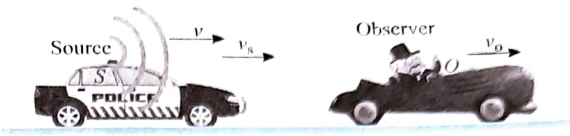
Here v and v_s are velocity of the wave and source respectively and T and n_0 are the time period and frequency of the wave.



- The Doppler effect does not depends on the distance between the source and the observer. Although the intensity of a sound varies as the distance changes, the apparent frequency depends only on the relative speed of source and observer. As you listen to an approaching source, you will detect increasing intensity but constant frequency. As the source passes, you will hear the frequency suddenly drop to a new constant value and the intensity begin to decrease.

SOURCE AND OBSERVER BOTH IN MOTION

Let us consider the situation when both source and observer are moving in same direction as shown in figure at speeds v_s and v_o, respectively.



The relative velocity of sound waves w.r.t. moving observer O is $v' = v - v_0$

The apparent wavelength of the waves received by the observer is $\lambda' = \dfrac{(v - v_s)}{n_o}$

Now, this wavelength will approach the observer at relative speed $v - v_0$ .Thus the apparent frequency of sound heard by the observer is given as

$$n_{app} = \frac{v'}{\lambda'} = \frac{(v - v_0)}{(v - v_s)/n_o} = n_o\left(\frac{v - v_0}{v - v_s}\right) \qquad \text{...(i)}$$

Above equation can be used as a general relation for finding the apparent frequency heard by a observer in any type of situation. Equation(i) can be written as

$$n_{app} = n_o\left(\frac{v \pm v_0}{v \pm v_s}\right) \qquad \text{...(ii)}$$

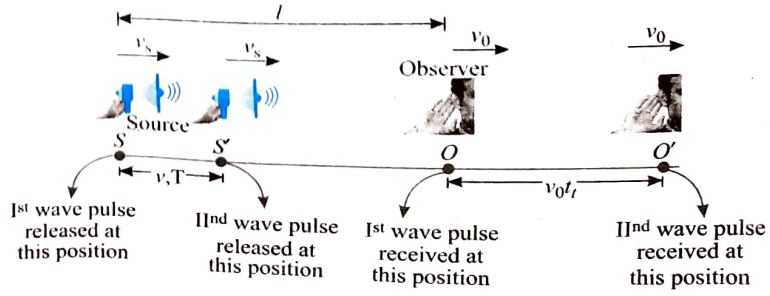
Here + and – signs are chosen according to the direction of motion of source and observer. The sign convention related to the direction of motion can be stated as follows:

(i) For both source and observer, v_0 and v_s are taken in Eq. (ii) with –ve sign if they are moving in the direction of propagation of sound from source to observer.

(ii) For both source and observer, v_0 and v_s are taken in Eq. (ii) with +ve sign if they are moving in the direction opposite to the propagation of sound from source to observer.

SECOND APPROACH

Let us assume that the source and observer are moving along the same line with velocities v_s and v_0 respectively. The observer O is to the right of the source S. Suppose that at time $t = 0$, when the source and the observer are separated by a distance $SO = l$, the source emits a wave pulse (say p_1) that reaches the observer at a later time t_1. In that time the observer has moved a distance $v_0\,t_1$ and the total distance travelled by p_1 in the time t_1 has been $l + v_0\,t_1$. If v is the speed of sound, this distance is also vt_1.



Then, $vt_1 = l + v_0 t_1$

or $$t_1 = \frac{l}{v - v_0} \quad \ldots(i)$$

If the time period of the wave is T, it means next pulse will be emitted after time T. At time $t = T$, the source is at S' and the next wave pulse (say p_2) emitted at this time will reach the observer at a time say t_2, measured from the same time $t = 0$, as before. The total distance travelled by p_2 till it is received by the observer (measured from S') is $(l - v_S T) + v_0 t_2$. The actual travel time for p_2 is $(t_2 - T)$ and the distance travelled is $v(t_2 - T)$. Therefore,

$$v(t_2 - T) = (l - v\ T) + v_0 t_2$$

or $$t_2 = \frac{l + (v - v_S)T}{v - v_0} \quad \ldots(ii)$$

The time interval between two pulses as observed by the observer between the two pulses emitted by the source at S and at S' is

$$T' = t_2 - t_1 = \left(\frac{v - v_S}{v - v_0}\right) T$$

This is nothing but the changed time period as observed by the observer. Hence, the new frequency is

$$n_{app} = \frac{1}{T'} = n_o \left(\frac{v - v_O}{v - v_S}\right)$$

This is the same result which we derived by previous method.

Important Points:

- If medium (air) is also moving with v_m velocity in direction of source to observer. Then velocity of sound relative to observer will be $v \pm v_m$ (–ve sign, if v_m is opposite to sound velocity).

  So, $n_{app} = n_o \left(\frac{v \pm v_m \pm v_o}{v \pm v_m \mp v_s}\right)$

- If medium moves in a direction opposite to the direction of propagation of sound, then $n_{app} = n_o \left(\frac{v - v_m - v_o}{v - v_m - v_S}\right)$

Apparent Frequency and Apparent Wavelength in Different Situations

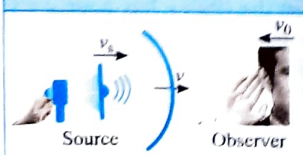
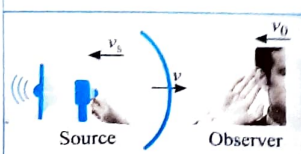
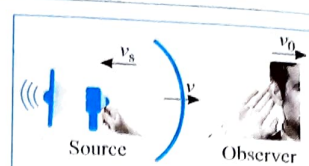
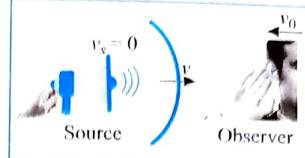
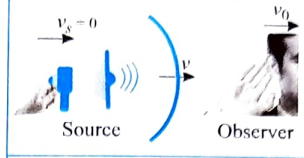
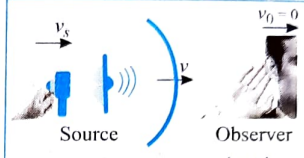
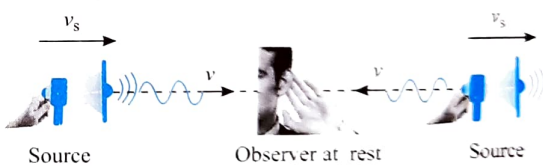
Situation	Apparent frequency	Apparent wavelength
	$n_{app} = n_o \left(\frac{v + v_0}{v - v_s}\right)$	$\lambda_{app} = \left(\frac{v - v_s}{n_o}\right) = \lambda_o \left(\frac{v - v_s}{v}\right)$
	$n_{app} = n_o \left(\frac{v + v_0}{v + v_s}\right)$	$\lambda_{app} = \left(\frac{v - v_s}{n_o}\right) = \lambda_o \left(\frac{v - v_s}{v}\right)$
	$n_{app} = n_o \left(\frac{v - v_0}{v + v_s}\right)$	$\lambda_{app} = \left(\frac{v + v_s}{n_o}\right) = \lambda_o \left(\frac{v + v_s}{v}\right)$
	$n_{app} = n_o \left(\frac{v + v_0}{v}\right)$	No change in wavelength
	$n_{app} = n_o \left(\frac{v - v_0}{v}\right)$	No change in wavelength
	$n_{app} = n_o \left(\frac{v}{v - v_s}\right)$	$\lambda_{app} = \left(\frac{v - v_s}{n_o}\right) = \lambda_o \left(\frac{v - v_s}{v}\right)$

Illustration 8.27

A source of frequency f is moving towards the observer along the line with a constant velocity v_s as shown in figure. Plot apparent frequency observed by the observer versus time graph.

Sol. Let a source of frequency f is moving along the straight line with velocity v_b. and observer is on the same line at rest as shown in the figure.



Apparent frequency during approach, $f_1 = f\left(\frac{v}{v - v_s}\right) \Rightarrow f_1 > f$.
f_1 is constant with time.

Apparent frequency during recede $f_2 = f\left(\frac{v}{v + v_s}\right) \Rightarrow f_2 < f$.
f_2 is constant with time.

Hence the graph of frequency versus time as shown in the figure.

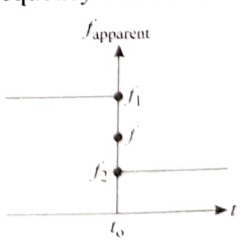
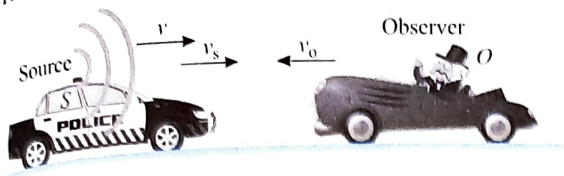


Illustration 8.28

When both source and observer approach each other with a speed equal to the half the speed of sound, then determine the percentage change in frequency of sound as detected by the listener.

Here we can use general formula of Doppler effect.



$$f_{app} = f_n\left(\frac{v - v_o}{v - v_s}\right) \quad \ldots(i)$$

The direction of sound wave is always from source to observer. this case both the source and observer are moving towards h other. The direction of motion of observer is opposite to the ocity of sound. So equation (i) becomes

$$f_{app} = f_n\left(\frac{v-(-v_o)}{v - v_s}\right) = f_n\left(\frac{v + v_o}{v - v_s}\right)$$

The speed of source and observer is equal to half the speed of ound.

$$\text{Hence } f_{app} = f_n\left(\frac{v + \frac{v}{2}}{v - \frac{v}{2}}\right) = 3f_n$$

ercentage change in frequency

$$\% \text{ change} = \frac{f_{app} - f_n}{f_n} \times 100 = \frac{3f_n - f_n}{f_n} \times 100 = 200\%$$

ILLUSTRATION 8.29

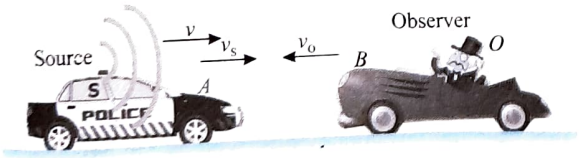
Two cars A and B are travelling in opposite directions at 30 m/s each, cross each other while car A is whistling. If the frequency of the node is 3600 Hz find the apparent frequency as heard by an observer in car B:

(a) Before the cars cross each other

(b) After the car have crossed each other. $(v_{sound} = 330\,\text{m/sec})$

Sol.

(a) Before the cars cross each other



Here we will use general formula of Doppler effect

$$f_{app} = f_n\left(\frac{v - v_o}{v - v_s}\right)$$

In this case velocity of car A is in the direction of sound velocity while the velocity of car B is in the direction opposite to velocity of the sound.

$$f_{app} = f_n\left(\frac{v-(-v_o)}{v - v_s}\right) = f_n\left(\frac{v + v_o}{v - v_s}\right)$$

$$\Rightarrow \quad f_{app} = 3600\left(\frac{330 + 30}{330 - 30}\right) = 4320 \text{ Hz}$$

(b) After the cars have crossed each other.

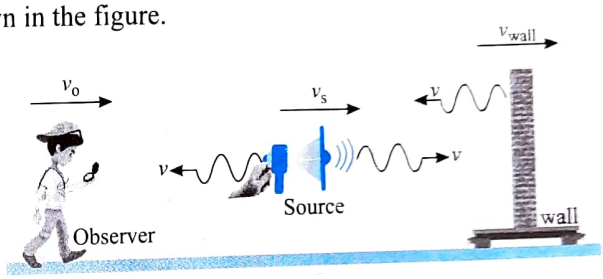
Now in this case the velocity of car B is in the direction of travelling of sound while car A (sound) moves in the direction opposite to sound. In this case the general formula of Doppler effect becomes

$$f_{app} = f_n\left(\frac{v - v_o)}{v-(-v_s)}\right) = f_n\left(\frac{v - v_o}{v + v_s}\right)$$

$$\Rightarrow \quad f_{app} = 3600\left(\frac{330 - 30}{330 + 30}\right) = 3000 \text{ Hz}$$

DOPPLER EFFECT IN REFLECTED SOUND

Let a source of sound is moving with velocity v_S towards a wall. which itself moving in same direction with velocity v_{wall}. An observer is also moving in the same direction with velocity n_0, as shown in the figure.



In this case the observer will notice two frequencies. One direct through the source and second through the reflection by moving wall.

The direct frequency received by the observer.

$$n_{direct} = n_o\left(\frac{v-(-v_o)}{v-(-v_s)}\right) = n_o\left(\frac{v + v_o)}{v + v_s)}\right) \quad \ldots(i)$$

Now let us analyse the frequency reflected by wall. In this case the reflected sound observe by the observer appears to be of relatively high pitch.

First we consider the wall as an observer. In this case the frequency received by wall is given by

$$n_1 = n_o\left(\frac{v - v_{wall}}{v - v_s}\right) \quad \ldots(ii)$$

Now the wall reflect this frequency and behave like a moving source. The situation is just like an observer and source moving towards each other. In this case the frequency received by the observer.

$$n_2 = n_1\left(\frac{v-(-v_o)}{v-(-v_{wall})}\right) = n_1\left(\frac{v + v_o}{v + v_{wall}}\right) \quad \ldots(iii)$$

from (ii) and (iii) we get, $n_2 = n_o\left(\frac{v - v_{wall}}{v - v_s}\right) \times \left(\frac{v + v_o}{v + v_{wall}}\right)$

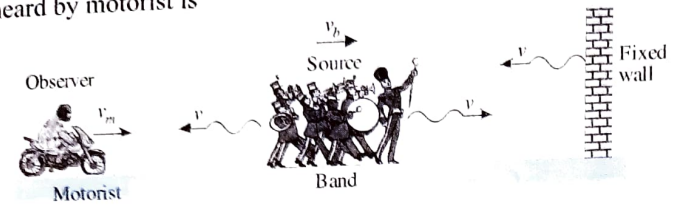
Hence the frequency of the reflected sound received by the observer

$$n_{reflected} = n_o\left(\frac{v + v_o}{v - v_s}\right)\left(\frac{v - v_{wall}}{v + v_{wall}}\right)$$

ILLUSTRATION 8.30

A band playing music at a frequency f is moving towards a wall at a speed v_b. A motorist is following the band with a speed v_m. If v is the speed of sound, obtain an expression for the beat frequency heard by the motorist.

Sol. Actual frequency of band music is f. The direct frequency heard by motorist is



$$f_1 = \left(\frac{v-(-v_m)}{v-(-v_b)}\right) f = \left(\frac{v+v_m}{v+v_b}\right) f$$

Now consider the reflected sound. First the wall will act as an observer.

The frequency heard by observer at wall is

$$f' = \left(\frac{v-0}{v-v_b}\right) f = \left(\frac{v}{v-v_b}\right) f$$

The reflection does not cause any change in frequency. Therefore, frequency f' is reflected as such and the motorist is approaching the wall. Hence, the frequency of reflected sound heard by motorist is

$$f_2 = \left(\frac{v-(-v_m)}{v-0}\right) f' = \left(\frac{v+v_m}{v}\right) \times \left(\frac{v}{v-v_b}\right) f = \left(\frac{v+v_m}{v-v_b}\right) f$$

Therefore, the beat frequency is

$$n = f_2 - f_1 = \left(\frac{v+v_m}{v-v_b}\right) f - \left(\frac{v+v_m}{v+v_b}\right) f$$

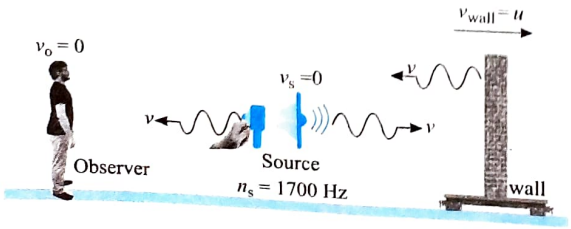
$$= \frac{(v+v_b)-(v-v_b)}{v^2-v_b^2}(v+v_m) f$$

$$\Rightarrow \quad n = \frac{2v_b(v+v_m)f}{v^2-v_b^2}$$

ILLUSTRATION 8.31

A source of sonic oscillations with frequency $n = 1700$ Hz and a receiver are located on the same normal to a wall. Both the source and receiver are stationary, and the wall recedes from the source with velocity $u = 6.0$ cm/s. Find the beat frequency registered by the receiver. The velocity of sound is equal to $v = 340$ m/s.

Sol. In this case the observer will notice two frequencies. One direct through the source and second through the reflection by moving wall.



As both source and receiver are at rest the frequency of sound waves which directly reach to receiver, will be 1700 Hz.

Hence $n_{direct} = 1700$ Hz. The frequency of sound which wall will receive as a moving observer is

$$n_1 = n_0\left[\frac{v-u}{v-0}\right] = n_0\left[\frac{v-u}{v}\right] \quad \ldots(i)$$

The wall now behaves as a moving source of frequency n_1 which when received by receiver, the frequency observed is

$$n_2 = n_1\left[\frac{v-0}{v-(-u)}\right] = n_1\left[\frac{v}{v+u}\right] = n_0\left(\frac{v-u}{v+u}\right) \quad \ldots(ii)$$

$$= 1700\left[\frac{340-0.06}{340+0.06}\right] = 1700 \times \frac{339.94}{340.06} = 1699.4 \text{ Hz}$$

Thus beat frequency received by detector is

$$\Delta n = 1700 - 1699.4 = 0.6 \text{ Hz}$$

DOPPLER EFFECT FOR ACCELERATED MOTION

For the case of moving source and a moving observer, we know that the apparent frequency can be given as

$$n_{ap} = n_0\left[\frac{v \pm v_o}{v \pm v_S}\right] \quad \ldots(i)$$

Here v is the velocity of sound and v_o and v_S are the velocity of observer and source, respectively. When a source or observer has accelerated or retarded motion, then in Eq. (i) we use that value of v_o at which the observer receives the sound and for source, we use that value of v_s at which it has emitted the wave.

The alternative method of solving this case is by the traditional method of compressing or expending wavelength of sound by motion of source and using relative velocity of sound with respect to observer.

ILLUSTRATION 8.32

Two trains A and B simultaneously start moving along parallel tracks from a station along same direction. A starts with constant acceleration 2 m/s² from rest, while B with the same acceleration but with initial velocity of 40 m/s. Twenty seconds after the start, passenger of A hears whistle of B. If frequency of whistle is 1194 Hz and velocity of sound in air is 322 m/s, calculate frequency observed by the passenger.

Sol. Since both the trains move in same direction with same acceleration, therefore, their relative acceleration is zero. Initial velocity of train A is zero while that of train B is 40 m/s. Therefore, train A or observer is behind the train B or source when whistle is heard by passenger of A. Let this sound be produced at time t. Considering motion of train B (sound source) up to this instant,

$$u = 40 \text{ m/s}, \quad a = 2 \text{ m/s}^2, \quad \text{time} = t$$

$$v = v_B = ? \quad s = s_B = ? \quad \ldots(i)$$

We have, $v = u + at$,

$$\therefore \quad v_B = 40 + 2t \quad \ldots(ii)$$

$$s = ut + \frac{1}{2}at^2$$

$$s_B = 40t + t^2$$

But this sound is heard by the observer at $t = 20$ s. Considering his motion (motion of train A) up to this instant,

$u = 0, \quad a = 2 \text{ m/s}^2, \quad t = 20$ s

$v = v_A = ? \quad s = s_A = ?$

We have, $v = u + at$,

$$\therefore \quad v_A = 40 \text{ m/s}$$

$$s = ut + \frac{1}{2}at^2$$

$$\therefore \quad s_A = 400 \text{ m}$$

When sound was produced, train B (source) was at a distance s_B from initial point while observer receives this sound at $t = 20$ s and at a distance s_A from initial point. It means that sound waves travel a distance $(s_B - s_A)$ in air and take time $(20 - t)$ to travel it.

$$s_B - s_A = (20 - t)v \qquad \ldots(iii)$$

where $v = 322$ m/s (speed of sound). Substituting values of s_A and s_B in Eq. (iii),

$t = 18$ s

When sound was produced, source was moving with velocity

$$v_B = (40 + 2t) = 76 \text{ m/s}$$

(away from observer) while observer receives these sound waves at $t = 20$ s, when he was moving with velocity $v_A = 40$ m/s (towards the source). Hence, the observed frequency is

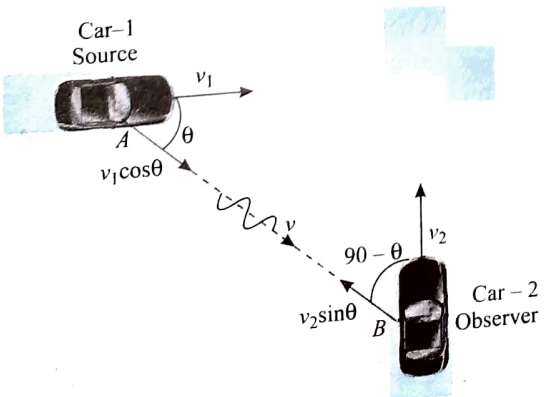
$$n = n_0 \frac{v + v_A}{v + v_B}$$

where n_0 = 1194 Hz (natural frequency of source) and so $n = 1086$ Hz.

Doppler's Effect When Source and Observer are not in Same Line of Motion

Let us consider the situation shown in figure. Two cars 1 and 2 are moving along two roads crossing perpendicularly at speeds v_1 and v_2. When car–1 sound a horn of frequency n_0 it emits sound pulse and car–2 receive receives the sound at the position as shown in the figure. The direction of sound is from source position to observation position. It means Doppler effect is applicable along the line AB. In such cases we use velocity components of the cars along the line joining the source and observer thus the apparent frequency of sound heard by car–2 can be given as

$$n_{ap} = n_0 \left[\frac{v - (-v_2 \cos(90 - \theta))}{v - v_1 \cos\theta}\right] = n_0 \left[\frac{v + v_2 \sin\theta}{v - v_1 \cos\theta}\right]$$

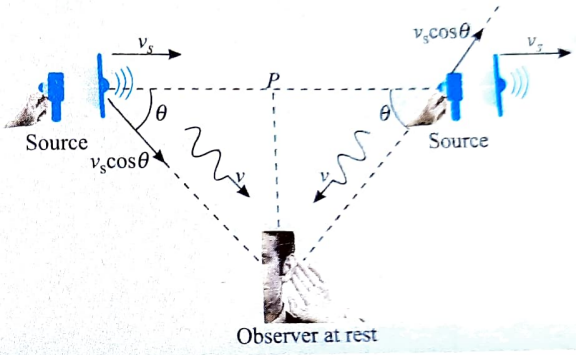


Conditions When Doppler's Effect is not Observed for Sound Waves

- When the source of sound and observer both are at rest then Doppler effect is not observed.
- When the source and observer both are moving with same velocity in same direction.
- When the source and observer are moving mutually in perpendicular directions.
- When the medium only is moving.
- When the distance between the source and observer is constant.

Illustration 8.33

A source of frequency f is moving along line with a constant velocity v_s and observer is separated from the line of motion of the source as shown in figure. Plot apparent frequency observed by the observer versus time graph.



Sol. Let a source of frequency f is moving along the straight line with velocity v_s.

$$f_1 = f\left(\frac{v}{v - v_s\cos\theta}\right) \qquad \ldots(i)$$

As source moves, θ changes. Hence f_1 is not constant which is variable.

But, at a large distance, $\theta \to 0°$ or $\cos\theta \to 1$

The apparent frequency $f_1 = f\left(\frac{v}{v - v_s}\right)$ and this is the maximum value of f_1 (say f_{max})

At point P, $\theta = 90°$, hence $f_1 = f$

During receding (to the right of P), $f_2 = f\left(\frac{v}{v + v_s\cos\theta}\right)$

Again, at large distance from P, $\theta \to 0°$ or $\cos\theta \to 1$

$$f_2 = f\left(\frac{v}{v + v_s}\right)$$

This time, this is the minimum value of f_2 (say f_{min})

Hence, f_2 varies from f_{max} to f_{min}, with $f_2 = f$ at the time of crossing P.

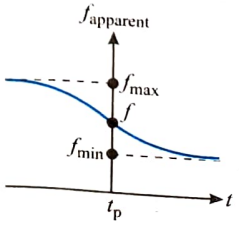


ILLUSTRATION 8.34

A train approaching a railway crossing at a speed of 120 km/h sounds a short whistle at frequency 640 Hz when it is 300 m away from the crossing. The speed of sound in air is 340 m/s. What will be the frequency heard by a person standing on a road perpendicular to the track through the crossing at a distance of 400 m from the crossing?

Sol. The observer A is at rest with respect to the air and the source is travelling at a velocity of 120 km/h, i.e., (100/3) m/s. As is clear from the figure, the person receives the sound of the whistle in a direction BA making an angle θ with the track where $\cos\theta = 300/500 = 3/5$. The component of the velocity of the source (i.e., of the train) along the direction AB is (100/3) × (3/5) m/s = 20 m/s. As the source is approaching the person with this component, the frequency heard by the observer is

$$n' = \frac{v}{v - u\cos\theta} n = \frac{340}{340 - 20} \times 640 \text{ Hz} = 680 \text{ Hz}$$

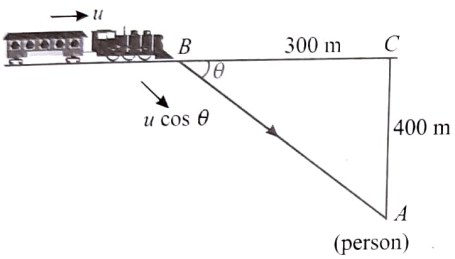


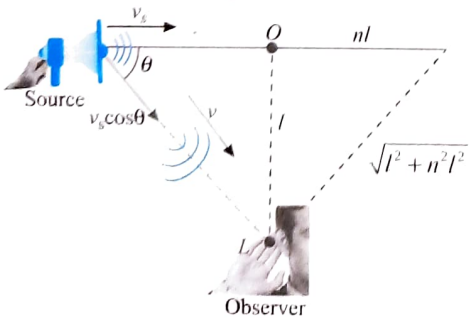
ILLUSTRATION 8.35

A source of sound having natural frequency f_o = 1800 Hz is moving uniformly along a line separated from a stationary listener by a distance l = 250 m. The velocity of source is $n = 0.80$ times the velocity of sound. Find:

(a) the frequency of sound received by the listener at the moment when the source gets closest to him.

(b) the distance between the source and the listener at the moment when the listener receives actual frequency of the source.

Sol. As sound waves takes finite time in reaching from one position to other position. The sound waves emitted by the source when it is closest to the listener (at O) does not reach instantaneously. The frequency heard at this instant should be emitted from some previous position of source. Let this sound (or disturbance) was created at position 'S'.



The direction of velocity vector of source makes an angle θ with the line joining the disturbance position (source position) and observer position as shown in figure. The time in which sound reaches the listener along SL the source reaches point O, closest to the listener. From figure,

$$\cos\theta = \frac{SO}{SL} = \frac{v_S t}{vt} = \frac{v_S}{v} = n \quad \ldots(i)$$

(a) Now using the formula for apparent frequency received by observer

$$f_{app} = f_{original}\left(\frac{v - v_o}{v - v_S}\right)$$

As Doppler effect is applicable only along the line joining disturbance position (source position) and observer position, hence we should substitute the component of velocity of the source along this line 'SL'.

Hence the frequency heard by the listener is

$$f_{app} = f_o\left(\frac{v - 0}{v - v_S\cos\theta}\right) \quad \ldots(ii)$$

$$\Rightarrow \quad f_{app} = f_o\left(\frac{v}{v - (nv)n}\right) = \frac{f_S}{1 - n^2} \quad \ldots(iii)$$

On substituting numerical values, we get f_{app} = 5000 Hz

(b) When the source is at O, the velocity component of source towards listener is $v_s \cos 90° = 0$.

Hence from eqn. (ii), $f_{app} = f_o\left(\frac{v}{v - v_S\cos 90°}\right) = f_o$

Time taken by sound waves to reach the listener $= \frac{l}{v}$

Distance moved by source in this time $= \left(\frac{l}{v}\right)v_S = \left(\frac{l}{v}\right)nv = nl$

Distance of source from the listener $d = \sqrt{l^2 + n^2 l^2}$

$$\Rightarrow \quad d = l\sqrt{1 + n^2} = 250\sqrt{1 + (0.8)^2} = 320 \text{ m}$$

CONCEPT APPLICATION EXERCISE 8.3

1. Is there a Doppler effect in the case of sound when the observer or the source moves at right angles to the line joining them? How then can we determine the Doppler effect when the motion has a component at right angles to this line?
2. A source moves away from an observer with a certain speed, and the ratio of actual to the apparent frequency as heard by the observer is η. If the two approach each other with the same speed, then find the ratio.
3. A railroad train is travelling at 30 m/s in still air. The frequency of the note emitted by locomotive whistle is 500 Hz. What is the frequency of the sound waves heard by a stationary listener **(a)** in front of the train and **(b)** behind the train? (speed of sound is 345 m/s)
4. The ratio of the apparent frequencies of a car when approaching and receding a stationary observer is 11:9. What is the speed of the car, if the velocity of sound in air is 300 m/s?
5. A bat flies perpendicularly towards a wall with a speed of 6 m/s, emitting sound of frequency 450 kHz. What is the frequency of the wave reflected from the wall that it will hear? Given, c = 340 m/s.

9. A whistle emitting a sound of frequency 440 Hz is tied to a string of 1.5 m length and rotated with an angular velocity of 20 rad/s in the horizontal plane. Calculate the range of frequencies heard by an observer stationed at a large distance from the whistle (speed of sound is 330 m/s).

10. An engine blowing a whistle of frequency 133 Hz moves with a velocity of 60 m/s towards a hill from which an echo is heard. Calculate the frequency of the echo heard by the driver. (Velocity of sound in air is 340 m/s.)

11. A siren emitting a sound of frequency 2000 Hz moves away from you towards a cliff at a speed of 8 m/s.
(a) What is the frequency of the sound you hear coming directly from the siren.
(b) What is the frequency of sound you hear reflected off the cliff. Speed of sound in air is 330 m/s.

12. Two tuning forks A and B having a frequency of 500 Hz each are placed with B to the right of A. An observer in between the forks is moving towards B with a speed of 25 m/s. The speed of sound is 345 m/s and the wind speed is 5 m/s from A to B. Calculate the difference in the two frequencies heard by observer.

13. A source of sound of frequency 1000 Hz moves to the right with a speed of 32 m/s relative to the ground. To its right there is a reflecting surface moving to the left with a speed of 64 m/s relative to the ground. Take the speed of sound in air to be 332 m/s and find
(a) the wavelength of the sound emitted in air by the source,
(b) the number of waves per second arriving at the reflecting surface,
(c) the speed of the reflected waves and
(d) the wavelength of the reflected waves.

14. A stationary source emits sound of frequency $\upsilon = 1200$ Hz. If a wind blows at the speed of $0.1c$, deduce (a) the percentage change in the wavelength and (b) the change in the frequency for a stationary observer on the wind-side of the source. What happens when there is no wind, but the observer moves at speed $0.1\,c$ towards the source?

15. A stationary observer receives sound waves from two tuning forks, one of which approaches and the other recedes with the same velocity. As this takes place, the observer hears beats of frequency $\upsilon = 2$ Hz. Find the velocity of each tuning fork if their oscillation frequency is $\upsilon_0 = 680$ Hz and the velocity of sound in air is $\upsilon_s = 340$ m/s.

ANSWERS

1. No, It is done by considering the component of the velocity of the source along the line joining it with the observer.
2. $\frac{2}{\eta} - 1$ 3. (a) 547.62 Hz, (b) 460 Hz 4. 30 m/s 5. 466 kHz
6. 403 Hz, 484 Hz 7. 190 Hz 8. (a)1953 Hz (b) 2050 Hz
9. 72.5 Hz 10. (a) 0.3 m (b) 1320 Hz (c) 332 ms^{-1} (d) 0.2 m
12. 0.5 m/s

Solved Examples

EXAMPLE 8.1

The sound level at a distance of 3.00 m from a source is 120 dB. At what distance will the sound level be **(a)** 100 dB and **(b)** 10.0 dB?

Sol. A reduction of 20 dB means reducing the intensity by a factor of 10^2, so we expect the radial distance to be 10 times larger, namely 30 m. A further reduction of 90 dB may correspond to an extra factor of $10^{4.5}$ in distance, to about 30×30000 m, or about 1000 km. We use the definition of the decibel scale and the inverse square law for sound intensity. From the definition of sound level,

$$\beta = 10 \log \left(\frac{1}{10^{-12}\ \text{W/m}^2} \right)$$

We can compute the intensities corresponding to each of the levels mentioned as

$$I = [10^{\beta/10}]10^{-12}\ \text{W/m}^2$$

They are $I_{120} = 1\ \text{W/m}^2$, $I_{100} = 10^{-2}\ \text{W/m}^2$ and $I_{10} = 10^{-11}\ \text{W/m}^2$.

(a) The power passing through any sphere around the source is

$$P = 4\pi r^2 I$$

If we ignore absorption of sound by the medium, conservation of energy requires that

$$r_{120}^2 I_{120} = r_{100}^2 I_{100} = r_{10}^2 I_{10}$$

Then, $r_{100} = r_{120}\sqrt{\frac{I_{120}}{I_{100}}} = (3.00\ \text{m})\sqrt{\frac{1\ \text{W/m}^2}{10^{-2}\ \text{W/m}^2}} = 30.0\ \text{m}$

(b) $r_{10} = r_{120}\sqrt{\frac{I_{120}}{I_{10}}} = (3.00\ \text{m})\sqrt{\frac{1\ \text{W/m}^2}{10^{-11}\ \text{W/m}^2}} = 9.49 \times 10^5\ \text{m}$

At 949 km away, the faint 10 dB sound would not be identifiable among many other soft and loud sounds produced by other sources across the continent. And 949 km of air is such a thick screen that it may do a significant amount of sound absorption.

EXAMPLE 8.2

Two train whistles have identical frequencies of 180 Hz. When one train is at rest in the station and the other is moving nearby, a passenger standing on the station platform hears beats with a frequency of 2.00 beats/s when the whistles operate together. What are the two possible speeds and directions the moving train can have? Speed of sound is 343 m/s.

Sol. A Doppler shift in the frequency received from the moving train is the source of the beats. We consider velocities of approach and of recession separately in the Doppler equation, after we observes from our beat equation $f_b = |f_1 - f_2| = |f - f'|$ that the moving train must have an apparent frequency of either $f' = 182$ Hz or $f'' = 178$ Hz. We let v_s represent the magnitude of

the train's velocity. If the train is moving away from the station, the apparent frequency is 178 Hz lower, as described by

$$f' = \frac{v}{v+v_t} f$$

and the train is moving away at

$$v_t = \left(\frac{f}{f'} - 1\right) v = (343 \text{ m/s}) \left(\frac{180 \text{ Hz}}{178 \text{ Hz}} - 1\right) = 3.85 \text{ m/s}$$

If the train is pulling into the station, then the apparent frequency is 182 Hz. Again from the Doppler shift,

$$f' = f \frac{v}{v - v_s}$$

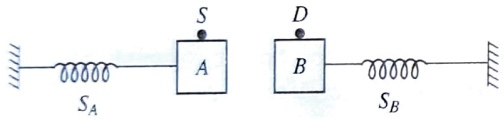
The train is approaching at

$$v_s = v \left(1 - \frac{f}{f'}\right) = (343 \text{ m/s}) \left(1 - \frac{180 \text{ Hz}}{182 \text{ Hz}}\right) = 3.77 \text{ m/s}$$

The Doppler effect does not measure distance, but it can be a remarkably sensitive measure of speed.

EXAMPLE 8.3

A source S emitting sound of 300 Hz is fixed on block A which is attached to free end of a spring S_A as shown in the figure. The detector D fixed on block B attached to the free end of spring S_B detects this sound.



The blocks A and B are simultaneously displaced towards each other through a distance of 1.0 m and then left to vibrate. Find the maximum and minimum frequencies of sound detected by D if the vibrational frequency of each block is 2 Hz (velocity of sound is 340 m/s).

Sol. The motion of A and B is synchronized. Hence the maximum frequencies detected by D will be

$$f_{max} = \left(\frac{v + v_D}{v - v_S}\right) f$$

(when S and D are approaching each other while at their equilibrium position.)

$$= \left(\frac{340 + A\omega}{340 - A\omega}\right) f = \left(\frac{340 + 12.56}{340 - 12.56}\right) \times 300 = 323 \text{ Hz}$$

Similarly, the apparent frequency will be minimum when the source and the detector recede from each other with maximum speed ($= A\omega$)

$$f_{min} = \left(\frac{v - A\omega}{v + A\omega}\right) f = \left(\frac{340 - 12.56}{340 + 12.56}\right) \times 300 = 278.6 \text{ Hz}$$

EXAMPLE 8.4

An observer standing on a railway crossing receives frequencies 2.2 kHz and 1.8 kHz when the train approaches and recedes from the observer. Find the velocity of the train (speed of sound in air is 300 m/s).

Sol. Let $v = 300$ m/s be the velocity of sound and u the velocity of train. When the train approaches the observer, the frequency heard is

$$n_1 = \frac{v}{v - u} n \quad \ldots(i)$$

When the train recedes from the observer, the frequency heard is

$$n_2 = \frac{v}{v + u} n \quad \ldots(ii)$$

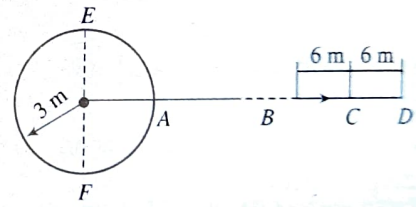
Dividing Eqs. (i) and (ii), we get $\frac{n_1}{n_2} = \frac{v+u}{v-u} \Rightarrow u = \frac{n_1 - n_2}{n_1 + n_2} v$

Given $n_1 = 2.2 \text{ kHz} = 2.2 \times 10^3 \text{ Hz}$.

$$\therefore \quad u = \frac{(2.2 - 1.8) \times 10^3}{(2.2 + 1.8) \times 10^3} \times 300 \text{ m/s} = \frac{0.4 \times 300}{4} = 30 \text{ m/s}$$

EXAMPLE 8.5

A source of sound is moving along a circular orbit of radius 3 m with an angular velocity of 10 rad/s. A sound detector located far away from the source is executing linear simple harmonic motion along the line BD with an amplitude $BC = CD = 6$ m. The frequency of oscillation of the detector is $5/\pi$ s^{-1}. The source is at the point A when the detector is at the point B. If the source emits a continuous sound wave of frequency 340 Hz, find the maximum and the minimum frequencies recorded by the detector.



Sol. Speed of source, $v_s = r\omega = 3 \times 10 = 30$ m/s

Maximum speed of detector,

$$v_{max} = v_d = A\omega' = A\,2\pi f' = 6 \times 2\pi \times \frac{5}{\pi} = 60 \text{ m/s}$$

The speed of sound is $v = 330$ m/s

The frequency recorded by detector is maximum when source and detector approach each other (i.e. source at F and detector at C after times $3T/4$, and $3T'/4$, respectively, T and T' being time periods of source and detector). Then maximum frequency is given by

$$n_{max} = \frac{v + v_d}{v - v_s} n = \frac{330 + 60}{330 - 30} \times 340 = \frac{390}{300} \times 340 = 442 \text{ Hz}$$

The frequency recorded by detector is minimum when source and detector recede from each other (i.e., source at E and detector at C after time $T/4$), so that the minimum frequency is

$$n_{min} = \frac{v - v_d}{v + v_s} n = \frac{330 - 60}{330 + 30} \times 340 = \frac{270}{360} \times 340 = 255 \text{ Hz}$$

EXAMPLE 8.6

Two tuning forks with natural frequencies 340 Hz each move relative to a stationary observer. One fork moves away from the observer, while the other moves towards the observer at the same speed. The observer hears beats of frequency 3 Hz. Find the speed of the tuning forks (speed of sound is 340 m/s).

Sol. Let v be the speed of sound and v_s the speed of forks. The apparent frequency of fork which moves towards the observer is

$$n_1 = \frac{v}{v - v_s} n$$

The apparent frequency of fork which moves away from observer is

$$n_2 = \frac{v}{v + v_s} n$$

If x is the number of beats heard per second, then

$$x = n_1 - n_2 \quad \Rightarrow x = \frac{v}{v - v_s} n - \frac{v}{v + v_s} n$$

or $$x = \frac{v + v_s - (v - v_s)}{v^2 - v_s^2} (vn)$$

or $$\frac{2vv_s n}{v^2 - v_s^2} = x$$

$$2\left(\frac{v_s}{v}\right) n = x \qquad [\because \text{if } v_s \ll v]$$

or $$v_s = \frac{vx}{2n}$$

Given $v = 340$ m/s, $x = 3$, $n = 340$ Hz.

$$\therefore \quad v_s = \frac{340 \times 3}{2 \times 340} = 1.5 \text{ m/s}$$

EXAMPLE 8.7

A boat is travelling in a river with a speed of 10 m/s along the stream flowing with a speed of 2 m/s. From this boat, a sound transmitter is lowered into the river through a rigid support. The wavelength of the sound emitted from the transmitter inside the water is 14.45 mm. Assume that attenuation of sound in water and air is negligible.

(a) What will be the frequency detected by a receiver kept inside downstream?

(b) The transmitter and the receiver are now pulled up into air. The air is blowing with a speed of 5 m/s in the direction opposite the river stream. Determine the frequency of the sound detected by the receiver.

(Temperature of the air and water is 20°C; density of river water is 10^3 kg/m^3; bulk modulus of the water is 2.088×10^9 Pa; gas constant $R = 8.31$ J/mol-K; mean molecular mass of air is 28.8×10^{-3} kg/mol; C_p/C_v for air is 1.4.)

Sol.

(a) Speed of boat $u = 10$ m/s.

Speed of river stream $w = 2$ m/s.

Wavelength of sound wave inside water is 14.45 mm.

Speed of sound inside water is

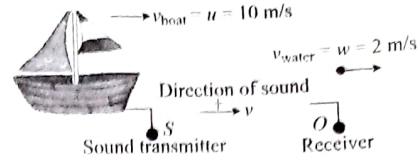
$$v = \sqrt{\frac{B}{d}} = \sqrt{\left\{\frac{2.088 \times 10^9}{10^3}\right\}}$$

$$= \sqrt{2.088} \times 10^3 \text{ m/s} = 1445 \text{ m/s}$$

Frequency of wave inside water is

$$n = \frac{v}{\lambda} = \frac{1445}{14.45 \times 10^{-3}} = 10^5 \text{ Hz}$$

In this case source and medium both are in motion.



Apparent frequency is given by

$$n' = \frac{(v + w)}{(v + w) - u} n = \frac{(1445 + 2)}{(1445 + 2) - 10} \times 10^5$$

$$= \frac{1447}{1437} \times 10^5 \text{ Hz} = 1.00696 \times 10^5 \text{ Hz}$$

(b) Speed of sound in air,

$$v = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{1.4 \times 8.3 \times 293}{28.8 \times 10^{-3}}} \equiv 344 \text{ m/s}$$

Now speed of air $w = 5$ m/s

$$n' = \frac{(v - w)}{(v - w) - u} n = \frac{344 - 5}{344 - 5 - 10} \times 10^5$$

$$= \frac{339}{329} \times 10^5 \text{ Hz} = 1.03040 \times 10^5 \text{ Hz}$$

EXAMPLE 8.8

A sonometer wire under tension of 64 N vibrating in its fundamental mode is in resonance with a vibrating tuning fork. The vibrating portion of that sonometer wire has a length of 10 cm and a mass of 1 g. The vibrating tuning fork is now moved away from the vibrating wire with a constant speed and an observer standing near the sonometer hears one beat per second. Calculate the speed with which the tuning fork is moved if the speed of sound in air is 300 m/s.

Sol. The fundamental frequency of wire is equal to the frequency of the stationary fork, which is given by

$$n = \frac{1}{2l}\sqrt{\left(\frac{T}{m}\right)}$$

Here, $m = \frac{1 \text{ g}}{10 \text{ cm}} = \frac{1 \times 10^{-3} \text{ kg}}{10 \times 10^{-2} \text{ m}} = 10^{-2}$ kg/m,

$T = 64$ N, $\quad l = 10$ cm $= 0.1$ m.

$$\therefore \quad n = \frac{1}{2 \times 0.1} \times \sqrt{\left(\frac{64}{10^{-2}}\right)} = 400 \text{ Hz}$$

When the fork moves away from the source, its apparent frequency decreases. As the number of beats heard per second is 1, apparent frequency of moving fork is 400 – 1 = 399. The apparent frequency due to receding source is given by

$$n' = \frac{v}{v + v_s} n$$

or $$399 = \frac{300}{300 + v_s} \times 400$$

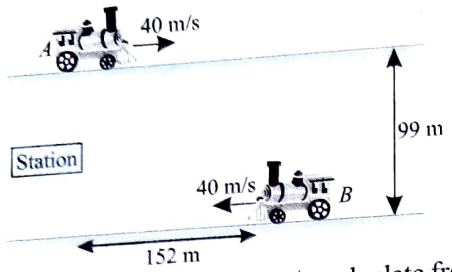
or $$399(300 + v_s) = 300 \times 400$$

or $$399\, v_s = 300$$

or $$v_S = \frac{300}{399} = 0.752 \text{ m/s}$$

EXAMPLE 8.9

A train A crosses a station with a speed of 40 m/s and whistles a short pulse of natural frequency $n_0 = 596$ Hz. Another train B is approaching towards the same station with the same speed along a parallel track. Two tracks are $d = 99$ m apart. When train A whistles, train B is 152 m away from the station as shown in the figure.



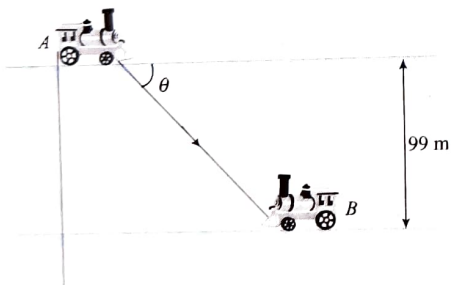
If velocity of sound in air is $v = 330$ m/s, calculate frequency of the pulse heard by driver of train B.

Sol. When train A whistles, sound pulse starts to travel in air from train A to train B. During this interval train B moves some distance towards the station. Let sound pulse take time t to travel from train A to train B. Distance moved by train B during this interval is 40 t. Therefore, the distance of train B from station when its driver hears the pulse is $152 - 40t$. Hence, the distance travelled by the pulse is $\sqrt{(152-40t)^2+(99)^2}$. But it is equal to $vt = 330t$.

$$\therefore \quad \sqrt{(152-40t)^2+(99)^2} = 330t$$

or $\quad t = 0.5$ s

Therefore, driver of train B hears the pulse when his train is $152 - 40t = 132$ m from the station. Hence, path of pulse will be as shown in the figure. Its inclination θ with track is given by



$$\tan\theta = \frac{99}{132} \quad \Rightarrow \theta = 37^\circ$$

Velocity component of train A along path of the pulse,

$v_s = 40 \cos 37^\circ = 32$ m/s

Velocity component of train B along path of the pulse,

$v_0 = 40 \cos 37^\circ = 32$ m/s

Hence, the frequency of pulse heard by driver of train B is

$$n = n_0\left(\frac{v+v_0}{v-v_s}\right) = 724 \text{ Hz}$$

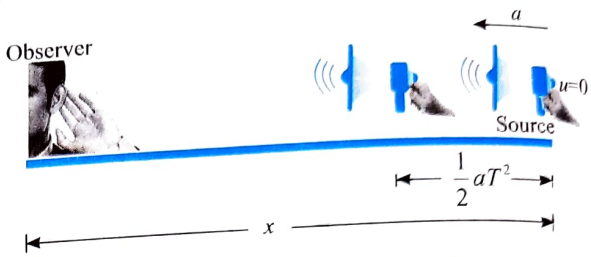
EXAMPLE 8.10

A source of sonic oscillations with frequency 'f' is placed at a large distance from a detector. The source starts moving towards the detector with a constant acceleration 'a'. Find the frequency corresponding to the wave emitted just after the source starts. The speed of sound in the medium is 'v'.

Sol. Let the initial separation between source and detector be x.

The first wave pulse arrives at detector at time $t_1 = \frac{x}{v}$

The second similar wave pulse is emitted after a time period $T = \frac{1}{f}$



The distance moved by source in this time $\frac{1}{2}aT^2$

The separation between second wave pulse and source is

$$x - \frac{1}{2}aT^2$$

Time taken by second wave pulse to reach detector $= \dfrac{x - \frac{1}{2}aT^2}{v}$

The time when the second wave pulse reaches detector.

$$t_2 = T + \frac{x - \frac{1}{2}aT^2}{v}$$

Time interval between successive wave pulse,

$$\Delta t = t_2 - t_1 = T - \frac{1}{2}\frac{a}{v}T^2$$

Hence, frequency received by detector,

$$f_{app} = \frac{1}{\Delta t} = \frac{1}{T - \frac{1}{2}\frac{aT^2}{v}} = \frac{2vf^2}{2vf - a}$$

EXAMPLE 8.11

A 3 m long organ pipe open at both ends is driven to third harmonic standing wave. If the amplitude of pressure oscillations is 1 per cent of mean atmospheric pressure ($p_0 = 10^5$ Nm^2). Find the amplitude of particle displacement and density oscillations. Speed of sound $v = 332$ m/s and density of air $\rho = 1.03$ kg/m^3.

Sol. Given length of pipe, $l = 3$m.

Third harmonic implies that $3\left(\frac{\lambda}{2}\right) = l$

or $\quad \lambda = \frac{2l}{3} = \frac{2\times 3}{3} = 2$ m

The angular frequency is $\omega = 2\pi f = \frac{2\pi v}{\lambda} = \frac{(2\pi)(332)}{2}$

or $\quad \omega = 332\pi$ rad/s

The particle displacement y(x,t) can be written as

$$y(x,t) = A\cos kx \sin \omega t$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{(2l/3)} = \frac{3\pi}{l}$$

$$\omega = kv = \frac{3\pi v}{l} \quad \left(v = \frac{\omega}{k}\right)$$

$$y(x,t) = A\cos\left(\frac{3\pi x}{l}\right)\cdot\sin\left(\frac{3\pi v}{l}\right)t$$

The longitudinal oscillations of an air column can be viewed as oscillations of particle displacement or pressure wave or density wave. Pressure variation is related to particle displacement as

$$P(x,t) = -B\frac{\partial y}{\partial x} \quad (B = \text{bulk modulus})$$

$$= \left(\frac{3BAx}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi v}{l}\right)t$$

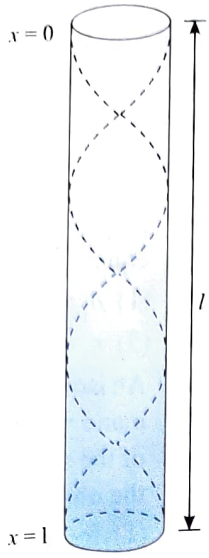
The amplitude of pressure variation is

$$P_{max} = \frac{3BA\pi}{l} \Rightarrow v = \sqrt{\frac{B}{\rho}} \quad \text{or} \quad B = \rho v^2$$

$$P_{max} = \frac{3\rho v^2 A\pi}{l}$$

$$A = \frac{P_{max} l}{3\rho v^2 \pi}$$

Here, $p_{max} = 1\%$



or $p_0 = 10^3 \text{ N/m}^2$

Substituting the values, $A = \frac{(10^3)(3)}{(3)(1.03)(332)^2\pi} = 0.0028 \text{ m}$

or $A = 0.28 \text{ m}$

According to definition of Bulk modulus (B)

$$B = \frac{-dP}{(dV/V)} \qquad \ldots(i)$$

$$\text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

or $V = \frac{m}{\rho}$

or $dV = -\frac{m}{\rho^2}d\rho = -\frac{Vd\rho}{\rho}$

or $\frac{dV}{V} = -\frac{d\rho}{\rho}$

Substituting in equation (i), we get $d\rho = \frac{\rho(dP)}{B}$

or amplitude of density oscillation is $d\rho_{max} = \frac{\rho}{B}p_{max} = \frac{p_{max}}{v^2}$

$$\left(\frac{\rho}{B} = \frac{1}{v^2}\right)$$

$$d\rho_{max} = \frac{10^3}{(332)^2} = 9\times10^{-3} \text{ kg/m}^3$$

Exercises

Single Correct Answer Type

1. In expressing sound intensity, we take 10^{-12} W/m^2 as the reference level. For ordinary conversation, the intensity level is about 10^{-6} W/m^2. Expressed in decibel, this is

(1) 10^6 (2) 6

(3) 60 (4) $\log_e(10^6)$

2. The intensity level of two sounds are 100 dB and 50 dB. What is the ratio of their intensities?

(1) 10^1 (2) 10^3

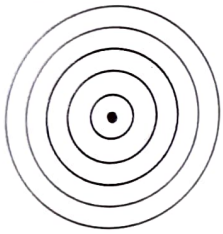
(3) 10^5 (4) 10^{10}

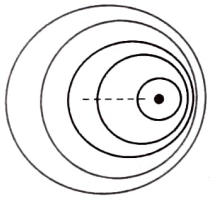
3. When a person wears a hearing aid, the sound intensity level increases by 30 dB. The sound intensity increases by

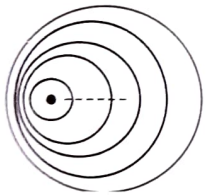
(1) e^3 (2) 10^3

(3) 30 (4) 10^2

4. If the source is moving towards right; wavefront of sound waves get modified to

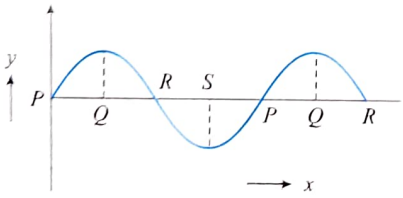
(1) 

(2) 

(3) 

(4) None of these

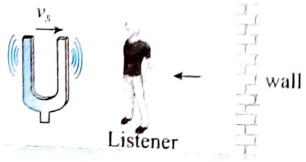
5. Given figure represents the displacement y versus distance x along the direction of propagation of a longitudinal wave. The pressure is maximum at position marked



(1) P (2) Q

(3) R (4) S

6. A tuning fork of frequency 380 Hz is moving towards a wall with a velocity of 4 m/s. Then the number of beats heard by a stationary listener between direct and reflected sounds will be (velocity of sound in air is 340 m/s)



(1) 0 (2) 5

(3) 7 (4) 10

7. A vehicle, with a horn of frequency n, is moving with a velocity of 30 m/s in a direction perpendicular to the straight line joining the observer and the vehicle. The observer perceives the sound to have a frequency $n + n_1$. Then n_1 is equal to (take velocity of sound in air as 330 m/s)

(1) $n_1 = 10n$ (2) $n_1 = -n$

(3) $n_1 = 0.1n$ (4) $n_1 = 0$

8. An isotropic stationary source is emitting waves of frequency n and wind is blowing due north. An observer A is on north of the source while observer B is on south the source. If both the observers are stationary, then

(1) frequency received by A is greater than n

(2) frequency received by B is less than n

(3) frequency received by A equals to that received by B

(4) frequencies received by A and B cannot be calculated unless velocity of waves in still air and velocity of wind are known

9. A train is moving with a constant speed along a circular track. The engine of the train emits a sound of frequency f. The frequency heard by the guard at the rear end of the train

(1) is less than f

(2) is greater than f

(3) is equal to f

(4) may be greater than, less than or equal to f depending on factors like speed of train, length of train and radius of circular track

10. An observer moves towards a stationary source of sound with a velocity one-fifth of the velocity of sound. What is the percentage increase in the apparent frequency?

(1) 5% (2) 20%

(3) 0% (4) 0.5%

11. An engine running at speed $v/10$ sounds a whistle of frequency 600 Hz. A passenger in a train coming from the opposite side at speed $v/15$ experiences this whistle to be of frequency f. If v is speed of sound in air and there is no wind, f is nearest to

(1) 711 Hz (2) 630 Hz

(3) 580 Hz (4) 510 Hz

12. A source of sound produces waves of wavelength 60 cm when it is stationary. If the speed of sound in air is 320 m/s and source moves with speed 20 m/s, the wavelength of sound in the forward direction will be nearest to

(1) 56 cm (2) 60 cm

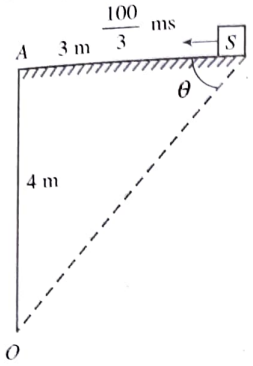
(3) 64 cm (4) 68 cm

13. The apparent frequency of the whistle of an engine changes in the ratio of 6:5 as the engine passes a stationary observer. If the velocity of sound is 330 m/s, then the velocity of the engine is

(1) 3 m/s (2) 30 m/s

(3) 0.33 m/s (4) 660 m/s

4. A source of sound S is travelling at 100/3 m/s along a road, towards a point A. When the source is 3 m away from A, a person standing at a point O on a road perpendicular to AS hears a sound of frequency ν'. The distance of O from A at that time is 4 m. If the original frequency is 640 Hz, then the value of ν' is (velocity of sound is 340 m/s)



(1) 620 Hz (2) 680 Hz
(3) 720 Hz (4) 840 Hz

5. A source of sound is travelling with a velocity of 30 m/s towards a stationary observer. If actual frequency of source is 1000 Hz and the wind is blowing with velocity 20 m/s in a direction at 60° with the direction of motion of source, then the apparent frequency heard by observer is (speed of sound is 340 m/s)
(1) 1011 Hz (2) 1000 Hz
(3) 1094 Hz (4) 1086 Hz

6. A man is watching two trains, one leaving and the other coming in with equal speed of 4 m/s. If they sound their whistles, each of frequency 240 Hz, the number of beats heard by the man (velocity of sound in air is 320 m/s) will be equal to
(1) 6 (2) 3
(3) 0 (4) 12

7. The intensity of a sound wave gets reduced by 20% on passing through a slab. The reduction in intensity on passage through two such consecutive slabs is
(1) 40% (2) 36 %
(3) 30 % (4) 50%

8. Two factories are sounding their sirens at 800 Hz. A man goes from one factory to the other at a speed of 2 m/s. The velocity of sound is 320 m/s. The number of beats heard by the person in 1 s will be
(1) 2 (2) 4
(3) 8 (4) 10

9. Two sources A and B are sounding notes of frequency 680 Hz. A listener moves from A to B with a constant velocity u. If the speed of sound is 340 m/s, what must be the value of u so that he hears 10 beats per second?
(1) 2.0 m/s (2) 2.5 m/s
(3) 30 m/s (4) 3.5 m/s

10. One train is approaching an observer at rest and another train is receding from him with the same velocity 4 m/s. Both the trains blow whistles of same frequency of 243 Hz. The beat frequency in Hz as heard by the observer is (speed of sound in air is 320 m/s)
(1) 10 (2) 6
(3) 4 (4) 1

21. Two sound sources are moving in opposite directions with velocities v_1 and v_2 ($v_1 > v_2$). Both are moving away from a stationary observer. The frequency of both the sources is 900 Hz. What is the value of $v_1 - v_2$ so that the beat frequency observed by the observer is 6 Hz? Speed of sound $v = 300$ m/s. Given that v_1 and $v_2 \ll v$.
(1) 1 m/s (2) 2 m/s
(3) 3 m/s (4) 4 m/s

22. The frequency changes by 10% as the source approaches a stationary observer with constant speed v_s. What should be the percentage change in frequency as the source recedes from the observer with the same speed? Given that $v_s \ll v$ (v is the speed of sound in air).
(1) 14.3 % (2) 20%
(3) 16.7 % (4) 10%

23. Speed of sound wave is v. If a reflector moves towards a stationary source emitting waves of frequency f with speed u, the frequency of reflected waves will be
(1) $\frac{v-u}{v+u}f$ (2) $\frac{v+u}{v}f$
(3) $\frac{v+u}{v-u}f$ (4) $\frac{v-u}{v}f$

24. An observer moves towards a stationary source of sound with a speed (1/5)th of the speed of sound. The wavelength and frequency of the source emitted are λ and f, respectively. The apparent frequency and wavelength recorded by the observer are, respectively,
(1) $1.2f$ and λ
(2) f and 1.2λ
(3) $0.8f$ and 0.8λ
(4) $1.2f$ and 1.2λ

25. A source of sound S is moving with a velocity 50 m/s towards a stationary observer. He measures the frequency of the source as 1000 Hz. What will be the apparent frequency of the sound when it is moving away from the observer after crossing him? The velocity of the sound in the medium is 350 m/s.
(1) 750 Hz (2) 857 Hz
(3) 1143 Hz (4) 1333 Hz

26. A whistle emitting a sound of frequency 440 Hz is tied to a string of 1.5 m length and rotated with an angular velocity of 20 rad/s in the horizontal plane. Then the range of frequencies heard by an observer stationed at a large distance from the whistle will be (v = 330 m/s)
(1) 400.0 Hz to 484.0 Hz
(2) 403.3 Hz to 480.0 Hz
(3) 400.0 Hz to 480.0 Hz
(4) 403.3 Hz to 484.0 Hz

27. A man standing on a platform hears the sound of frequency 605 Hz coming from a frequency 550 Hz from a train whistle moving towards the platform. If the velocity of sound is 330 m/s, then what is the speed of train?

(1) 30 m/s (2) 35 m/s

(3) 40 m/s (4) 45 m/s

28. A source of sound emits 200π W power which is uniformly distributed over a sphere of radius 10 m. What is the loudness of sound on the surface of the sphere?

(1) 70 dB (2) 107 dB

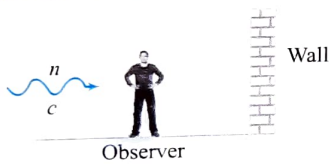
(3) 80 dB (4) 117 dB

29. Two cars are moving on two perpendicular roads towards a crossing with uniform speeds of 72 km/h and 36 km/h. If second car blows horn of frequency 280 Hz, then the frequency of horn heard by the driver of first car when the line joining the cars makes angle of 45° with the roads, will be (velocity of sound is 330 m/s)

(1) 321 Hz (2) 298 Hz

(3) 289 Hz (4) 280 Hz

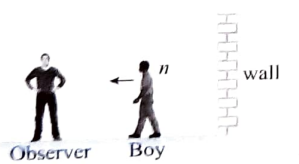
30. A sound wave of frequency n travels horizontally to the right with speed c. It is reflected from a broad wall moving to the left with speed v. The number of beats heard by a stationary observer to the left of the wall is



(1) zero (2) $\frac{n(c+v)}{c-v}$

(3) $\frac{nv}{c-v}$ (4) $\frac{2nv}{c-v}$

31. A boy is walking away from a wall at a speed of 1.0 m/s in a direction at right angles to the wall. The boy blows a whistle steadily. An observer towards whom the boy is moving hears 4 beats/s. If the speed of sound is 340 m/s, the frequency of whistle is



(1) 480 Hz (2) 680 Hz

(3) 840 Hz (4) 1000 Hz

32. Due to a point isotropic sonic source, loudness at a point is $L = 60$ dB. If density of air is $\rho = (15/11)$ kg/m^3 and velocity of sound in air is $v = 33$ m/s, the pressure oscillation amplitude at the point of observation is [$I_0 = 10^{-12}$ W/m^2]

(1) 0.3 N/m^2 (2) 0.03 N/m^2

(3) 3×10^{-3} N/m^2 (4) 3×10^{-4} N/m^2

33. The frequency of a car horn is 400 Hz. If the horn is honked as the car moves with a speed $u_s = 34$ m/s through still air towards a stationary receiver, the wavelength of the sound passing the receiver is [velocity of sound is 340 m/s]

(1) 0.765 m (2) 0.850 m

(3) 0.935 m (4) 0.425 m

34. Spherical sound waves are emitted uniformly in all directions from a point source. The variation in sound level SL as a function of distance 'r' from the source can be written as

(1) SL = $-b \log r^a$ (2) SL = $a - b (\log r)^2$

(3) SL = $a - b \log r$ (4) SL = $a - b/r^2$

where a and b are positive constants.

35. A motorcycle starts from rest and accelerates along a straight line at 2.2 m/s^2. The speed of sound is 330 m/s. A siren at the starting point remains stationary. When the driver hears the frequency of the siren at 90% of when the motorcycle is stationary, the distance travelled by the motorcyclist is

(1) 123.75 m (2) 247.5 m

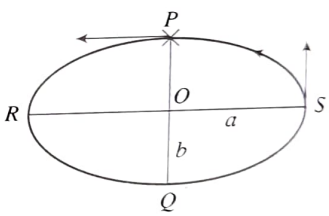
(3) 495 m (4) 990 m

36. The difference in the speeds of sound in air at –5°C, 60 cm pressure of mercury and 30°C, 75 cm pressure of mercury is (velocity of sound in air at 0°C is 332 m/s)

(1) 15.25 m/s (2) 21.35 m/s

(3) 18.3 m/s (4) 3.05 m/s

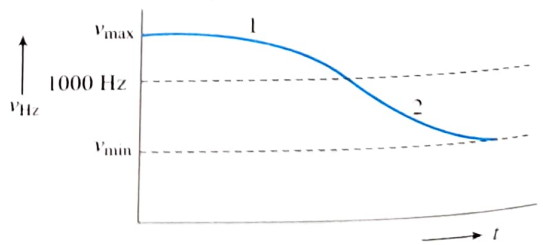
37. A train is moving in an elliptical orbit in anticlockwise sense with a speed of 110 m/s. Guard is also moving in the given direction with same speed as that of train. The ratio of the length of major and minor axes is 4/3. Driver blows a whistle of 1900 Hz at P, which is received by guard at S. The frequency received by guard is (velocity of sound v = 330 m/s)



(1) 1900 Hz (2) 1800 Hz

(3) 2000 Hz (4) 1500 Hz

38. A stationary observer receives a sound from a sound of frequency ν_0 moving with a constant velocity $v_s = 30$ m/s. The apparent frequency varies with time as shown in figure. Velocity of sound $v = 300$ m/s. Then which of the following is incorrect?



(1) The minimum value of apparent frequency is 889 Hz.

(2) The natural frequency of source is 1000 Hz.

(3) The frequency–time curve corresponds to a source moving at an angle to the stationary observer.

(4) The maximum value of apparent frequency is 1111 Hz.

40. The sound from a very high burst of fireworks takes 5 s to arrive at the observer. The burst occurs 1662 m above the observer and travels vertically through two stratifier layers of air, the top one of thickness d_1 at 0°C and the bottom one of thickness d_2 at 20°C. Then (assume velocity of sound at 0°C is 330 m/s)

(1) $d_1 = 342$ m (2) $d_2 = 1320$ m

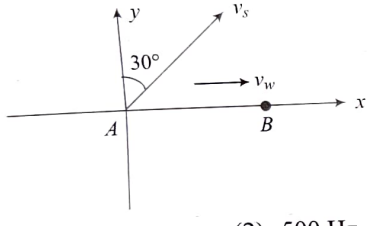
(3) $d_1 = 1485$ m (4) $d_2 = 342$ m

41. A car emitting sound of frequency 500 Hz speeds towards a fixed wall at 4 m/s. An observer in the car hears both the source frequency as well as the frequency of sound reflected from the wall. If he hears 10 beats per second between the two sounds, the velocity of sound in air will be

(1) 330 m/s (2) 387 m/s

(3) 404 m/s (4) 340 m/s

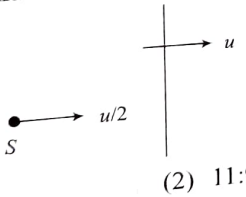
42. In the figure shown, a source of sound of frequency 510 Hz moves with constant velocity $v_s = 20$ m/s in the direction shown. The wind is blowing at a constant velocity $v_w = 20$ m/s towards an observer who is at rest at point B. Corresponding to the sound emitted by the source at initial position A, the frequency detected by the observer is equal to (speed of sound relative to air is 330 m/s)



(1) 510 Hz (2) 500 Hz

(3) 525 Hz (4) 550 Hz

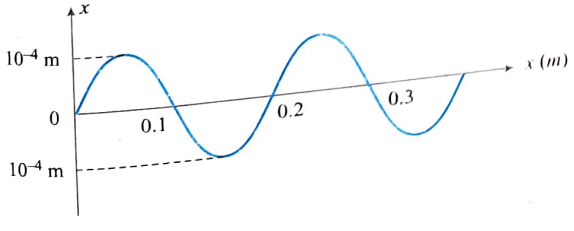
43. A wall is moving with velocity u and a source of sound moves with velocity $u/2$ in the same direction as shown in the figure. Assuming that the sound travels with velocity $10u$, the ratio of incident sound wavelength on the wall to the reflected sound wavelength by the wall is equal to

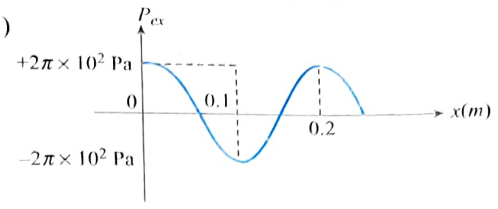


(1) 9:11 (2) 11:9

(3) 4:5 (4) 5:4

44. For a sound wave travelling towards $+x$ direction, sinusoidal longitudinal displacement ξ at a certain time is given as a function of x (see figure). If bulk modulus of air is $B = 5 \times 10^5$ N/m^2, the variation of pressure excess will be



(1)


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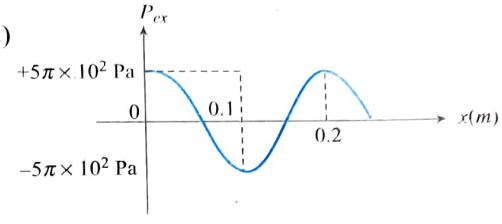

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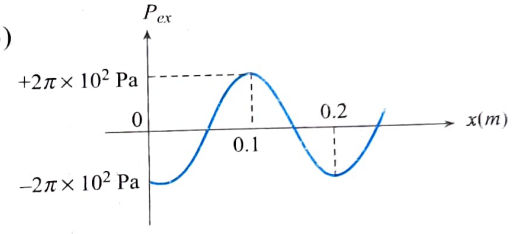

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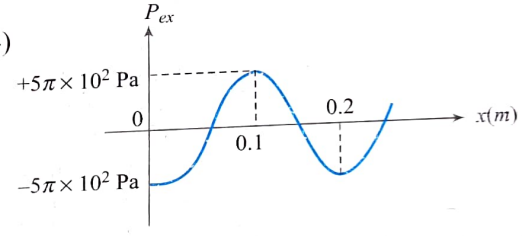


44. Consider a source of sound S, and an observer/detector D. The source emits a sound wave of frequency f_0. The frequency observed by D is found to be

(i) f_1, if D approaches S and S is stationary

(ii) f_2, if S approaches D and D is stationary

(iii) f_3, if both S and D approach each other with the same speed.

In all three cases, relative velocity of S wrt D is the same. For this situation which is incorrect?

(1) $f_1 \neq f_2 \neq f_3$ (2) $f_1 < f_2$

(3) $f_3 < f_0$ (4) $f_1 < f_3 < f_2$

45. When source and detector are stationary but the wind is blowing at speed v_w, the apparent wavelength λ' on the wind side is related to actual wavelength λ by [take speed of sound in air as v]

(1) $\lambda' = \lambda$ (2) $\lambda' = \frac{v_w}{v}\lambda$

(3) $\lambda' = \frac{v_w + v}{v}\lambda$ (4) $\lambda' = \frac{v}{v - v_w}\lambda$

46. The driver of a car approaching a vertical wall notices that the frequency of the horn of his car changes from 400 Hz to 450 Hz after being reflected from the wall. Assuming speed

of sound to be 340 m/s, the speed of approach of car towards the wall is

(1) 10 m/s (2) 20 m/s
(3) 30 m/s (4) 40 m/s

47. The difference between the apparent frequencies of a source of sound as perceived by a stationary observer during its approach and recession is 2% of the actual frequency of the source. If the speed of sound is 300 m/s, the speed of source is

(1) 1.5 m/s (2) 3 m/s
(3) 6 m/s (4) 12 m/s

48. The frequency of a radar is 780 MHz. After getting reflected from an approaching aeroplane, the apparent frequency is more than the actual frequency by 2.6 kHz. The aeroplane has a speed of

(1) 2 km/s (2) 1 km/s
(3) 0.5 km/s (4) 0.25 km/s

49. A train moves towards a stationary observer with a speed of 34 m/s. The train sounds a whistle and its frequency registered by the observer is f_1. If the train's speed is reduced to 17 m/s, the frequency registered is f_2. If the speed of sound is 340 m/s, then the ratio f_1/f_2 is

(1) 18/19 (2) 1/2
(3) 2 (4) 19/18

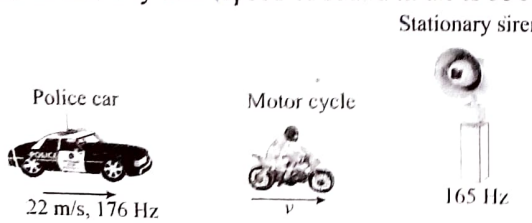
50. A siren placed at a railway platform is emitting sound of frequency 5 kHz. A passenger sitting in a moving train A records a frequency of 5.5 kHz while the train approaches the siren. During his return journey in a different train B he records a frequency of 6.0 kHz while approaching the same siren. The ratio of velocity of train B to that of train A is

(1) 242/252 (2) 2
(3) 5/6 (4) 11/6

51. A person speaking normally produces a sound of intensity 40 dB at a distance of 1 m. If the threshold intensity for reasonable audibility is 20 dB, the maximum distance at which he can be heard clearly is

(1) 4 m (2) 5 m
(3) 10 m (4) 20 m

52. A police car moving at 22 m/s chases a motorcyclist. The police man sounds his horn of frequency 176 Hz, while both of them move towards a stationary siren of frequency 165 Hz. Calculate the speed of motorcyclist if it is given that he does not hear any beat (speed of sound in air is 330 m/s).



(1) 33 m/s (2) 22 m/s
(3) 11 m/s (4) zero

53. In sports meet the timing of a 200 m straight dash is recorded at the finish point by starting an accurate stop watch on hearing the sound of starting gun fired at the starting point. The time recorded will be more accurate

(1) in winter (2) in summer
(3) in all seasons (4) none of these

54. When a source moves away from a stationary observer, th frequency is 6/7 times the original frequency. Given: spee of sound = 330 m/s. The speed of the source is

(1) 40 m/s (2) 55 m/s
(3) 330 m/s (4) 165 m/s

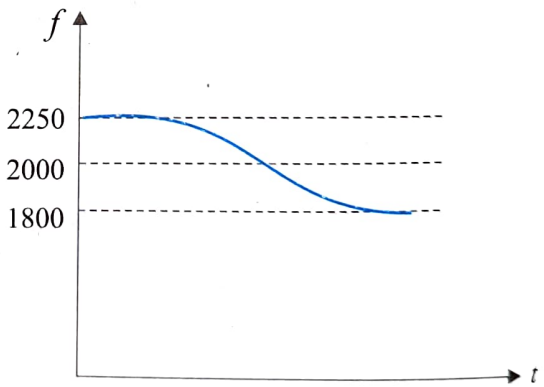
55. Length of a string of density ρ and Young's modulus Y unde tension is increased by $1/n$ times of its original length. I the velocity of transverse and longitudinal vibrations of th string is same, find the value of such velocity.

(1) $\sqrt{\frac{Y}{\rho n}}$ (2) $\sqrt{\frac{Y}{\rho n^{1/2}}}$
(3) $\sqrt{\frac{Y}{\rho}}$ (4) $\sqrt{\frac{Y}{\rho n^{3/2}}}$

56. Source and observer start moving simultaneously along x and y-axis, respectively. The speed of source is twice the speed of observer, V_0. If the ratio of observed frequency to the frequency of the source is 0.75, find the velocity of sound.

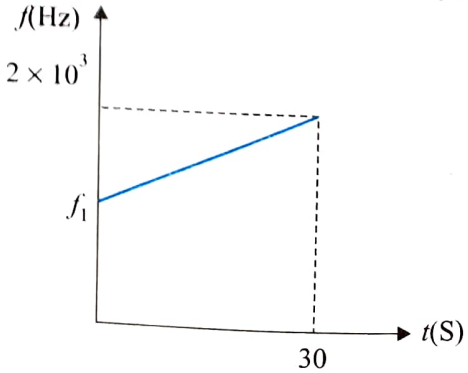
(1) $\frac{11}{\sqrt{5}}V_0$ (2) $\frac{17}{\sqrt{5}}V_0$
(3) $\frac{16}{\sqrt{5}}V_0$ (4) $\frac{19}{\sqrt{5}}V_0$

57. A stationary observer receives a sound of frequency 2000 Hz. The variation of apparent frequency and time is shown. Find the speed of source; if velocity of sound is 300 m/s:



(1) 66.6 m/s (2) 33.3 m/s
(3) 27.3 m/s (4) 59.3 m/s

58. A source of sound of frequency f_1 is placed on the ground. A detector placed at a height is released from rest on this source. The observed frequency f(Hz) is plotted against time t(sec). The speed of sound in air is 300 m/s. Find f_1 (g = 10 m/s^2).



(1) 0.5×10^3 Hz (2) 1×10^3 Hz
(3) 0.25×10^3 (4) 0.2×10^3 Hz

A sound wave of frequency f travels horizontally to the right. It is reflected from a large vertical plane surface moving to left with a speed v. The speed of sound in medium is c. The incorrect statement is

(1) The number of wave striking the surface per second is $f\frac{(c+v)}{c}$.

(2) The wavelength of reflected wave is $\frac{c(c-v)}{f(c+v)}$.

(3) The frequency of reflected wave is $\frac{f(c+v)}{(c-v)}$.

(4) The number of beats heard by a stationary listener to the left to the reflecting surface is $\frac{vf}{(c-v)}$.

Multiple Correct Answers Type

Which of the following statements are correct?

(1) Changes in air temperature have no effect on the speed of sound.

(2) Changes in air pressure have no effect on the speed of sound.

(3) The speed of sound in water is higher than in air.

(4) The speed of sound in water is lower than in air.

An observer A is moving directly towards a stationary sound source while another observer B is moving away from the source with the same velocity. Which of the following statements are correct?

(1) Average of frequencies recorded by A and B is equal to natural frequency of the source.

(2) Wavelength of wave received by A is less than that of waves received by B.

(3) Wavelength of waves received by two observers will be same.

(4) Both the observers will observe the wave travelling with same speed.

Consider a source of sound S and an observer P. The sound source is of frequency n_0. The frequency observed by P is found to be n_1 if P approaches S at speed v and S is stationary; n_2 if S approaches P at a speed v and P is stationary and n_3 if each of P and S has speed $v/2$ towards one another. Now,

(1) $n_1 = n_2 = n_3$

(2) $n_1 < n_2$

(3) $n_3 > n_0$

(4) n_3 lies between n_1 and n_2

Which of the following statements are incorrect?

(1) Wave pulses in strings are transverse waves.

(2) Sound waves in air are transverse waves of compression and rarefaction.

(3) The speed of sound in air at 20°C is twice that at 5°C.

(4) A 60 dB sound has twice the intensity of a 30 dB sound.

5. A source S of sound wave of fixed frequency N and an observer O are located in air initially at the space points A and B, a fixed distance apart. State in which of the following cases, the observer will NOT see any Doppler effect and will receive the same frequency N as produced by the source.

(1) Both the source S and observer O remain stationary but a wind blows with a constant speed in an arbitrary direction.

(2) The observer remains stationary but the source S moves parallel to and in the same direction and with the same speed as the wind.

(3) The source remains stationary but the observer and the wind have the same speed away from the source.

(4) The source and the observer move directly against the wind but both with the same speed.

6. A vibrating tuning fork is first held in the hand and then its end is brought in contact with a table. Which of the following statement(s) is/are correct in respect of this situation?

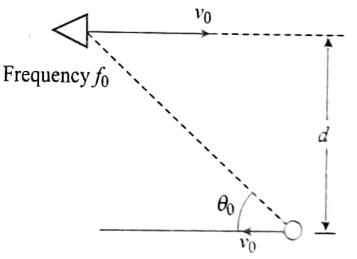
(1) The sound is louder when the tuning fork is held in hand.

(2) The sound is louder when the tuning fork is in contact with table.

(3) The sound dies away sooner when tuning fork is brought in contact with the table.

(4) The sound remains for a longer duration when turning fork is held in hand.

7. A source of sound and detector are moving as shown in the given figure at $t = 0$. Take velocity of sound wave to be v.



For this situation mark out the correct statement(s).

(1) The frequency received by the detector is always greater than f_0.

(2) Initially, frequency received by the detector is greater than f_0, becomes equal to f_0, and then decreases with the time.

(3) Frequency received by the detector is equal to f_0 at $t = d \cot \theta_0/(2\, v_0)$.

(4) Frequency received by the detector can never be equal to f_0.

8. Plane harmonic waves of frequency 500 Hz are produced in air with displacement amplitude of 10 μm. Given that density of air is 1.29 kg/m^3 and speed of sound in air is 340 m/s. Then

(1) the pressure amplitude is 13.8 N/m^2

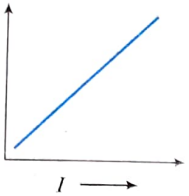
(2) the energy density is 6.4×10^{-4} J/m^3

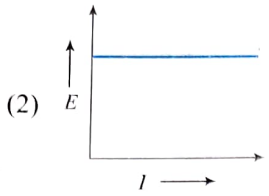
(3) the energy flux is 0.22 J/(m^2 s)

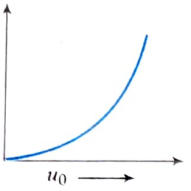
(4) only (a) and (c) are correct

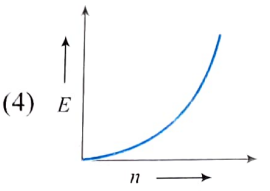
9. A sonic source, located in a uniform medium, emits waves of frequency n. If intensity, energy density (energy per unit

volume of the medium) and maximum speed of oscillations of medium particle are, respectively, I, E and u_0 at a point, then which of the following graphs are correct?

(1) 

(2) 

(3) 

(4) 

10. A driver in a stationary car blows a horn which produces monochromatic sound waves of frequency 1000 Hz normally towards a reflecting wall. The wall approaches the car with a speed of 3.3 m/s.

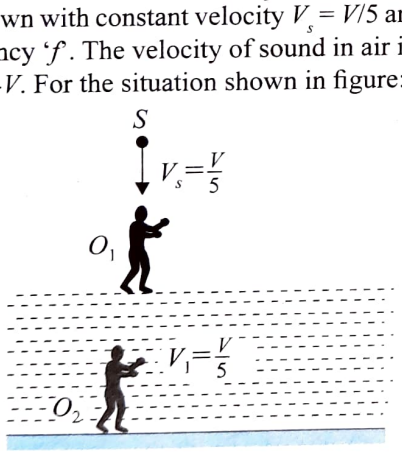
(1) The frequency of sound reflected from wall and heard by the driver is 1020 Hz.

(2) The frequency of sound reflected from wall and heard by the driver is 980 Hz.

(3) The percentage increase in frequency of sound after reflection from wall is 2%.

(4) The percentage decrease in frequency of sound after reflection from wall is 2%.

11. In the figure shown, an observer O_1 floats (static) on water surface with ears in air while another observer O_2 is moving upwards with constant velocity $V_1 = V/5$ in water. The source moves down with constant velocity $V_s = V/5$ and emits sound of frequency 'f'. The velocity of sound in air is V and that in water is $4V$. For the situation shown in figure:



(1) The wavelength of the sound received by O_1 is $\frac{4V}{5f}$

(2) The wavelength of the sound received by O_1 is $\frac{V}{f}$

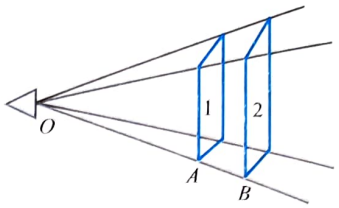
(3) The frequency of the sound received by O_2 is $\frac{21f}{16}$

(4) The wavelength of the sound received by O_2 is $\frac{16V}{5f}$

Linked Comprehension Type

For Problems 1–3

In the figure shown below, a source of sound having power 12×10^{-6} W is kept at O, which is emitting sound waves in the directions as shown. Two surfaces are labelled as 1 and 2 having areas $A_1 = 2 \times 10^3$ m^2 and $A_2 = 4 \times 10^3$ m^2, respectively.



1. Find the intensity at both the surfaces.

(1) $I_1 = 12 \times 10^{-6}$ W/m^2, $I_2 = 12 \times 10^{-6}$ W/m^2

(2) $I_1 = 6 \times 10^{-9}$ W/m^2, $I_2 = 12 \times 10^{-9}$ W/m^2

(3) $I_1 = 6 \times 10^{-9}$ W/m^2, $I_2 = 3 \times 10^{-9}$ W/m^2

(4) $I_1 = 12 \times 10^{-9}$ W/m^2, $I_2 = 3 \times 10^{-9}$ W/m^2

2. If two persons (having almost same physique) A and B are standing at the location of surfaces 1 and 2, respectively, then who will hear a quieter sound?

(1) Both will hear same sound.

(2) A will hear a quieter sound.

(3) B will hear a quieter sound.

(4) Information is not sufficient.

3. Let the areas of the eardrums of persons A and B be $A_A = 2$ mm^2 and $A_B = 4$ mm^2, respectively. Then, who will hear a quieter sound?

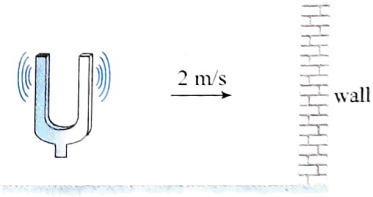
(1) A will hear a quieter sound.

(2) B will hear a quieter sound.

(3) Both will hear the same sound.

(4) Cannot say anything.

For Problems 4–7



As shown in figure a vibrating tuning fork of frequency 512 Hz is moving towards the wall with a speed 2 m/s. Take speed of sound as $v = 340$ m/s and answer the following questions.

4. Suppose that a listener is located at rest between the tuning fork and the wall. Number of beats heard by the listener per second will be

(1) 4 (2) 3

(3) 0 (4) 1

5. If the listener is at rest and located such that the tuning fork is moving between the listener and the wall, number of beats heard by the listener per second will be nearly

(1) 0 (2) 6

(3) 8 (4) 4

If the listener, along with the source, is moving towards the wall with the same speed, i.e., 2 m/s, such that the source remains between the listener and the wall, number of beats heard by the listener per second will be

(1) 4 (2) 8

(3) 0 (4) 6

If the listener, along with the source, is moving towards the wall with the same speed, i.e., 2 m/s, such that he (listener) remains between the source and the wall, number of beats heard by him will be

(1) 2 (2) 6

(3) 8 (4) 4

Problems 8–10

A source of sonic oscillations with frequency $n_0 = 600$ Hz moves away and at right angles to a wall with velocity 2 m/s. A stationary receiver is located on the line of source recession wall → source → receiver. If velocity of sound propagation is $v = 330$ m/s, then

8. The beat frequency recorded by the receiver is

(1) 110 Hz (2) 210 Hz

(3) 150 Hz (4) 220 Hz

9. The wavelength of direct waves received by the receiver is

(1) 50 cm (2) 100 cm

(3) 150 cm (4) 90 cm

10. The wavelength of reflected waves received by the receiver is

(1) 120 cm (2) 50 cm

(3) 90 cm (4) 60 cm

Problems 11–13

A small source of sound vibrating at frequency 500 Hz is rotated in a circle of radius $100/\pi$ cm at a constant angular speed of 5.0 revolutions per second. The speed of sound in air is 330 m/s.

11. For an observer situated at a great distance on a straight line perpendicular to the plane of the circle, through its centre, the apparent frequency of the source will be

(1) greater than 500 Hz

(2) smaller than 500 Hz

(3) always remain 500 Hz

(4) greater for half the circle and smaller during the other half

12. For an observer who is at rest at a great distance from the centre of the circle but nearly in the same plane, the minimum f_{min} and the maximum f_{max} of the range of values of the apparent frequency heard by him will be

(1) $f_{min} = 455$ Hz, $f_{max} = 535$ Hz

(2) $f_{min} = 485$ Hz, $f_{max} = 515$ Hz

(3) $f_{min} = 485$ Hz, $f_{max} = 500$ Hz

(4) $f_{min} = 500$ Hz, $f_{max} = 515$ Hz

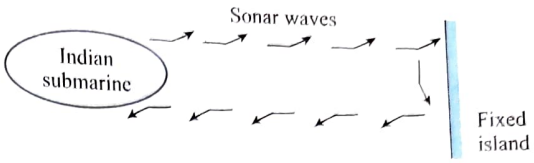
13. If the observer moves towards the source with a constant speed of 20 m/s, along the radial line to the centre, the fractional change in the apparent frequency over the frequency that the source will have if considered at rest at the centre will be

(1) 6% (2) 3%

(3) 2% (4) 9%

For Problems 14–17

An Indian submarine is moving in the Arabian Sea with a constant velocity. To detect enemy it sends out sonar waves which travel with velocity 1050 m/s in water. Initially the waves are getting reflected from a fixed island and the reflected waves are coming back to submarine. The frequency of reflected waves are detected by the submarine and found to be 10% greater than the sent waves.



Now an enemy ship comes in front, due to which the frequency of reflected waves detected by submarine becomes 21% greater than the sent waves.

14. The speed of Indian submarine is

(1) 10 m/s (2) 50 m/s

(3) 100 m/s (4) 20 m/s

15. The velocity of enemy ship should be

(1) 50 m/s towards Indian submarine

(2) 50 m/s away from Indian submarine

(3) 100 m/s towards Indian submarine

(4) 100 m/s away from Indian submarine

16. If the wavelength received by enemy ship is λ' and wavelength of reflected waves received by submarine is λ'', then (λ'/λ'') equals

(1) 1 (2) 1.1

(3) 1.2 (4) 2

17. Bulk modulus of sea water should be approximately $(\rho_{water} = 1000 \text{ kg/m}^3)$

(1) 10^8 N/m^2 (2) 10^9 N/m^2

(3) 10^{10} N/m^2 (4) 10^{11} N/m^2

For Problems 18–19

Due to a point isotropic sound source, the intensity at a point is observed as 40 dB. The density of air is $\rho = (15/11)$ kg/m^3 and velocity of sound in air is 330 m/s. Based on this information answer the following questions.

18. The pressure amplitude at the observation point is

(1) 3 N/m^2

(2) 3×10^3 N/m^2

(3) 3×10^{-3} N/m^2

(4) 6×10^{-2} N/m^2

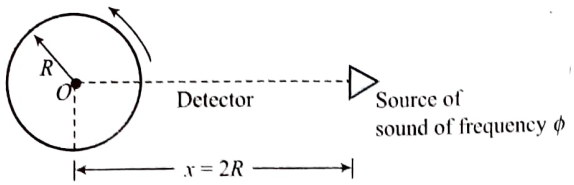
19. The ratio of displacement amplitude of wave at observation point to wavelength of sound waves is

(1) 3.22×10^{-6} (2) 3.22×10^{-12}

(3) 3.22×10^{-9} (4) 1.07×10^{-10}

For Problems 20–22

A source of sound and detector are arranged as shown in figure. The detector is moving along a circle with constant angular speed ω. It starts from the shown location in anti-clockwise direction at $t = 0$.



(Take velocity of sound in air as v.)

Based on this information answer the following questions:

20. What is the frequency as received by detector, when it rotates by an angle $\pi/2$?

(1) f (2) $\dfrac{v-\omega R}{v}\times f$

(3) $\dfrac{v-\omega R/2}{v}\times f$ (4) $\dfrac{v-\omega R\times 2/\sqrt{5}}{v}\times f$

21. Find the time at which the detector will hear the maximum frequency for the first time.

(1) $\pi/(3\omega)$ (2) $5\pi/(3\omega)$

(3) $4\pi/(3\omega)$ (4) π/ω

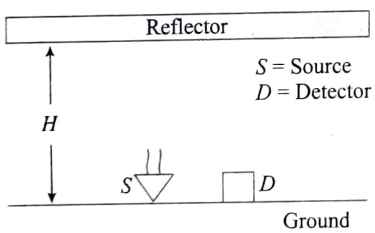
22. Find the time interval between minimum and maximum frequency as received by the detector.

(1) $\pi/(3\omega)$ (2) $5\pi/(3\omega)$

(3) $4\pi/(3\omega)$ (4) π/ω

For Problems 23–25

A source of sound and a detector are placed at the same place on ground. At $t = 0$, the source S is projected towards reflector with velocity v_0 in vertical upward direction and reflector starts moving down with constant velocity v_0. At $t = 0$, the vertical separation between the reflector and source is $H(>v_0^2/2g)$. The speed of sound in air is v ($\gg v_0$). Take f_0 as the frequency of sound waves emitted by source.



Based on above information answer the following questions:

23. Frequency of sound waves emitted by source at $t = v_0/2g$ is

(1) f_0 (2) $f_0\left[\dfrac{v}{v+\dfrac{v_0}{2}}\right]$

(3) $f_0\left[\dfrac{v-v_0/2}{v}\right]$ (4) $f_0\left[\dfrac{v-v_0/2}{v+v_0/2}\right]$

24. Wavelength of sound waves as received by detector before reflection at $t = v_0/2g$ is

(1) $\dfrac{v}{f_0}$ (2) $\dfrac{v+v_0/2}{f_0}$

(3) $\dfrac{(v-v_0/2)^2}{vf_0}$ (4) None of these

25. Frequency of sound received by detector after being reflected by reflector at $t = v_0/2g$ is

(1) $\dfrac{2f_0(v+v_0)}{2v-v_0}$ (2) $\dfrac{2f_0v}{v-v_0}$

(3) $2f_0\left[\dfrac{v+v_0}{2v-v_0}\right]\times\left[\dfrac{v}{v-v_0}\right]$ (4) $2f_0\times\dfrac{v+v_0}{v-v_0}$

Matrix Match Type

1. For transverse wave on a string,

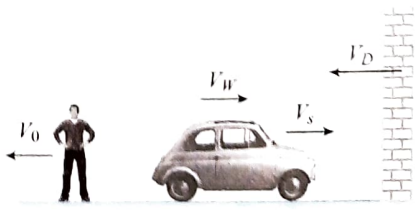
Column I	Column II
i. if amplitude increases,	**a.** maximum instantaneous power increases
ii. if frequency increases,	**b.** average power increases
iii. if amplitude decreases,	**c.** maximum instantaneous power decreases
iv. if frequency decreases,	**d.** average power decreases

2. A loudspeaker diaphragm 0.2 m in diameter is vibrating at 1 kHz with an amplitude of 0.01×10^{-3} m. Assume that the air molecules in the vicinity have the same amplitude of vibration. Density of air is 1.29 kg/m^3. Then match the item given in column I to that in column II. Take velocity of sound = 340 m/s.

Column I	Column II
i. Pressure amplitude immediately in front of the diaphragm (in N/m^2)	**a.** 2.7×10^{-2}
ii. Sound intensity in front of the diaphragm (in W/m^2)	**b.** 2.15×10^{-5}
iii. The acoustic power radiated (in W)	**c.** 27.55
iv. Intensity at 10 m from the loud speaker (in W/m^2)	**d.** 0.865

3. S, O and W represent source of sound (of frequency f), observer and wall, respectively. V_O, V_S, V_D, V are velocity of observer, source, wall and sound (in still air) respectively. V_W is the velocity of wind. They are moving as shown. then match the following

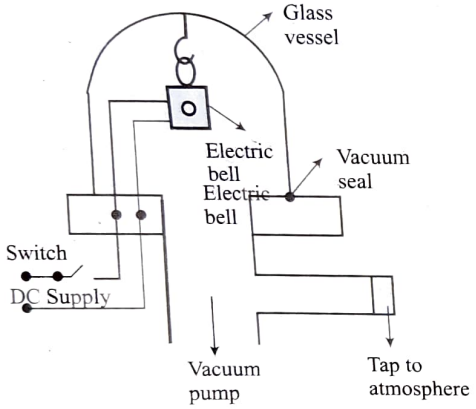
where $f_r = (V + V_W + V_D/V + V_W - V_S)f$



Column I	Column II
i. The wavelength of the waves coming towards the observer from source.	**a.** $(V - V_W - V_D)/f_r$

ii. The wavelength of the waves incident on the wall.	b. $(V - V_W - V_O/V - V_W - V_D) f_r$
iii. The wavelength of the waves coming towards observer from the wall.	c. $(V - V_W + V_S)/f$
iv. Frequency of the waves (as detected by O) coming from wall after reflection.	d. $(V + V_W - V_S)/f$

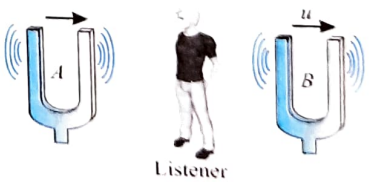
4. The diagram below shows the apparatus which could be used to demonstrate that the transmission of sound wave requires a material medium. The electric bell in the figure consists of a striker and is a steel hemispherical type structure. Vacuum pump is used to create the vacuum in vessel and when the tap to atmosphere is opened the jar will be filled with air. In column I some operations carried out are mentioned, while in column II, the effect, i.e., what is observed and heard along with the conclusions are mentioned. Match the entries of column I with the entries of column II.



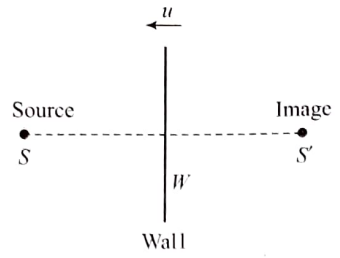
Column I	Column II
i. Switch is closed and tap is opened	a. sound heard
ii. Switch is closed, tap is closed and vacuum pump is ON	b. striker seen vibrating
iii. Switch is closed and vacuum pump is OFF and tap opened	c. sound is not heard
iv. Switch is opened and tap is closed	d. light passes through vacuum but sound cannot

Numerical Value Type

1. The average power transmitted across a cross-section by two sound waves moving in the same direction are equal. The wavelengths of two sound waves are in the ratio of 1 : 2, then find the ratio of their pressure amplitudes.
2. The resultant loudness at a point P is n dB higher than the loudness of S_1 which is one of the two identical sound sources S_1 and S_2 reaching at that point in phase. Find the value of n.
3. The intensity of sound from a point source is 1.0×10^{-8} W/m^2 at a distance of 5.0 m from the source. What will be the intensity at a distance of 25 m from the source? (in $\times 10^{-10}$ W/m^2)
4. If the intensity of sound is doubled, by how many decibels does the sound level increase? (in dB)
5. Two sound sources are moving away from a stationary observer in opposite directions with velocities V_1 and V_2 ($V_1 > V_2$). The frequency of both the sources is 900 Hz. V_1 and V_2 are both quite less than speed of sound, $V = 300$ m/s. Find the value of $(V_1 - V_2)$ so that beat frequency observed by observer is 9 Hz. (in m/s.).
6. Loudness of sound from an isotropic point source at a distance of 70 cm is 20 dB. What is the distance (in m) at which it is not heard.
7. An ambulance sounding a horn of frequency 264 Hz is moving towards a vertical wall with a velocity of 5 ms^{-1}. If the speed of the sound is 330 ms^{-1}, how many beats per second will be heard by an observer standing a few meters behind the ambulance?
8. In a car race sound signals emitted by two cars are detected by the detector on the straight track at the end point of the race. Frequency observed is 330 Hz and 360 Hz and the original frequency is 300 Hz of both cars. Race ends with the separation of 100 m between the cars. Assume both cars move with constant velocity and velocity of sound is 330 m/s. Find the time taken (in sec) by wining car.
9. Airport authority has made the regulations that maximum allowable intensity level detected by a microphone situated at the end of 1630 m long runway can be 100 dB. An aeroplane when flying at a height of 200 m produces an intensity level of 100 dB on ground. While taking off, this aeroplane makes an angle of 30° with horizontal. Find the maximum distance (in m) this aeroplane can cover on the runway, so that the regulations are not violated (assume no reflection).
10. The sound level at a point is increased by 30 dB. What factor is the pressure amplitude increased?
11. Two tuning forks A and B are vibrating at the same frequency 256 Hz. A listener is standing midway between the forks. If both tuning forks move to the right with a velocity of 5 m/s, find the number of beats heard per second by the listener (speed of sound in air is 330 m/s).



12. A driver in a stationary car horns which produces monochromatic sound waves of frequency n = 1000 Hz, normally towards a reflecting wall. If the wall approaches the car will a velocity u = 3.3 m/s, calculate the frequency of sound reflected from wall and heard by the driver. What is the percentage change of sound frequency on reflection from the wall?



13. A source of sound of frequency 256 Hz is moving towards a wall with a velocity of 5 m/s. How many beats per second will be heard by an observer O standing in such a position that the source S is between O and wall? (c = 330 m/s)

14. A vibrating tuning fork tied to the end of a string 1.988 m long is whirled round a circle. If it makes two revolutions in a second, calculate the ratio of the frequencies of the highest and the lowest notes heard by an observer situated in the plane of the tuning fork. Velocity of sound is 350 m/s.

15. A source of sonic oscillations with frequency n = 1700 H and a receiver are located on the same normal to a wall. Bot the source and receiver are stationary, and the wall recede from the source with velocity u = 6.0 cm/s. Find the bea frequency (in Hz) registered by the receiver. The velocity o sound is v = 340 m/s.

16. A locomotive approaching a crossing at a speed of 80 mi/h sounds a whistle of frequency 400 Hz when 1 mi from the crossing. There is no wind, and the speed of sound i air is 0.200 mi/s. What frequency is heard by an observe 0.60 mi from the crossing on the straight road which crosse the railroad at right angles?

17. A whistle of frequency f_0 = 1300 Hz is dropped from a height H = 505 m above the ground. At the same time, a detector is projected upwards with velocity v = 50 ms^{-1} along the same line. If the velocity of sound is v = 300 ms^{-1}, find the frequency (in Hz) detected by the detector after t = 5 s.

Archives

JEE Main

Single Correct Answer Type

1. A motor cycle starts from rest and accelerates along a straight path at 2 m/s^2. At the starting point of the motor cycle, there is a stationary electric siren. How far has the motor cycle gone when the driver hears the frequency of the siren at 94% of its value when the motor cycle was at rest? (Speed of sound is 330 m/s.)

(1) 49 m (2) 98 m
(3) 147 m (4) 196 m

(AIEEE 2009)

2. A train is moving on a straight track with speed 20 ms^{-1}. It is blowing its whistle at the frequency of 1000 Hz. The percentage change in the frequency heard by a person standing near the track as the train passes him is (speed of sound = 320 ms^{-1}) close to:

(1) 6% (2) 12%
(3) 18% (4) 24%

(JEE Main 2015)

JEE Advanced

Single Correct Answer Type

1. Two monatomic ideal gases 1 and 2 of molecular masses m_1 and m_2, respectively, are enclosed in separate containers kept at the same temperature. The ratio of the speed of sound in gas 1 to that in gas 2 is given by

(1) $\sqrt{\frac{m_1}{m_2}}$ (2) $\sqrt{\frac{m_2}{m_1}}$
(3) $\frac{m_1}{m_2}$ (4) $\frac{m_2}{m_1}$

(IIT-JEE 2010)

2. A police car with a siren of frequency 8 kHz is moving with uniform velocity 36 km/h towards a tall building which reflects the sound waves. The speed of sound in air is 320 m/s. The frequency of the siren heard by the car driver is

(1) 8.50 kHz (2) 8.25 kHz
(3) 7.75 kHz (4) 7.50 kHz

(IIT-JEE 2011)

Multiple Correct Answers Type

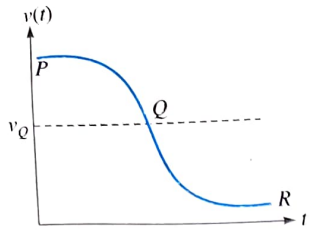
1. Two vehicles, each moving with speed u on the same horizontal straight road, are approaching each other. Wind blows along the road with velocity w. One of these vehicles blows a whistle of frequency f_1. An observer in the other vehicle hears the frequency of the whistle to be f_2. The speed of sound in still air is V. The correct statement(s) is (are)

(1) If the wind blows from the source to the observer, $f_2 > f_1$
(2) If the wind blows from the observer to the source, $f_2 > f_1$
(3) If the wind blows from observer to the source, $f_2 < f_1$
(4) If the wind blows from the source to the observer $f_2 < f_1$

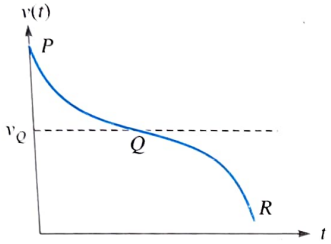
(JEE Advanced 2013)

2. Two loudspeakers M and N are located 20 m apart and emit sound at frequencies 118 Hz and 121 Hz, respectively. A car is initially at a point P, 1800 m away from the midpoint Q of the line MN and moves towards Q constantly at 60 km/hr along the perpendicular bisector of MN. It crosses Q and eventually reaches a point R, 1800 m away from Q. Let $v(t)$ represent the beat frequency measured by a person sitting in the car at time t. Let v_P, v_Q and v_R be the beat frequencies measured at locations P, Q and R, respectively. The speed of sound in air is 330 ms^{-1}. Which of the following statement(s) is (are) true regarding the sound heard by the person?

(1) The plot below represents schematically the variation of beat frequency with time



(2) The plot below represents schematically the variations of beat frequency with time



(3) The rate of change in beat frequency is maximum when the car passes through Q

(4) $v_P + v_R = 2v_Q$ **(JEE Advanced 2016)**

Numerical Value Types

1. A stationary source is emitting sound at a fixed frequency f_0, which is reflected by two cars approaching the source. The difference between the frequencies of sound reflected from the cars is 1.2% of f_0. What is the difference in the speeds of the cars (in km per hour) to the nearest integer? The cars are moving at constant speeds much smaller than the speed of sound which is 330 ms^{-1}. **(IIT-JEE 2010)**

2. A stationary source emits sound of frequency $f_0 = 492$ Hz. The sound is reflected by a large car approaching the source with a speed of 2 ms^{-1}. The reflected signal is received by the source and superposed with the original.

   What will be the beat frequency of the resulting signal in Hz? (Given that the speed of sound in air is 330 ms^{-1} and the car reflects the sound at the frequency it has received).

   (JEE Advanced 2017)

3. Two men are walking along a horizontal straight line in the same direction. The man in front walks at a speed 1.0 ms^{-1} and the man behind walks at a speed 2.0 ms^{-1}. A third man is standing at a height 12 m above the same horizontal line such that all three men are in a vertical plane. The two walking men are blowing identical whistles which emit a sound of frequency 1430 Hz. The speed of sound in air is 330 ms^{-1}. At the instant, when the moving men are 10 m apart, the stationary man is equidistant from them. The frequency of beats in Hz, heard by the stationary man at this instant, is

   (JEE Advanced 2018)

Answers Key

EXERCISES

Single Correct Answer Type

1. (3)	2. (3)	3. (2)	4. (2)	5. (3)
6. (1)	7. (4)	8. (3)	9. (3)	10. (2)
11. (1)	12. (1)	13. (2)	14. (2)	15. (3)
16. (1)	17. (2)	18. (4)	19. (2)	20. (2)
21. (2)	22. (4)	23. (3)	24. (1)	25. (1)
26. (4)	27. (1)	28. (4)	29. (2)	30. (4)
31. (2)	32. (2)	33. (1)	34. (3)	35. (2)
36. (2)	37. (2)	38. (1)	39. (4)	40. (3)
41. (3)	42. (1)	43. (4)	44. (3)	45. (3)
46. (2)	47. (2)	48. (3)	49. (4)	50. (2)
51. (3)	52. (2)	53. (2)	54. (2)	55. (2)
56. (3)	57. (2)	58. (2)	59. (4)	

Multiple Correct Answers Type

1. (2),(3)	2. (1),(3)	3. (2),(3),(4)
4. (2),(3),(4)	5. (1),(4)	6. (2),(3),(4)
7. (2),(3)	8. (1),(2),(3)	9. (1),(3),(4)
10. (1),(3)	11. (1),(3),(4)	

Linked Comprehension Type

1. (3)	2. (3)	3. (3)	4. (3)	5. (2)
6. (4)	7. (2)	8. (1)	9. (1)	10. (4)
11. (3)	12. (2)	13. (1)	14. (2)	15. (1)
16. (2)	17. (2)	18. (3)	19. (3)	20. (4)
21. (2)	22. (3)	23. (1)	24. (2)	25. (3)

Matrix Match Type

1. i. → a., b.; ii. → a., b.; iii. → c., d.; iv. → c., d.
2. i. → c.; ii. → d.; iii. → a.; iv. → b.
3. i. → c.; ii. → d.; iii. → a.; iv. → b.
4. i. → a., b.; ii. → b., c., d.; iii. → a., b.; iv. → c., d.

Numerical Value Type

1. (1)	2. (6)	3. (4)	4. (3)	5. (3)
6. (7)	7. (8)	8. (4)	9. (1230)	10. (32)
11. (8)	12. (2)	13. (7.7)	14. (1.154)	15. (0.6)
16. (442)	17. (1500)			

ARCHIVES

JEE Main

Single Correct Answer Type

1. (2) 2. (2)

JEE Advanced

Single Correct Answer Type

1. (2) 2. (1)

Multiple Correct Answers Type

1. (1),(2) 2. (1),(3),(4)

Numerical Value Type

1. (7) 2. (6) 3. (5)